PRINCIPIORUM
CALCULI DIFFERENTIALIS
ET
INTEGRALIS
EXPOSITIO ELEMENTARIS

AD NORMAM DISSERTATIONIS AB ACADEMIA SCIENT. REG. PRUSSICA ANNO 1786. PRÆMII HONORE DECORATÆ ELABORATA

AUCTORE

SIMONE L'HUILIER

ACADEMIÆ SCIENT. REGIÆ PRUSSICÆ ET SOCIETATIS REGIÆ LONDINENSIS SOCIO ACADEMIÆ IMPERIALIS PETROPOLITANÆ CORRESPONDENTE.

L'Infini est le goufre où se perdent nos pensees.

BAILLY Hist. de l'Astron. mod.

TUBINGÆ

APUD JOH. GEORG. COTTAM.

1795.

ILLUSTRISSIMO EXCELLENTISSIMO

DOMINO

COMITI AB HERZBERG

REGIÆ SCIENTIARUM ARTIUMQUE LIBERALIUM ACADEMIÆ PRUSSICÆ CURATORI
OMNIUM BONARUM LITERARUM FAUTORI ET ARBITRO PERITISSIMO
STUDIOSISSIMO

VIRIS ILLUSTRIBUS DOCTISSIMIS CELEBERRIMIS
REGIÆ SCIENTIARUM ELEGANTIORUMQUE LITERARUM ACADEMIÆ
PRUSSICÆ MEMBRIS

OPUS HOC

QUOD SUFFRAGIIS IPSORUM EXCITATUS ACCURATIUS ELABORARE
EORUMQUE ATTENTIONE DIGNIUS REDDERE
ANNIXUS EST

D. D. D.

AUCTOR.

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INTRODUCTIO.

EXPOSITIO.

Caput

INTRO-

INTRODUCTIO.

Quo amplior eſt ſcientiæ alicujus ambitus, & quo uberiores gravioresque ſunt ejus applicationes; eo magis neceſſarium eſt principiis certis ac dilucidis ipſam ſuperſtruere, & conſequentiarum cum principiis nexum ab omni dubio ac difficultate tutum atque immunem præſtare. Priorem prærogativam doctrinis mathematicis competere in confeſſo eſt; ut altera gaudeant, qui eas meditantur atque exponunt, debent curare. Mathematici veteres, methodi rigoroſæ, qua ipſorum opera nitent, ſedulo tenaces, exemplar nobis præbuerunt modi in ſcientiis his ſtabiliendis atque explicandis adhibendi, ut nomen exactarum mereantur. Recentiores, ſtadium ab antiquis panſum ingreſſi, veſtigiis ipſorum non ſemper inſtiterunt; neque progreſſus ſuos omnino ad normam ab illis conſtitutam compoſuerunt. Speciatim veteres quantitatem ſemper, notioni ipſius conformiter, tanquam augmenti & decrementi capacem, proinde ut ineptam recipiendo ultimo cuidam magnitudinis vel parvitatis termino, conſiderarunt. Recentiores contra, quantitatem in utroque hoc ſtatu extremo ratiociniis & calculis ſubjici poſſe rati, quantitates (etiamnum ſic dictas) infinite magnas & infinite parvas adoptarunt; peculiaremque effinxerunt infiniti ſcientiam, cui partes matheſeos fundamentis ſolidioribus nixæ opitulentur quidem, ſed quæ ab illis ſit tam objecto, quam principiis diverſa: quo audaci conatu doctrinas ab antiquis transmiſſas mirifice auxerunt.

Difficultas, vel impoſſibilitas potius ſtatus illos quantitatis (etiamnum ſic dictæ) definiendi impoſſibilitatem prodit exiſtentiæ eorum & conceptus, quo apprehendantur. Non aggrediar hoc loco definitiones, quæ propoſitæ a variis fuerunt, diſcutere. Interim aſſerere auſim, eas vel contradictiones implicare,

quæ

quæ dubiis obnoxias reddant consequentias ex ipsis deductas; vel adeo vagas esse & indeterminatas, ut juxta eas nonnisi propemodum ac relate ad propositum quendam scopum peculiarem vera essent, quæ per analysin, quam vocant, infinitesimalem eruerentur.

Sententiam hanc de indole statuum illorum quantitatis profiteri mihi visa est illustris Academia scientiarum Berolinensis in Programmate, quo theoriam infiniti mathematici præmio ab se anno 1786. condecorandam promulgavit. Quare ab scopo ipsius alienum haud fore existimavi, si mathesin infinito carere posse ostenderem, vastissimamque ac maxime sublimem ejus partem ad eadem principia reducerem, quibus veterum inventa nituntur; quorum vero vel opem recentiores plerique justo vilius penderunt, vel fœcundati nimium diffisi sunt.

Evincere igitur institui: methodum exhaustionis seu limitum ab antiquis excultam, & in Euclidis præsertim atque Archimedis, quæ ad nos pervenere, scriptis traditam, si congruenter extendatur, novorum calculorum principiis solide ac dilucide stabiliendis sufficere. Honor, quem illustris Academia conatui meo detulit, utilem eum esse mihi persuasit; atque ad objectum hoc arduum novis curis tractandum, Dissertationemque & tempore nimis brevi & loco minus commodo conscriptam perficiendam me exstimulavit.

Scriptum meum, quod novum appellare posse credo, juris publici eo confidentius facio, quod judici non incongruo fuit probatum. Turbas inter satis proh dolor notas, quibus patria mea fuit exagitata, tranquillitate animi ad meditandum necessaria frui non poteram. Amicus meus, Dn. Pfleiderer, Phys. & Math. Prof. in Universitate Tubingensi, refugium mihi secum obtulit: ubi studiis ad levandos dolores intentus, quæ ad novam Dissertationis meæ editionem, dudum necessariam visam, sparsim præparaveram, in ordinem redegi, & cum amico communicavi, ipsiusque observationes consului; quo factum esse spero, ut, quod publico nunc offero, scriptum utile sit nec ejus attentione indignum.

Juxta ejusdem amici per literas antea jam mecum communicatas, nec non Dni. Prevost, Prof. Philos. Genevensis, in Dissertationem meam præmio or-

natam

natam animadverſiones amplius extendi primum ejus Caput; quod, cum fundamenta exhibeat totius ſyſtematis calculorum ſuperiorum, præcipua cura erat tractandum. Speciatim notio limitis ibidem tradita nimis erat anguſta; &, ut omnes complecteretur caſus, debebat, uti nunc fit §. 13, extendi.

Ita factum eſt, ut Caput hoc fundamentale augmentum ſatis inſigne nanciſceretur; neque rationem habendam eſſe exiſtimavi cenſuræ prolixitatis horum præliminarium. Adeo perſuaſum habeo, principia calculorum ſuperiorum debere ad methodum limitum reduci, nec poſſe alio modo ſolide ſtabiliri; ut neceſſarium eſſe putaverim, methodum illam in formam doctrinæ redigere. Quæ cum, mea quidem ſententia, partem conſtituere debeat Curſus matheſeos elementaris, eo, quo oportet, rigore & ambitu concinnati; ſuperſediſſem huic operæ, ſi quem noſſem, ad quem remittere potuiſſem. Ceterum comparatio attenta docebit, nullam in Capite hoc, utut amplo, tradi propoſitionem, quin ad applicationes ſequentes neceſſaria ſit.

Deſiderio ſolo utile quid præſtandi motus non reformido reprehenſionem; quod ſententia, quam ſuſtineo, nova non ſit, ſed ab variis jam mathematicis qropoſita. Gnarus ſcriptorum d'Alembertii, Cousini, Kæstneri, Karstenii, Tempelhoffii, Pasquichii, præſertim diſſertationum egregiarum Robinsii, & tractatus ſolidi Maclaurini, in animum haud induxiſſem meditationes meas de arduo hoc objecto publice exponere; niſi quæſtio ab Societate literaria adeo illuſtri promulgata attentionem meam excitaſſet, mihique perſuaſiſſet, quæ jam præſtita ſint, nondum omnibus deſideriis ſatisfacere ipſi videri.

Dubium non eſt, quin Newtonus doctrinas ſuas iisdem principiis ſuperſtruere, ſpeciatim per rationes primas & ultimas, tam frequenter in Principiis ſuis uſurpatas, limites rationum intelligi voluerit; cum ipſe in Scholio Lemmati XI. Libri I. Princip. ſubjuncto monuerit: *Ultimæ rationes illæ, quibuscum „quantitates evaneſcunt, revera non ſunt rationes quantitatum ultimarum; ſed limites, ad „quos quantitatum ſine limite decreſcentium rationes ſemper appropinquant, & quos pro„pius aſſequi poſſunt quam pro data quavis differentia.*

 Quam-

Quamvis Leibnitius modis loquendi magis audacibus uti affueverit: eos tamen explicatione rigidandos effe præcepit (Act. Erud. Lipf. 1712. p. 167.); & momentum demonftrationum exactarum agnovit atque commendavit. Ad Joh. Bernoullium fcribens (31 Dec. 1700.) de Nieuwentitii, Rollii, Cluverii, & aliorum adverfus calculum differentialem objectionibus: *Perutile* „*eft*, inquit, *os illis occludi per reductionem ad demonftrationes veterum more formatas.* (Leibnitii & Bernoullii Commerc. philos. & math. T. II. Ep. 108.)

Et quamvis Wolfius, veftigiis Leibnitii infiftens, infinitum profufe in fcriptis fuis fparferit; methodo tamen Archimedeæ, quam jure fibi vindicat, prærogativam tribuit: „Ipfius (Archimedis) demonftrandi methodo principia „methodi infinitefimalis rigidantur." (Elem. Mathes. univ. T. I. Geom. Theor. 86. Schol.)

Pluribus aliis auctoritatibus poffem inftitutum meum munire; atque ex. gr. confenfum ejus cum methodo indivifibilium, fano fenfu explicationibusque fagaciffimi ipfius auctoris conformiter intellecta, docere: verum hæc longius me ab fcopo propofito abducerent. Quare, illis miffis, ad partem Introductionis hujus mathematicam, fequentibus quatuor Capitibus traditam, progredior.

Tubingæ, 1 Mart. 1795.

Caput

Caput A.

Theorematis binomialis Newtoni *demonſtratio generalis elementaris.*

Theorema binomiale Newtoni univerſim ad quoscunque exponentes, poſitivos & negativos, integros & fractos, extenſum adeo uberis eſt per totam matheſin uſus, ut demonſtrationem ejus firmam & elementarem ſummi momenti eſſe cenſeam; eoque magis, quod Vir celeberrimus, a quo denominari conſuevit, & plurimi inſignes mathematici poſt ipſum, ſola, ut videtur, analogia fulti, formulam, de exponentibus integris ac poſitivis tantum eo, quo par eſt, rigore demonſtratam, ad exponentes fractos & negativos applicare acquieverunt. In Diſſertatione mea, inſcripta: *Expoſition elementaire des principes des calculs ſuperieurs*, p. 26. ejusmodi demonſtrationis compotem me eſſe aſſerui; ſed brevitatis ſtudio eam omittere conſultum duxi. Quæ cum fuerit, ut non omnino facilis, in Diario litterario Gœttingenſi (Nro. 165. 16 Oct. 1788.) deſiderata; in hac introductione illam exponere non dubitavi. Conſentit ea præter leviores quasdam mutationes & illuſtrationes, quas neceſſarias eſſe cenſui, cum methodo, quam in Commentariis Acad. Reg. Berol. ad annum 1777. tradidit illuſtris Segnerus; &, quo debet, modo evoluta mea quidem ſententia tam propoſito ſatisfacit, quam ea ſe brevitate commendat, cujus demonſtratio mere elementaris capax eſſe poteſt.

§. a. *Theorema.* Sint duæ quantitates P & P', quales ſequuntur

$$P = a^n + \frac{n}{1}a^{n-1}b + \frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2 + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}a^{n-3}b^3 + \frac{n}{1}\ldots\frac{n-3}{4}a^{n-4}b^4 + \frac{n}{1}\ldots\frac{n-4}{5}a^{n-5}b^5 + \ldots$$

$$P' = a^{n'} + \frac{n'}{1}a^{n'-1}b + \frac{n'}{1}\cdot\frac{n'-1}{2}a^{n'-2}b^2 + \frac{n'}{1}\cdot\frac{n'-1}{2}\cdot\frac{n'-2}{3}a^{n'-3}b^3 + \frac{n'}{1}\ldots\frac{n'-3}{4}a^{n'-4}b^4 + \frac{n'}{1}\ldots\frac{n'-4}{5}a^{n'-5}b^5 + \ldots$$

Dico: productum ex hisce duabus quantitatibus conflatum ejusdem eſſe formæ, cujus utraque illarum, ſi loco τȣ n aut n' illarum ſumma $n+n'$ ſubſtituatur. Nempe

$$PP' = a^{n+n'} + \frac{n+n'}{1}a^{n+n'-1}b + \frac{n+n'}{1}\cdot\frac{n+n'-1}{2}a^{n+n'-2}b^2 + \frac{n+n'}{1}\ldots\frac{n+n'-2}{3}a^{n+n'-3}b^3 + \frac{n+n'}{1}\ldots\frac{n+n'-3}{4}a^{n+n'-4}b^4 + \frac{n+n'}{1}\ldots\frac{n+n'-4}{5}a^{n+n'-5}b^5 + \ldots$$

Et primo quidem exponentes τῦ a in ſucceſſivis producti terminis progreſſionem arithmeticam decreſcentem ſequuntur, cujus primus terminus eſt $n+n'$, & communis terminorum differentia eſt unitas. Exponentes autem τῦ b ſequuntur progreſſionem numerorum naturalium creſcentem inde ab unitate.

Determinanda ſupereſt lex coëfficientium.

1°. Coëfficiens primi termini eſt unitas.

2°. Coëfficiens ſecundi termini eſt $\frac{n}{1}+\frac{n'}{1}=\frac{n+n'}{1}$.

3°. Coëfficiens termini tertii eſt

$$\left.\begin{array}{l} \frac{n}{1}\cdot\frac{n-1}{2} \\ +\frac{n}{1}\cdot\frac{n'}{1} \\ +\frac{n'}{1}\cdot\frac{n'-1}{2}\end{array}\right\} = \left\{\begin{array}{l} \frac{n}{1}\cdot\frac{n-1}{2} \\ +2\frac{n}{1}\cdot\frac{n'}{2} \\ +\frac{n'}{1}\cdot\frac{n'-1}{2}\end{array}\right. = \left\{\begin{array}{l} \frac{n}{1}\cdot\frac{n-1}{2}+\frac{n}{1}\cdot\frac{n'}{2} \\ +\frac{n}{1}\cdot\frac{n'}{2}+\frac{n'}{1}\cdot\frac{n'-1}{2}\end{array}\right. = \left\{\begin{array}{l} \frac{n+n'-1}{2}\cdot\frac{n}{1} \\ +\frac{n+n'-1}{2}\cdot\frac{n'}{1}\end{array}\right. = \frac{n+n'}{1}\cdot\frac{n+n'-1}{2} \quad (2°.)$$

4°. Coëfficiens termini quarti eſt

$$\left.\begin{array}{l} \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3} \\ +\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{1} \\ +\frac{n}{1}\cdot\frac{n'}{1}\cdot\frac{n'-1}{2} \\ +\frac{n'}{1}\cdot\frac{n'-1}{2}\cdot\frac{n'-2}{3}\end{array}\right\} = \left\{\begin{array}{ll} \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3} = \frac{n+n'-2}{3} & \left(\frac{n}{1}\cdot\frac{n-1}{2}\right. = \frac{n+n'}{1}\cdot\frac{n+n'-1}{2}\cdot\frac{n+n'-2}{3} \quad (3°.) \\ +3\cdot\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{3} & +2\cdot\frac{n}{1}\cdot\frac{n'}{2} \\ +3\cdot\frac{n}{1}\cdot\frac{n'}{2}\cdot\frac{n'-1}{3} & \left.+1\cdot\frac{n'}{1}\cdot\frac{n'-1}{2}\right) \\ +\frac{n'}{1}\cdot\frac{n'-1}{2}\cdot\frac{n'-2}{3} & \end{array}\right.$$

5°. Coëfficiens termini quinti eſt

$$\left.\begin{array}{l} \frac{n}{1}\cdots\cdots\frac{n-3}{4} \\ +\frac{n}{1}\cdots\frac{n-2}{3}\cdot\frac{n'}{1} \\ +\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{1}\cdot\frac{n'-1}{2} \\ +\frac{n}{1}\cdot\frac{n'}{1}\cdots\frac{n'-2}{3} \\ +\frac{n'}{1}\cdots\cdots\frac{n'-3}{4}\end{array}\right\} = \left\{\begin{array}{ll} \frac{n}{1}\cdots\cdots\frac{n-3}{4} = \frac{n+n'-3}{4} & \left(\frac{n}{1}\cdots\cdots\frac{n-2}{3}\right. = \frac{n+n'}{1}\cdots\frac{n+n'-3}{4} \quad (4°.) \\ +4\cdot\frac{n}{1}\cdots\frac{n-3}{3}\cdot\frac{n'}{4} & +3\cdot\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{3} \\ +6\cdot\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{1}\cdot\frac{n'-1}{2} & +3\cdot\frac{n}{1}\cdot\frac{n'}{2}\cdot\frac{n'-1}{3} \\ +4\cdot\frac{n}{1}\cdot\frac{n'}{1}\cdots\frac{n'-2}{3} & \left.+\frac{n'}{1}\cdots\cdots\frac{n'-2}{3}\right) \\ +1\cdot\frac{n'}{1}\cdots\cdots\frac{n'-3}{4} & \end{array}\right.$$

6°. Coëffi-

6°. Coëfficiens termini sexti est

$$\left.\begin{array}{l}
\frac{n}{1}\cdots\cdot\frac{n-4}{5}\\
+\frac{n}{1}\cdots\frac{n-3}{4}\cdot\frac{n'}{1}\\
+\frac{n}{1}\cdots\frac{n-2}{3}\cdot\frac{n'}{1}\cdot\frac{n'-1}{2}\\
+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{1}\cdots\frac{n'-2}{3}\\
+\frac{n}{1}\cdot\frac{n'}{1}\cdots\frac{n'-3}{4}\\
+\frac{n'}{1}\cdots\cdot\frac{n'-4}{5}
\end{array}\right\}=\left\{\begin{array}{l}
\frac{n}{1}\cdots\cdots\frac{n-4}{5}\\
+5.\frac{n}{1}\cdots\frac{n-3}{4}\cdot\frac{n'}{5}\\
+10.\frac{n}{1}\cdots\frac{n-2}{3}\cdot\frac{n'}{4}\cdot\frac{n'-1}{5}\\
+10.\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{3}\cdots\frac{n'-2}{5}\\
+5.\frac{n}{1}\cdot\frac{n'}{2}\cdots\cdot\frac{n'-3}{5}\\
+1.\frac{n'}{1}\cdots\cdot\frac{n'-4}{5}
\end{array}\right.
=\frac{n+n'-4}{5}\left[\begin{array}{l}
\frac{n}{1}\cdots\frac{n-3}{4}\\
+4.\frac{n}{1}\cdots\frac{n-2}{3}\cdot\frac{n'}{4}\\
+6.\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{3}\cdot\frac{n'-1}{4}\\
+4.\frac{n}{1}\cdot\frac{n'}{2}\cdots\frac{n'-2}{4}\\
+\frac{n'}{1}\cdots\cdot\frac{n'-3}{4}
\end{array}\right]=\frac{n+n'}{1}\cdots\frac{n+n'-4}{5}\quad(5°.)$$

Generatim, ostenso, quod lex valeat pro termino quocunque m^{to}, scilicet quod coëfficiens m^{ti} termini est $\frac{n+n'}{1}\cdot\frac{n+n'-1}{2}\cdots\cdot\frac{n+n'-(m-2)}{m-1}$: dico, eandem legem etiam valere pro termino sequenti $m+1^{mo}$; scilicet coëfficientem hujus termini esse $\frac{n+n'}{1}\cdot\frac{n+n'-1}{2}\cdots\cdot\frac{n+n'-(m-1)}{m}$.

Etenim coëfficiens $m+1^{ti}$ termini est

$$\left.\begin{array}{l}
\frac{n}{1}\cdot\frac{n-1}{2}\cdots\cdot\frac{n-(m-1)}{m}\\
+\frac{n}{1}\cdot\frac{n-1}{2}\cdots\frac{n-(m-2)}{m-1}\cdot\frac{n'}{1}\\
+\frac{n}{1}\cdot\frac{n-1}{2}\cdots\frac{n-(m-3)}{m-2}\cdot\frac{n'}{1}\cdot\frac{n'-1}{2}\\
+\frac{n}{1}\cdot\frac{n-1}{2}\cdots\frac{n-(m-4)}{m-3}\cdot\frac{n'}{1}\cdots\frac{n'-2}{3}\\
+\frac{n}{1}\cdot\frac{n-1}{2}\cdots\frac{n-(m-5)}{m-4}\cdot\frac{n'}{1}\cdots\frac{n'-3}{4}\\
\vdots\\
+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{1}\cdots\frac{n'-(m-3)}{m-2}\\
+\frac{n}{1}\cdot\frac{n'}{1}\cdots\cdot\frac{n'-(m-2)}{m-1}\\
+\frac{n'}{1}\cdots\cdot\frac{n'-(m-1)}{m}
\end{array}\right\}=\left\{\begin{array}{l}
\frac{n}{1}\cdots\cdots\frac{n-(m-1)}{m}\\
+\frac{m}{1}\cdot\frac{n}{1}\cdots\cdots\frac{n-(m-2)}{m-1}\cdot\frac{n'}{m}\\
+\frac{m}{1}\cdot\frac{m-1}{2}\cdot\frac{n}{1}\cdots\frac{n-(m-3)}{m-2}\cdot\frac{n'}{m-1}\cdot\frac{n'-1}{m}\\
+\frac{m}{1}\cdots\frac{m-2}{3}\cdot\frac{n}{1}\cdots\frac{n-(m-4)}{m-3}\cdot\frac{n'}{m-2}\cdots\frac{n'-2}{m}\\
+\frac{m}{1}\cdots\frac{m-3}{4}\cdot\frac{n}{1}\cdots\frac{n-(m-5)}{m-4}\cdot\frac{n'}{m-3}\cdots\frac{n'-3}{m}\\
\vdots\\
+\frac{m}{1}\cdot\frac{m-1}{2}\cdot\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{3}\cdots\cdot\frac{n'-(m-3)}{m}\\
+\frac{m}{1}\cdot\frac{n}{1}\cdot\frac{n'}{2}\cdots\cdots\frac{n'-(m-2)}{m}\\
+\frac{n'}{1}\cdots\cdots\frac{n'-(m-1)}{m}
\end{array}\right.$$

$= n+n$

$$
\begin{aligned}
= \frac{n+n'-(m-1)}{m} \Bigg\{ & \frac{n}{1}\cdot\frac{n-1}{2}\cdots\cdots\frac{n-(m-2)}{m-1} \\
& + \frac{m-1}{1}\cdot\frac{n}{1}\cdot\frac{n-1}{2}\cdots\cdots\frac{n-(m-3)}{m-2}\cdot\frac{n'}{m-1} \\
& + \frac{m-1}{1}\cdot\frac{m-2}{2}\cdot\frac{n}{1}\cdot\frac{n-1}{2}\cdots\frac{n-(m-4)}{m-3}\cdot\frac{n'}{m-2}\cdot\frac{n'-1}{m-1} \\
& + \frac{m-1}{1}\cdots\frac{m-3}{3}\cdot\frac{n}{1}\cdot\frac{n-1}{2}\cdots\frac{n-(m-5)}{m-4}\cdot\frac{n'}{m-3}\cdots\frac{n'-2}{m-1} \\
& \quad\vdots \\
& + \frac{m-1}{1}\cdot\frac{m-2}{2}\cdot\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{3}\cdots\frac{n'-(m-4)}{m-1} \\
& + \frac{m-1}{1}\cdot\frac{n}{1}\cdot\frac{n'}{2}\cdots\cdots\frac{n'-(m-3)}{m-1} \\
& + 1\cdot\frac{n'}{1}\cdots\cdots\frac{n'-(m-2)}{m-1} \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
= \frac{n+n'-(m-1)}{m} \Bigg\{ & \frac{n}{1}\cdot\frac{n-1}{2}\cdots\cdots\frac{n-(m-2)}{m-1} \\
& + \frac{n}{1}\cdot\frac{n-1}{2}\cdots\cdots\frac{n-(m-3)}{m-2}\cdot\frac{n'}{1} \\
& + \frac{n}{1}\cdot\frac{n-1}{2}\cdots\cdots\frac{n-(m-4)}{m-3}\cdot\frac{n'}{1}\cdot\frac{n'-1}{2} \\
& + \frac{n}{1}\cdot\frac{n-1}{2}\cdots\cdots\frac{n-(m-5)}{m-4}\cdot\frac{n'}{1}\cdots\frac{n'-2}{3} \\
& \quad\vdots \\
& + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{1}\cdots\cdots\frac{n'-(m-4)}{m-3} \\
& + \frac{n}{1}\cdot\frac{n'}{1}\cdots\cdots\frac{n'-(m-3)}{m-2} \\
& + \frac{n'}{1}\cdots\cdots\frac{n'-(m-2)}{m-1} \Bigg)
\end{aligned}
$$

$$
= \frac{n+n'}{1}\cdot\frac{n+n'-1}{2}\cdot\frac{n+n'-2}{3}\cdots\cdots\frac{n+n'-(m-1)}{m} \text{ (per hyp.).}
$$

Ergo

Ergo si lex coëfficientium valet pro quocunque producti termino, eadem quoque valet pro coëfficiente termini sequentis. Atqui valere demonstrata fuit in termino secundo, tertio, quarto, quinto, sexto: valet igitur etiam in septimo, atque hinc rursus in octavo, tum in nono, & sic consequenter in universa serie.

§. *b.* *Applicatio prima.* 1°. Quadratum quantitatis hujus formæ

$$a^n + \frac{n}{1}a^{n-1}b + \frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2 + \frac{n}{1}\ldots\frac{n-2}{3}a^{n-3}b^3 + \frac{n}{1}\ldots\frac{n-3}{4}a^{n-4}b^4 + \ldots\ldots$$

est $a^{2n}+\frac{2n}{1}a^{2n-1}b+\frac{2n}{1}\cdot\frac{2n-1}{2}a^{2n-2}b^2+\frac{2n}{1}\ldots\frac{2n-2}{3}a^{2n-3}b^3+\frac{2n}{1}\ldots\frac{2n-3}{4}a^{2n-4}b^4+\ldots\ldots$

2°. Hinc cubus ejusdem quantitatis est

$$a^{3n}+\frac{3n}{1}a^{3n-1}b+\frac{3n}{1}\cdot\frac{3n-1}{2}a^{3n-2}b^2+\frac{3n}{1}\ldots\frac{3n-2}{3}a^{3n-3}b^3+\frac{3n}{1}\ldots\frac{3n-3}{4}a^{3n-4}b^4+\ldots\ldots$$

Hinc iterum 3°. potentia quarta prædictæ quantitatis est

$$a^{4n}+\frac{4n}{1}a^{4n-1}b+\frac{4n}{1}\cdot\frac{4n-1}{2}a^{4n-2}b^2+\frac{4n}{1}\ldots\frac{4n-2}{3}a^{4n-3}b^3+\frac{4n}{1}\ldots\frac{4n-3}{4}a^{4n-4}b^4+\ldots\ldots$$

Et universim, quicunque numerus integer positivus fuerit m, potentia m^{ta} ejusdem quantitatis est

$$a^{mn}+\frac{mn}{1}a^{mn-1}b+\frac{mn}{1}\cdot\frac{mn-1}{2}a^{mn-2}b^2+\frac{mn}{1}\ldots\frac{mn-2}{3}a^{mn-3}b^3+\frac{mn}{1}\ldots\frac{mn-3}{4}a^{mn-4}b^4+\ldots\ldots$$

§. *c.* *Applicatio secunda.* Vicissim, denotante m numerum quemcunque integrum positivum, radix m^{ta} quantitatis

$$a^n+\frac{n}{1}a^{n-1}b+\frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2+\frac{n}{1}\ldots\frac{n-2}{3}a^{n-3}b^3+\frac{n}{1}\ldots\frac{n-3}{4}a^{n-4}b^4+\ldots\ldots$$

est $a^{\frac{n}{m}}+\frac{\frac{n}{m}}{1}a^{\frac{n}{m}-1}b+\frac{\frac{n}{m}}{1}\cdot\frac{\frac{n}{m}-1}{2}a^{\frac{n}{m}-2}b^2+\frac{\frac{n}{m}}{1}\ldots\frac{\frac{n}{m}-2}{3}a^{\frac{n}{m}-3}b^3+\frac{\frac{n}{m}}{1}\ldots\frac{\frac{n}{m}-3}{4}a^{\frac{n}{m}-4}b^4+\ldots\ldots$

Etenim (per applicationem præcedentem) prior quantitas est potentia m^{ta} posterioris: quare vicissim posterior quantitas est radix m^{ta} prioris.

§. *d.* *Applicatio tertia.* Sint m & n numeri quilibet integri positivi. Dico esse

$$(a+b)^{\frac{n}{m}}=a^{\frac{n}{m}}+\frac{\frac{n}{m}}{1}a^{\frac{n}{m}-1}b+\frac{\frac{n}{m}}{1}\cdot\frac{\frac{n}{m}-1}{2}a^{\frac{n}{m}-2}b^2+\frac{\frac{n}{m}}{1}\ldots\frac{\frac{n}{m}-2}{3}a^{\frac{n}{m}-3}b^3+\frac{\frac{n}{m}}{1}\ldots\frac{\frac{n}{m}-3}{4}a^{\frac{n}{m}-4}b^4+\ldots\ldots$$

Etenim

Etenim $(a+b)^{\frac{n}{m}} = \sqrt[m]{(a+b)^n}$

$$= \sqrt[m]{\left(a^n + \frac{n}{1}a^{n-1}b + \frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2 + \frac{n}{1}\cdots\frac{n-2}{3}a^{n-3}b^3 + \frac{n}{1}\cdots\frac{n-3}{4}a^{n-4}b^4 + \ldots\right)} \quad (\text{§. } b.)$$

$$= a^{\frac{n}{m}} + \frac{\frac{n}{m}}{1}a^{\frac{n}{m}-1}b + \frac{\frac{n}{m}}{1}\cdot\frac{\frac{n}{m}-1}{2}a^{\frac{n}{m}-2}b^2 + \frac{\frac{n}{m}}{1}\cdots\frac{\frac{n}{m}-2}{3}a^{\frac{n}{m}-3}b^3 + \frac{\frac{n}{m}}{1}\cdots\frac{\frac{n}{m}-3}{4}a^{\frac{n}{m}-4}b^4 + \ldots \quad (\text{§. } c.)$$

§. *e.* *Corollarium.* Univerſim igitur, ſi n fuerit numerus rationalis poſitivus quicunque, ſive integer, ſive fractus,

$$(a+b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2 + \frac{n}{1}\cdots\frac{n-2}{3}a^{n-3}b^3 + \frac{n}{1}\cdots\frac{n-3}{4}a^{n-4}b^4 + \ldots$$

§. *f.* *Applicatio quarta.* Sint duæ quantitates

$$a^n + \frac{n}{1}a^{n-1}b + \frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2 + \frac{n}{1}\cdots\frac{n-2}{3}a^{n-3}b^3 + \frac{n}{1}\cdots\frac{n-3}{4}a^{n-4}b^4 + \ldots$$

$$a^{-n} + \frac{-n}{1}a^{-n-1}b + \frac{-n}{1}\cdot\frac{-n-1}{2}a^{-n-2}b^2 + \frac{-n}{1}\cdots\frac{-n-2}{3}a^{-n-3}b^3 + \frac{-n}{1}\cdots\frac{-n-3}{4}a^{-n-4}b^4 + \ldots$$

Productum ex illis factum reducitur ad primum terminum $a^n \times a^{-n} = 1$: cum terminorum omnium ſequentium coëfficientes ingrediatur factor $n - n = 0$.

§. *g.* *Corollarium.* Ergo

$$a^{-n} + \frac{-n}{1}a^{-n-1}b + \frac{-n}{1}\cdot\frac{-n-1}{2}a^{-n-2}b^2 + \frac{-n}{1}\cdots\frac{-n-2}{3}a^{-n-3}b^3 + \frac{-n}{1}\cdots\frac{-n-3}{4}a^{-n-4}b^4 + \ldots$$

$$= \frac{1}{a^n + \frac{n}{1}a^{n-1}b + \frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2 + \frac{n}{1}\cdots\frac{n-2}{3}a^{n-3}b^3 + \frac{n}{1}\cdots\frac{n-3}{4}a^{n-4}b^4 + \ldots}$$

Atqui, n denotante numerum quemcunque rationalem poſitivum, denominator hujus fractionis eſt $(a+b)^n$ (§. *e*). Ergo

$$\frac{1}{(a+b)^n} \text{ ſeu } (a+b)^{-n} = a^{-n} + \frac{-n}{1}a^{-n-1}b + \frac{-n}{1}\cdot\frac{-n-1}{2}a^{-n-2}b^2 + \frac{-n}{1}\cdots\frac{-n-2}{3}a^{-n-3}b^3 + \frac{-n}{1}\cdots\frac{-n-3}{4}a^{-n-4}b^4 + \ldots$$

$$= \frac{1}{a^n} - \frac{n}{1}\,\frac{b}{a^{n+1}} + \frac{n}{1}\cdot\frac{n+1}{2}\,\frac{b^2}{a^{n+2}} - \frac{n}{1}\cdots\frac{n+2}{3}\,\frac{b^3}{a^{n+3}} + \frac{n}{1}\cdots\frac{n+3}{4}\,\frac{b^4}{a^{n+4}} - \ldots$$

Eadem itaque lex, quæ valet de exponentibus rationalibus poſitivis, pariter applicatur ad exponentes rationales negativos.

§. *h.*

§. *h.* *Obſervatio.* Quod ad exponentes irrationales attinet: cum per extractionem radicum, vero propius ſemper propiusque accedentem, quantitates ſurdæ exprimi poſſint per quantitates rationales, quæ a vero illarum valore minus quam quantitate data differant; ad exponentes ſurdos applicare licet, quæ de exponentibus rationalibus demonſtrata fuerunt.

Obſervandum præterea: calculos, quos quantitates exponentibus ſurdis affectæ ingrediuntur, nonniſi per analogiam juxta regulas exponentium rationalium peragi.

Quod ipſum tanto etiam magis ad exponentes imaginarios pertinet; cum mera ſint ſigna, facilitatis & univerſalitatis calculi cauſa ab mathematicis introducta.

CAPUT B.

De differentiis quantitatum mutabilium.

§. *i.* *Definitio.* Sit aliqua quantitas mutabilis, quomodocunque expreſſa per alteram quantitatem mutabilem, ſolam, vel cum quantitatibus conſtantibus combinatam. Prior quantitas dicitur *functio* poſterioris.

Quod attinet ad functionum diviſionem in integras & fractas, uniformes & multiformes, rationales & irrationales, algebraicas & transcendentes; videatur inter alios EULERI *Introductio*, Cap. I.

§. *k.* Sit P functio quantitatis variabilis x uniformiter creſcentis vel decreſcentis. Denotentur incrementa vel decrementa ejus ſucceſſiva ſignis

Δx, $2\Delta x$, $3\Delta x$, $4\Delta x$, ... $(n-1)\Delta x$, $n\Delta x$.

Tunc, qui valoribus ipſius ſucceſſivis	$x+\Delta x$,	$x+2\Delta x$,	$x+3\Delta x$,	$x+4\Delta x \dots$	$x+(n-1)\Delta x$,	$x+n\Delta x$
reſpondent functionis P valores, deſignentur per	P',	P'',	P''',	$P^{iv} \dots$	P^{N-1},	P^N,
Et valorum horum mutationes ſucceſſivæ ita notentur	ΔP,	$\Delta P'$,	$\Delta P''$,	$\Delta P''' \dots$	$\Delta P^{N-\text{II}}$,	$\Delta P^{N-\text{I}}$.

Erunt ideo

$$
\begin{aligned}
P' &= P + \Delta P \\
P'' &= P' + \Delta P' \\
P''' &= P'' + \Delta P'' \\
P^{iv} &= P''' + \Delta P''' \\
&\vdots \\
P^{N-\text{I}} &= P^{N-\text{II}} + \Delta P^{N-\text{II}} \\
P^{N} &= P^{N-\text{I}} + \Delta P^{N-\text{I}}.
\end{aligned}
$$

Quantitates ΔP, $\Delta P'$, $\Delta P''$, $\Delta P'''$ .... ΔP^{N-II}, ΔP^{N-I} vocantur *differentiæ primi ordinis* quantitatis P.

§. *l.* Si hæ differentiæ non fuerint conſtantes: eædem componentur ex quantitate conſtante Δx, & ex variabili x; adeoque pariter ſunt functiones variabilis x. Succeſſivæ harum functionum mutationes, uniformi quantitatis x mutationi reſpondentes, deſignentur per $\Delta^2 P$, $\Delta^2 P'$, $\Delta^2 P''$, $\Delta^2 P'''$ ... $\Delta^2 P^{N-II}$, $\Delta^2 P^{N-I}$.

Hæ differentiæ differentiarum primi ordinis vocantur *differentiæ ſecundi ordinis* functionis P; & eodem quo prius modo obtinentur æquationes ſequentes:

$$
\begin{aligned}
\Delta P' &= \Delta P + \Delta^2 P \\
\Delta P'' &= \Delta P' + \Delta^2 P' \\
\Delta P''' &= \Delta P'' + \Delta^2 P'' \\
&\vdots \\
\Delta P^{N-I} &= \Delta P^{N-II} + \Delta^2 P^{N-II} \\
\Delta P^{N} &= \Delta P^{N-I} + \Delta^2 P^{N-I}.
\end{aligned}
$$

§. *m.* Si differentiæ ſecundi ordinis non fuerint conſtantes: pariter componentur ex quantitate conſtanti Δx^2 & variabili x; ideoque functiones erunt quantitatis x. Succeſſivæ earum mutationes, uniformi ipſius x mutationi reſpondentes, deſignentur per $\Delta^3 P$, $\Delta^3 P'$, $\Delta^3 P''$, $\Delta^3 P'''$ . . . $\Delta^3 P^{N-II}$, $\Delta^3 P^{N-I}$. Differentiæ iſtæ differentiarum ſecundi ordinis vocantur *differentiæ tertii ordinis*; & rurſus ſunt

$$
\begin{aligned}
\Delta^2 P' &= \Delta^2 P + \Delta^3 P \\
\Delta^2 P'' &= \Delta^2 P' + \Delta^3 P' \\
\Delta^2 P''' &= \Delta^2 P'' + \Delta^3 P'' \\
\Delta^2 P^{IV} &= \Delta^2 P''' + \Delta^3 P''' \\
&\vdots \\
\Delta^2 P^{N-I} &= \Delta^2 P^{N-II} + \Delta^3 P^{N-II} \\
\Delta^2 P^{N} &= \Delta^2 P^{N-I} + \Delta^3 P^{N-I}.
\end{aligned}
$$

§. *n.* Si differentiæ tertii ordinis non ſunt conſtantes; progreſſus fit ad earundem differentias, ſeu ad *differentias quarti ordinis:* deinde ſimiliter ad *differentias quinti ordinis*, & hinc ad *differentias ſexti ordinis*, atque ita deinceps; donec (ſi fieri poſſit) perveniatur ad *differentias conſtantes*, ſeu quarum nullæ ſunt differentiæ.

Gene-

Generatim differentiæ differentiarum $m-1^{ti}$ ordinis vocantur *differentiæ* m^{ti} *ordinis*; & ſic deſignantur $\Delta^m P$, $\Delta^m P'$, $\Delta^m P''$, $\Delta^m P''' \ldots \Delta^m P^{N-II}$, $\Delta^m P^{N-I}$. Unde

$$\begin{aligned}
\Delta^{m-1}P' &= \Delta^{m-1}P &+ \Delta^m P \\
\Delta^{m-1}P'' &= \Delta^{m-1}P' &+ \Delta^m P'. \\
\Delta^{m-1}P''' &= \Delta^{m-1}P'' &+ \Delta^m P'' \\
\Delta^{m-1}P^{IV} &= \Delta^{m-1}P''' &+ \Delta^m P''' \\
&\vdots &\vdots \\
\Delta^{m-1}P^{N-I} &= \Delta^{m-1}P^{N-II} &+ \Delta^m P_{N-II} \\
\Delta^{m-1}P^{N} &= \Delta^{m-1}P^{N-I} &+ \Delta^m P^{N-I}.
\end{aligned}$$

§. o. Differentiæ omnium ordinum ſucceſſivorum functionis P immediate per ſeriem functionum P, P', P'', $P''' \ldots P^{N-1}$, P^N exprimi poſſunt modo ſequenti:

$$\begin{aligned}
& \Delta P &&= P' - P \\
& \Delta P' &&= P'' - P' \\
\text{Ergo } & \Delta P' - \Delta P \\
\text{ſeu } & \Delta^2 P &&= P'' - 2P' + P \\
& \Delta^2 P' &&= P''' - 2P'' + P' \\
\text{Ergo } & \Delta^3 P &&= P''' - 3P'' + 3P' - P \\
& \Delta^3 P' &&= P^{IV} - 3P''' + 3P'' - P' \\
\text{Ergo } & \Delta^4 P &&= P^{IV} - 4P''' + 6P'' - 4P' + P \\
& \Delta^4 P' &&= P^{V} - 4P^{IV} + 6P''' - 4P'' + P'. \\
\text{Ergo } & \Delta^5 P &&= P^{V} - 5P^{IV} + 10P''' - 10P'' + 5P' - P.
\end{aligned}$$

Generaliter

ſit $\Delta^m P = P^M - \frac{m}{1} P^{M-I} + \frac{m}{1} \cdot \frac{m-1}{2} P^{M-II} - \frac{m}{1} \ldots \frac{m-2}{3} P^{M-III} + \ldots \mp mP' \pm P$

& ideo

$\Delta^m P' = P^{M+I} - \frac{m}{1} P^M + \frac{m}{1} \cdot \frac{m-1}{2} P^{M-I} - \frac{m}{1} \ldots \frac{m-2}{3} P^{M-II} + \frac{m}{1} \ldots \frac{m-3}{4} P^{M-III} - \ldots \pm P'$

erit

$\Delta^{m+1} P = P^{M+I} - \frac{m+1}{1} P^M + \frac{m+1}{1} \cdot \frac{m}{2} P^{M-I} - \frac{m+1}{1} \ldots \frac{m-1}{3} P^{M-II} + \frac{m+1}{1} \ldots \frac{m-2}{4} P^{M-III} - \ldots \pm \frac{m+1}{1} P' \mp P$

Ergo

Ergo generaliter

$$\Delta^m p = p^M - \frac{m}{1} p^{M-1} + \frac{m}{1} \cdot \frac{m-1}{2} p^{M-II} - \frac{m}{1} \cdots \frac{m-2}{3} p^{M-III} \cdots \pm \frac{m}{1} \cdot \frac{m-1}{2} p'' \mp \frac{m}{1} p' \pm P.$$

§. p. Viciſſim terminus quilibet P^M ſeriei P, P', P'', P''' … P^{M-1}, P^M exprimi poteſt per functionem P, & per differentias ſucceſſivas omnium ordinum hujus functionis P.

Etenim quoniam

$$\begin{array}{lll} & P' & = P+\Delta P \\ & \Delta P' & = \quad \Delta P+\Delta^2 P \\ \text{ergo} & P''(=P'+\Delta P') & = P+2\Delta P+\Delta^2 P \\ \text{Hinc} & \Delta P'' & = \quad \Delta P+2\Delta^2 P+\Delta^3 P \\ \text{ergo} & P'''(=P''+\Delta P'') & = P+3\Delta P+3\Delta^2 P+\Delta^3 P \\ \text{Hinc} & \Delta P''' & = \quad \Delta P+3\Delta^2 P+3\Delta^3 P+\Delta^4 P \\ \text{ergo} & P^{IV}(=P'''+\Delta P''') & = P+4\Delta P+6\Delta^2 P+4\Delta^3 P+\Delta^4 P \\ \text{Hinc} & \Delta P^{IV} & = \quad \Delta P+4\Delta^2 P+6\Delta^3 P+4\Delta^4 P+\Delta^5 P \\ \text{ergo} & P^{V}(=P^{IV}+\Delta P^{IV}) & = P+5\Delta P+10\Delta^2 P+10\Delta^3 P+5\Delta^4 P+\Delta^5 P. \end{array}$$

Generatim ſit

$$P^M = P + \frac{m}{1}\Delta P + \frac{m}{1} \cdot \frac{m-1}{2}\Delta^2 P + \frac{m}{1} \cdots \frac{m-2}{3}\Delta^3 P + \frac{m}{1} \cdots \frac{m-3}{4}\Delta^4 P + \cdots$$
$$+ \frac{m}{1} \cdot \frac{m-1}{2}\Delta^{m-2} P + \frac{m}{1}\Delta^{m-1} P + \Delta^m P$$

hinc

$$\Delta P^M = \Delta P + \frac{m}{1}\Delta^2 P + \frac{m}{1} \cdot \frac{m-1}{2}\Delta^3 P + \frac{m}{1} \cdots \frac{m-2}{3}\Delta^4 P + \cdots$$
$$+ \frac{m}{1} \cdots \frac{m-2}{3}\Delta^{m-2} P + \frac{m}{1} \cdot \frac{m-1}{2}\Delta^{m-1} P + \frac{m}{1}\Delta^m P + \Delta^{m+1} P.$$

erit

$$P^{M+1}(=P^M+\Delta P^M) = P + \frac{m+1}{1}\Delta P + \frac{m+1}{1} \cdot \frac{m}{2}\Delta^2 P + \frac{m+1}{1} \cdots \frac{m-1}{3}\Delta^3 P + \frac{m+1}{1} \cdots \frac{m-2}{4}\Delta^4 P + \cdots$$
$$+ \frac{m+1}{1} \cdots \frac{m-1}{3}\Delta^{m-2} P + \frac{m+1}{1} \cdot \frac{m}{2}\Delta^{m-1} P + \frac{m+1}{1}\Delta^m P + \Delta^{m+1} P.$$

Ergo generatim

$$P^M = P + \frac{m}{1}\Delta P + \frac{m}{1} \cdot \frac{m-1}{2}\Delta^2 P + \frac{m}{1} \cdots \frac{m-2}{3}\Delta^3 P + \frac{m}{1} \cdots \frac{m-3}{4}\Delta^4 P + \cdots$$
$$+ \frac{m}{1} \cdot \frac{m-1}{2}\Delta^{m-2} P + \frac{m}{1}\Delta^{m-1} P + \Delta^m P.$$

Propoſitio hæc magni eſt momenti ad applicationes, quæ deinceps tradentur.

§. q.

§. 9. Cum functio quaelibet quantitatis variabilis potentiis variabilis hujus (immediate aut mediate) exprimi possit: multum interest, potentias quantitatis variabilis & differentias omnium ordinum harum potentiarum accuratius expendere. Ex sequentibus patebit, quanti speciatim hoc respectu momenti sit contemplatio potentiarum exponentis integri positivi cujuscunque.

Theorema. Potentiae quantitatis variabilis, cujus exponens est numerus integer positivus, differentia ordinis, cujus index idem est cum exponente potestatis, aequatur producto continuo numerorum naturalium inde ab unitate usque ad hunc exponentem, ducto in differentiae quantitatis variabilis potentiam ejusdem exponentis; proindeque differentiae ordinum superiorum ejusdem potentiae quantitatis variabilis evanescunt.

Sit $P = x^m$, exponente m existente numero integro positivo.

Dico esse $\Delta^m P = 1.2.3.4 \ldots m \Delta x^m$; & proinde $\Delta^{m+1} P = 0$.

Ad demonstrandum hoc theorema ostendam primo: illud verum esse pro minoribus exponentis m valoribus, quales sunt 1, 2, 3, 4 .... Tum evincam: quod, si theorema verum fuerit pro quolibet exponente dato, idem etiam valeat pro exponente sequenti, qui priorem unitate superet.

Exemplum 1. Sit $P = x$

$$\Delta P = \Delta x = 1.\Delta x$$

Corollarium. $\Delta^{1+1} x = 0.$

Exemplum 2. Sit $P = x^2$

$$\Delta P = 2x\Delta x + \Delta x^2$$

$$\Delta^2 P = 2\Delta x \times \Delta x + \Delta x^2 \times \Delta 1$$

$$= 2\Delta x^2 = 1.2.\Delta x^2.$$

Coroll. $\Delta^{1+2} x = 0.$

Exemplum 3. Sit $P = x^3$

$$\Delta P = 3xx\Delta x + 3x\Delta x^2 + \Delta x^3$$

$$\Delta^3 P (= \Delta^2 \Delta P) = 3\Delta x.\Delta^2 xx + 3\Delta x^2.\Delta^2 x + \Delta x^3.\Delta^2 1$$

$$= 3\Delta x.\Delta^2 xx = 1.2.3\Delta x^3$$

Coroll. $\Delta^{1+3} x^3 = 0.$

Exem-

Exemplum 4. Sit

$$\begin{aligned}
P &= x^4\\
\Delta P &= 4x^3\Delta x+6x^2\Delta x^2+4x\Delta x^3+\Delta x^4\\
\Delta^4 P(=\Delta^3.\Delta P) &= 4\Delta x\times\Delta^3x^3+6\Delta x^2.\Delta^3x^2+4\Delta x^3.\Delta^3x+4\Delta x^4.\Delta^3.1\\
&= 4\Delta x\times\Delta^3x^3 = 1.2.3.4\Delta x^4.
\end{aligned}$$

Coroll. $\Delta^{1+4}x^4 = 0.$

Generatim sit m numerus integer positivus, & $P = x^m$. Probatum fuerit, quod $\Delta^m P = 1.2.3.4\ldots m\Delta x^m$ & $\Delta^m x^{m-1} = 0.$ Dico, quod $\Delta^{m+1}x^{m+1} = 1.2.3\ldots(m+1)\Delta x^{m+1}$ & $\Delta^{m+1+1}x^{m+1} = 0.$

Etenim

$$\begin{aligned}
\Delta x^{m+1} = {} & \frac{m+1}{1}x^m\Delta x\\
&+\frac{m+1}{1}.\frac{m}{2}x^{m-1}\Delta x^2\\
&+\frac{m+1}{1}\ldots\frac{m-1}{3}x^{m-2}\Delta x^3\\
&+\frac{m+1}{1}\ldots\frac{m-2}{4}x^{m-3}\Delta x^4\\
&\quad\vdots\\
&+\frac{m+1}{1}.\frac{m}{2}x^2\Delta x^{m-1}\\
&+\frac{m+1}{1}x\Delta x^m\\
&+\Delta x^{m+1}
\end{aligned}$$

Ergo

$$\begin{aligned}
\Delta^{m+1}x^{m+1} = {} & m+1.\Delta x\times\Delta^m x^m\\
&+\frac{m+1}{1}.\frac{m}{2}\Delta x^2\times\Delta^m x^{m-1}\\
&+\frac{m+1}{1}\ldots\frac{m-1}{3}\Delta x^3\times\Delta^m x^{m-2}\\
&+\frac{m+1}{1}\ldots\frac{m-2}{4}\Delta x^4\times\Delta^m x^{m-3}\\
&\quad\vdots\\
&+\frac{m+1}{1}.\frac{m}{2}\Delta x^{m-1}\times\Delta^m x^2\\
&+\frac{m+1}{1}\Delta x^m\times\Delta^m x\\
&+\Delta x^{m+1}\times\Delta^m 1\\
= {} & (m+1)\Delta x\times\Delta^m x^m \quad \text{(hyp.)}\\
= {} & (m+1)\Delta x\times 1.2.3.4\ldots m-1.m\Delta x^m \text{(hyp.)}\\
= {} & 1.2.3.4\ldots(m+1)\Delta x^{m+1}
\end{aligned}$$

Corollarium. $\Delta^{m+1+1}x^{m+1} = 0.$

Ergo generaliter $\Delta^m x^m = 1.2.3.4\ldots m\Delta x^m$

& $\Delta^{m+1}x^m = 0.$

§. 1. *Corollarium.* Speciatim potestatum numerorum naturalium successivorum, quarum exponentes sunt integri positivi, differentiæ, quarum indices ordinum sunt exponentibus his æquales, æquantur producto continuo numerorum naturalium ab unitate inde usque ad illum exponentem; earundemque potentiarum differentiæ, quarum index ordinis exponentem illum superat, evanescunt.

Scili-

Scilicet $\Delta^1 n^1 = 1$ $\qquad \Delta^{r+1} n^1 = 0$

$\Delta^2 n^2 = 1.2$ $\qquad \Delta^{r+2} n^2 = 0$

$\Delta^3 n^3 = 1.2.3$ $\qquad \Delta^{r+3} n^3 = 0$

$\Delta^4 n^4 = 1.2.3.4$ $\qquad \Delta^{r+4} n^4 = 0$

$\vdots$

$\Delta^m n^m = 1.2.3 \ldots . m$ $\qquad \Delta^{r+m} n^m = 0.$

Caſus particularis, corollario hoc traditi, eximiæ in ſequentibus applicationes occurrent.

§. 5. Eos inter caſus, quibus ad differentias conſtantes nunquam pervenitur, notandus inprimis eſt ille, quo ſeries propoſita eſt ſeries geometrica.

Theorema. Sit progreſſio geometrica quæcunque. Sumantur differentiæ omnium ordinum terminorum hujus progreſſionis: ſeries harum differentiarum pariter ſunt progreſſiones geometricæ, eundem, quem prima, exponentem habentes. Sumatur autem exceſſus, quo exponens iſte ſuperat unitatem, aut ab illa ſuperatur, prouti progreſſio eſt creſcens aut decreſcens; & ſumantur poteſtates hujus exceſſus, quarum exponens æqualis eſt ordini harum differentiarum: ſeries illæ erunt reſpective producta ex terminis primæ ſeriei per has poteſtates.

Sit $a^z,\ a^{2z},\ a^{3z},\ a^{4z},\ a^{5z}, \ldots a^{(n-2)z},\ a^{(n-1)z},\ a^{nz}$ progreſſio geometrica, cujus exponens a^z.

Series differentiarum primarum eſt

$$a^{2z}-a^z,\ a^{3z}-a^{2z},\ a^{4z}-a^{3z},\ a^{5z}-a^{4z}, \ldots a^{(n-1)z}-a^{(n-2)z},\ a^{nz}-a^{(n-1)z}$$

ſeu $(a^z-1)(a^z,\ a^{2z},\ a^{3z},\ a^{4z}, \ldots . a^{(n-2)z},\ a^{(n-1)z})$

Hinc ſeries differentiarum ſecundarum eſt

$$(a^z-1)^2(a^z,\ a^{2z},\ a^{3z},\ a^{4z}, \ldots . a^{(n-3)z},\ a^{(n-2)z})$$

Hinc rurſus ſeries differentiarum tertiarum eſt

$$(a^z-1)^3(a^z,\ a^{2z},\ a^{3z},\ a^{4z}, \ldots . a^{(n-4)z},\ a^{(n-3)z})$$

Item ſeries differentiarum quartarum eſt

$$(a^z-1)^4(a^z,\ a^{2z},\ a^{3z},\ a^{4z}, \ldots . a^{(n-5)z},\ a^{(n-4)z})$$

&c. &c. &c.

Generatim ſeries differentiarum m^{tarum} eſt

$$(a^z-1)^m(a^z,\ a^{2z},\ a^{3z},\ a^{4z}, \ldots . a^{n-(m-1)z},\ a^{(n-m)z}).$$

***

§. 6.

§. *t.* Notari etiam merentur omnium ordinum differentiæ, ſinuum & coſinuum arcuum in progreſſione arithmetica creſcentium aut decreſcentium; & inprimis (propter applicationes ſequentes) differentiæ ſinuum & coſinuum arcuum juxta ſeriem numerorum naturalium creſcentium.

Lemmata nota. 1°. Differentia ſinuum duorum arcuum æqualis eſt duplo producto coſinus dimidiæ ſummæ per ſinum dimidiæ differentiæ horum arcuum.

2°. Differentia coſinuum duorum arcuum æqualis eſt duplo producto ſinuum dimidiæ ſummæ & dimidiæ differentiæ horum arcuum.

Hoc eſt: 1°. $\text{ſin.}a - \text{ſin.}b = 2\text{coſ.}\frac{a+b}{2}\,\text{ſin.}\frac{a-b}{2}$

2°. $\text{coſ.}a - \text{coſ.}b = 2\text{ſin.}\frac{a+b}{2}\,\text{ſin.}\frac{b-a}{2}$.

Applicatio. Sint $\text{ſin.}a$, $\text{ſin.}2a$, $\text{ſin.}3a$, $\text{ſin.}4a$, $\text{ſin.}5a$, ... $\text{ſin.}na$ ſinus arcuum juxta arithmeticam numerorum naturalium progreſſionem creſcentium.

Series differentiarum primarum erit

$2\text{ſin.}\frac{1}{2}a(\text{coſ.}\frac{3}{2}a,\ \text{coſ.}\frac{5}{2}a,\ \text{coſ.}\frac{7}{2}a,\ \text{coſ.}\frac{9}{2}a,\ \text{coſ.}\frac{11}{2}a,\ \text{coſ.}\frac{13}{2}a\ \ldots)$

Series differentiarum ſecundarum erit

$-2^2\text{ſin.}^2\frac{1}{2}a(\text{ſin.}2a,\ \text{ſin.}3a,\ \text{ſin.}4a,\ \text{ſin.}5a,\ \text{ſin.}6a,\ \text{ſin.}7a\ \ldots)$

Series differentiarum tertiarum

$-2^3\text{ſin.}^3\frac{1}{2}a(\text{coſ.}\frac{5}{2}a,\ \text{coſ.}\frac{7}{2}a,\ \text{coſ.}\frac{9}{2}a,\ \text{coſ.}\frac{11}{2}a,\ \text{coſ.}\frac{13}{2}a,\ \text{coſ.}\frac{15}{2}a\ \ldots)$

Series differentiarum quartarum

$+2^4\text{ſin.}^4\frac{1}{2}a(\text{ſin.}3a,\ \text{ſin.}4a,\ \text{ſin.}5a,\ \text{ſin.}6a,\ \text{ſin.}7a,\ \text{ſin.}8a\ \ldots)$

Series differentiarum quintarum

$+2^5\text{ſin.}^5\frac{1}{2}a(\text{coſ.}\frac{7}{2}a,\ \text{coſ.}\frac{9}{2}a,\ \text{coſ.}\frac{11}{2}a,\ \text{coſ.}\frac{13}{2}a,\ \text{coſ.}\frac{15}{2}a,\ \text{coſ.}\frac{17}{2}a\ \ldots)$

Series differentiarum ſextarum

$-2^6\text{ſin.}^6\frac{1}{2}a(\text{ſin.}4a,\ \text{ſin.}5a,\ \text{ſin.}6a,\ \text{ſin.}7a,\ \text{ſin.}8a,\ \text{ſin.}9a\ \ldots)$.

Generatim ſeries differentiarum ordinis paris $2m$ eſt

$2^{2m}\text{ſin.}^{2m}\frac{1}{2}a(\text{ſin.}(m+1)a,\ \text{ſin.}(m+2)a,\ \text{ſin.}(m+3)a,\ \text{ſin.}(m+4)a,\ \text{ſin.}(m+5)a\ \ldots)$

cum alterutro ſigno $\pm$, prouti m eſt $\begin{matrix}\text{par}\\ \text{impar}\end{matrix}$.

Series differentiarum ordinis imparis $2m+1$ eſt

$2^{2m+1}\text{ſin.}^{2m+1}\frac{1}{2}a(\text{coſ.}\frac{2m+3}{2}a,\ \text{coſ.}\frac{2m+5}{2}a,\ \text{coſ.}\frac{2m+7}{2}a,\ \text{coſ.}\frac{2m+9}{2}a,\ \text{coſ.}\frac{2m+11}{2}a\ \ldots)$

cum alterutro ſigno $\pm$, prouti m eſt $\begin{matrix}\text{o vel par}\\ \text{impar}\end{matrix}$.

2°. Sint

2°. Sint $\text{coſ.}a$, $\text{coſ.}2a$, $\text{coſ.}3a$, $\text{coſ.}4a$, $\text{coſ.}5a$, $\text{coſ.}6a, \ldots \text{coſ.}na$, coſinus arcuum juxta ſeriem numerorum naturalium creſcentium.

Series differentiarum primarum eſt

$$-2\,\text{ſin.}\tfrac{1}{2}a\,(\text{ſin.}\tfrac{3}{2}a,\ \text{ſin.}\tfrac{5}{2}a,\ \text{ſin.}\tfrac{7}{2}a,\ \text{ſin.}\tfrac{9}{2}a,\ \text{ſin.}\tfrac{11}{2}a,\ \text{ſin.}\tfrac{13}{2}a\ \ldots)$$

Series differentiarum ſecundarum eſt

$$-2^2\,\text{ſin.}^2\tfrac{1}{2}a\,(\text{coſ.}2a,\ \text{coſ.}3a,\ \text{coſ.}4a,\ \text{coſ.}5a,\ \text{coſ.}6a,\ \text{coſ.}7a\ \ldots)$$

Series differentiarum tertiarum eſt

$$+2^3\,\text{ſin.}^3\tfrac{1}{2}a\,(\text{ſin.}\tfrac{5}{2}a,\ \text{ſin.}\tfrac{7}{2}a,\ \text{ſin.}\tfrac{9}{2}a,\ \text{ſin.}\tfrac{11}{2}a,\ \text{ſin.}\tfrac{13}{2}a,\ \text{ſin.}\tfrac{15}{2}a\ \ldots)$$

Series differentiarum quartarum eſt

$$2^4\,\text{ſin.}^4\tfrac{1}{2}a\,(\text{coſ.}3a,\ \text{coſ.}4a,\ \text{coſ.}5a,\ \text{coſ.}6a,\ \text{coſ.}7a,\ \text{coſ.}8a\ \ldots)$$

Series differentiarum quintarum eſt

$$-2^5\,\text{ſin.}^5\tfrac{1}{2}a\,(\text{ſin.}\tfrac{7}{2}a,\ \text{ſin.}\tfrac{9}{2}a,\ \text{ſin.}\tfrac{11}{2}a,\ \text{ſin.}\tfrac{13}{2}a,\ \text{ſin.}\tfrac{15}{2}a,\ \text{ſin.}\tfrac{17}{2}a\ \ldots)$$

Series differentiarum ſextarum eſt

$$-2^6\,\text{ſin.}^6\tfrac{1}{2}a\,(\text{coſ.}4a,\ \text{coſ.}5a,\ \text{coſ.}6a,\ \text{coſ.}7a,\ \text{coſ.}8a,\ \text{coſ.}9a\ \ldots)$$

Generatim ſeries differentiarum ordinis paris $2m$ eſt

$$2^{2m}\,\text{ſin.}^{2m}\tfrac{1}{2}a\,(\text{coſ.}(m+1)a,\ \text{coſ.}(m+2)a,\ \text{coſ.}(m+3)a,\ \text{coſ.}(m+4)a,\ \text{coſ.}(m+5)a\ \ldots)$$

cum alterutro ſigno $\pm$, prouti m eſt $\genfrac{}{}{0pt}{}{\text{par}}{\text{impar}}$.

Et ſeries differentiarum ordinis imparis $2m+1$ eſt

$$2^{2m+1}\,\text{ſin.}^{2m+1}\tfrac{1}{2}a\,(\text{ſin.}\tfrac{2m+3}{2}a,\ \text{ſin.}\tfrac{2m+5}{2}a,\ \text{ſin.}\tfrac{2m+7}{2}a,\ \text{ſin.}\tfrac{2m+9}{2}a,\ \text{ſin.}\tfrac{2m+11}{2}a\ \ldots)$$

cum alterutro ſigno $\pm$, prouti m eſt $\genfrac{}{}{0pt}{}{\text{impar}}{\text{o vel par}}$.

Scholium. Omiſſo factore $2^m\text{ſin.}^m\tfrac{1}{2}a$, ſeries harum differentiarum ſucceſſivarum, pariter atque ipſæ ſeries primitivæ, periodice in ſe redeunt, aut ſunt ſemper diverſæ; prouti arcus a & dimidia circumferentia commenſurabiles, aut incommenſurabiles ſunt.

Monenda de compendioſo differentiarum ſymbolismo.

In ſequentibus, compendii & majoris impreſſionis facilitatis cauſa, differentiæ ordinum ſucceſſivorum potentiarum numerorum naturalium, quarum exponens m, plerumque deſignabuntur, ut ſequitur:

 Diffe-

Differentiæ		Designatio compendiosa
$2^{dæ}$	$n^m - 2(n-1)^m + (n-2)^m$	$\Delta^{II}(n^m \dots (n-2)^m)$
$3^{tiæ}$	$n^m - 3(n-1)^m + 3(n-2)^m - (n-3)^m$	$\Delta^{III}(n^m \dots (n-3)^m)$
$4^{tæ}$	$n^m - 4(n-1)^m + 6(n-2)^m - 4(n-3)^m + (n-4)^m$	$\Delta^{IV}(n^m \dots (n-4)^m)$
$5^{tæ}$	$n^m - 5(n-1)^m + 10(n-2)^m - 10(n-3)^m + 5(n-4)^m - (n-5)^m$	$\Delta^{V}(n^m \dots (n-5)^m)$.
⋮		⋮
$p^{tæ}$	$n^m - \frac{p}{1}(n-1)^m + \frac{p}{1}.\frac{p-1}{2}(n-2)^m \dots \mp \frac{p}{1}(n-(p-1))^m \pm (n-p)^m$	$\Delta^{p}(n^m \dots (n-p)^m)$.

Caput C.

De summis potestatum numerorum æquidifferentium.

Summæ potestatum numerorum æquidifferentium (quos inter series numerorum naturalium primum tenet locum) adeo frequenter calculos superiores ingrediuntur, ut strictim de illis in hac introductione dicere consultum duxerim.

Plures mathematici variis modis eas indagarunt atque exposuerunt. Cum autem scopus in hac tractatione mihi præcipue propositus sit illius ad limites dictarum summarum applicatio: nulla methodus potior & huic fini melius accommodata se mihi obtulit, quam ea, qua usus est sagacissimus Pascal in eleganti opusculo, cui titulus: *Potestatum numericarum summa* (quod pag. 34 sqq. insertum est libro ipsius inscripto: *Traité du triangle arithmetique.* Paris 1665.); & qua potestatum cujuslibet ordinis summa ad summas potentiarum ordinum inferiorum reducitur.

§. *u.* *Lemma.* Sint a & $a+d$ duo numeri ad potentiam quamlibet evecti, cujus exponens r. Differentia $(a+d)^r - a^r$ harum potestatum (per §. *e.*) erit

$$\frac{r}{1}a^{r-1}d + \frac{r}{1}.\frac{r-1}{2}a^{r-2}d^2 + \frac{r}{1}\dots\frac{r-2}{3}a^{r-3}d^3 + \frac{r}{1}\dots\frac{r-3}{4}a^{r-4}d^4 + \dots + \frac{r}{1}.\frac{r-1}{2}a^2d^{r-2} + \frac{r}{1}ad^{r-1} + d^r.$$

§. *v.* Sit series terminorum æquidifferentium, quorum minimus sit a, differentia communis d, & numerus n.

Sit (compendii causa) $\int.na^r$ summa potestatum, quarum exponens r, terminorum illorum inde a primo a usque ad n^{tum} $a+(n-1)d$. Accipiatur etiam eadem potestas termini $a+nd$ ultimum sequentis.

Sumtis differentiis potestatum ejusdem exponentis quorumlibet duorum terminorum contiguorum, erit (§. *u.*)

$(a+d)^r$

$$(a+d)^r - a^r = \frac{r}{1}a^{r-1}d + \frac{r}{1}\cdot\frac{r-1}{2}a^{r-2}d^2 + \frac{r}{1}\ldots\frac{r-2}{3}a^{r-3}d^3 \;\ldots\ldots + \frac{r}{1}ad^{r-1} + d^r$$

$$(a+2d)^r - (a+d)^r = \frac{r}{1}(a+d)^{r-1}d + \frac{r}{1}\cdot\frac{r-1}{2}(a+d)^{r-2}d^2 + \frac{r}{1}\ldots\frac{r-2}{3}(a+d)^{r-3}d^3 \ldots + \frac{r}{1}(a+d)d^{r-1} + d^r$$

$$(a+3d)^r - (a+2d)^r = \frac{r}{1}(a+2d)^{r-1}d + \frac{r}{1}\cdot\frac{r-1}{2}(a+2d)^{r-2}d^2 + \frac{r}{1}\ldots\frac{r-2}{3}(a+2d)^{r-3}d^3 \ldots + \frac{r}{1}(a+2d)d^{r-1} + d^r$$

$$(a+4d)^r - (a+3d)^r = \frac{r}{1}(a+3d)^{r-1}d + \frac{r}{1}\cdot\frac{r-1}{2}(a+3d)^{r-2}d^2 + \frac{r}{1}\ldots\frac{r-1}{2}(a+3d)^{r-3}d^3 \ldots + \frac{r}{1}(a+3d)d^{r-1} + d^r$$

$$\vdots$$

$$(a+(n-2)d)^r - (a+(n-3)d)^r = \frac{r}{1}(a+(n-3)d)^{r-1}d + \frac{r}{1}\cdot\frac{r-1}{2}(a+(n-3)d)^{r-2}d^2 + \frac{r}{1}\ldots\frac{r-2}{3}(a+(n-3)d)^{r-3}d^3 \ldots + \frac{r}{1}(a+(n-3)d)d^{r-1} + d^r$$

$$(a+(n-1)d)^r - (a+(n-2)d)^r = \frac{r}{1}(a+(n-2)d)^{r-1}d + \frac{r}{1}\cdot\frac{r-1}{2}(a+(n-2)d)^{r-2}d^2 + \frac{r}{1}\ldots\frac{r-2}{3}(a+(n-2)d)^{r-3}d^3 \ldots + \frac{r}{1}(a+(n-2)d)d^{r-1} + d^r$$

$$(a+nd)^r - (a+(n-1)d)^r = \frac{r}{1}(a+(n-1)d)^{r-1}d + \frac{r}{1}\cdot\frac{r-1}{2}(a+(n-1)d)^{r-2}d^2 + \frac{r}{1}\ldots\frac{r-2}{3}(a+(n-1)d)^{r-3}d^3 \ldots + \frac{r}{1}(a+(n-1)d)d^{r-1} + d^r.$$

Summa omnium priorum membrorum harum æquationum eſt $(a+nd)^r - a^r$.

Et ſumma omnium poſteriorum membrorum eſt

$$\frac{r}{1}d\int.na^{r-1} + \frac{r}{1}\cdot\frac{r-1}{2}d^2\int.na^{r-2} + \frac{r}{1}\ldots\frac{r-2}{3}d^3\int.na^{r-3} + \frac{r}{1}\ldots\frac{r-3}{4}d^4\int.na^{r-4} + \ldots\ldots + \frac{r}{1}d^{r-1}\int.na + nd^r.$$

Hinc $\frac{r}{1}d\int.na^{r-1} = (a+nd)^r - a^r - \left(\frac{r}{1}\cdot\frac{r-1}{2}d^2\int.na^{r-2} + \frac{r}{1}\ldots\frac{r-2}{3}d^3\int.na^{r-3} + \ldots\ldots + \frac{r}{1}d^{r-1}\int.na + nd^r\right)$

unde $\frac{r+1}{1}d\int.na^r = (a+nd)^{r+1} - a^{r+1} - \left(\frac{r+1}{1}\cdot\frac{r}{2}d^2\int.na^{r-1} + \frac{r+1}{1}\ldots\frac{r-1}{3}d^3\int.na^{r-2} + \ldots\ldots + \frac{r+1}{1}d^r\int.na + nd^{r+1}\right).$

Et ſic ſumma poteſtatum cujuslibet ordinis per legem omnino regularem ad ſummas poteſtatum omnium ordinum inferiorum reducitur.

§. x. *Exemplum.* Series numerorum æquidifferentium ſit ſeries numerorum naturalium ab unitate usque ad n; ubi $a=1$, $d=1$. Et ſit $\int n^r$ ſumma poteſtatum ordinis r primorum n numerorum naturalium. Fit

$$\frac{r+1}{1}\int.n^r = (n+1)^{r+1} - 1 - \left(\frac{r+1}{1}\cdot\frac{r}{2}\int.n^{r-1} + \frac{r+1}{1}\ldots\frac{r-1}{3}\int.n^{r-2} + \frac{r+1}{1}\ldots\frac{r-2}{4}\int.n^{r-3} + \frac{r+1}{1}\ldots\frac{r-3}{5}\int.n^{r-4} \ldots\ldots + \frac{r+1}{1}\int.n + n\right).$$

Sit $r=1$. Fit $2\int.n = (n+1)^2 - 1 - n = n.n+1$

$r=2$ $\quad 3\int.n^2 = (n+1)^3 - 1 - \left(\frac{3.2}{1.2}\int.n + n\right) = \frac{n.n+1.\ 2n+1}{1.2}$

$$r=3 \qquad 4\int n^3=(n+1)^4-1-\left(\frac{4.3}{1.2}\int n^2+\frac{4.3.2}{1.2.3}\int n+n\right)=n^2.(n+1)^2$$

$$r=4 \qquad 5\int n^4=(n+1)^5-1-\left(\frac{5.4}{1.2}\int n^3+\frac{5\ldots3}{1\ldots3}\int n^2+\frac{5\ldots2}{1\ldots4}\int n+n\right)$$

$$r=5 \qquad 6\int n^5=(n+1)^6-1-\left(\frac{6.5}{1.2}\int n^4+\frac{6\ldots4}{1\ldots3}\int n^3+\frac{6\ldots3}{1\ldots4}\int n^2+\frac{6\ldots2}{1\ldots5}\int n+n\right)$$

$$r=6 \qquad 7\int n^6=(n+1)^7-1-\left(\frac{7.6}{1.2}\int n^5+\frac{7\ldots5}{1\ldots3}\int n^4+\frac{7\ldots4}{1\ldots4}\int n^3+\frac{7\ldots3}{1\ldots5}\int n^2+\frac{7\ldots2}{1\ldots6}\int n+n\right).$$

§. *y*. Ad scopum nostrum pertinet observare, quod

$$\int n = An^2+Bn \quad \text{- - - - - - - -} \quad \text{ubi } A=\tfrac{1}{2}$$
$$\int n^2 = An^3+Bn^2+Cn \quad \text{- - - - - - - -} \quad \text{ubi } A=\tfrac{1}{3}$$
$$\int n^3 = An^4+Bn^3+Cn^2+Dn \quad \text{- - - - - - -} \quad \text{ubi } A=\tfrac{1}{4}$$
$$\int n^4 = An^5+Bn^4+Cn^3+Dn^2+En \quad \text{- - - - -} \quad \text{ubi } A=\tfrac{1}{5}$$
$$\int n^5 = An^6+Bn^5+Cn^4+Dn^3+En^2+Fn \quad \text{- - - -} \quad \text{ubi } A=\tfrac{1}{6}$$
$$\int n^6 = An^7+Bn^6+Cn^5+Dn^4+En^3+Fn^2+Gn \quad \text{- - -} \quad \text{ubi } A=\tfrac{1}{7}$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$\int n^r = An^{r+1}+Bn^r+Cn^{r-1}+Dn^{r-2}+En^{r-3}+Fn^{r-4}+Gn^{r-5}\ldots+Ln^2+Mn, \quad \text{ubi } A=\frac{1}{r+1}.$$

Quod ad reliquos coëfficientes attinet: B est constans, nempe $=\frac{1}{2}$. Cæteri autem C, D, $E \ldots L$, M pendent ab exponente variabili r, juxta legem quamdam, cujus evolutio ad scopum præsentem non pertinet; & de qua videatur inter alios Jac. Bernoulli *Ars conjectandi* pag. 97.

Caput D.

De sinibus et cosinibus arcuum multiplorum, et de sinuum in factores resolutione.

§. *z*. *Lemma notum*. 1°. $\text{sin.}(a+b) = \text{sin.}\,a\,\text{cos.}\,b + \text{cos.}\,a\,\text{sin.}\,b$

2°. $\text{cos.}(a+b) = \text{cos.}\,a\,\text{cos.}\,b - \text{sin.}\,a\,\text{sin.}\,b$.

§. *aa*. Ex his duabus formulis deduci possunt expressiones sinuum & cosinuum arcuum, qui multipli sunt alicujus arcus propositi. Scilicet

1°. cos.

1°. $\cos.2a(=\cos.(a+a)) = \cos.^2 a - \sin.^2 a$ $\qquad$ $\sin.2a(=\sin.(a+a)) = 2\sin.a\cos.a.$

2°. $\cos.3a(=\cos.(2a+a))=\cos.2a\cos.a-\sin.2a\sin.a = \cos.^3 a - 3\cos.a\sin.^2 a$

$\sin.3a(=\sin.(2a+a))=\sin.2a\cos.a+\cos.2a\sin.a = 3\sin.a\cos.^2 a - \sin.^3 a.$

3°. $\cos.4a(=\cos.(3a+a))=\cos.3a\cos.a-\sin.3a\sin.a = \cos.4a - 6\cos.^2 a\sin.^2 a + \sin.^4 a$

$\sin.4a(=\sin.(3a+a))=\sin.3a\cos.a+\cos.3a\sin.a = 4\cos.^3 a\sin.a - 4\cos.a\sin.^3 a.$

4°. $\cos.5a(=\cos.(4a+a))=\cos.4a\cos.a-\sin.4a\sin.a = \cos.^5 a - 10\cos.^3 a\sin.^2 a + 5\cos.a\sin.^4 a$

$\sin.5a(=\sin.(4a+a))=\sin.4a\cos.a+\cos.4a\sin.a = 5\cos.^4 a\sin.a - 10\cos.^2 a\sin.^3 a + \sin.^5 a.$

5°. $\cos.6a(=\cos.(5a+a))=\cos.5a\cos.a-\sin.5a\sin.a = \cos.6a - 15\cos.^4 a\sin.^2 a + 15\cos.^2 a\sin.^4 a - \sin.^6 a.$

$\sin.6a(=\sin.(5a+a))=\sin.5a\cos.a+\cos.5a\sin.a = 6\cos.^5 a\sin.a - 20\cos.^3 a\sin.^3 a + 6\cos.a\sin.^5 a$

Exempla hæc sufficiunt ad indicandam legem harum expressionum, qua fit

$$\begin{aligned}\cos.2ma = {} & \cos.^{2m} a \\ & - \frac{2m}{1}\cdot\frac{2m-1}{2}\cos.^{2m-2} a\sin.^2 a \\ & + \frac{2m}{1}\cdots\frac{2m-3}{4}\cos.^{2m-4} a\sin.^4 a \\ & - \frac{2m}{1}\cdots\frac{2m-5}{6}\cos.^{2m-6} a\sin.^6 a \\ & + \;\text{- - - - -} \\ & \;\vdots \\ & \pm \frac{2m}{1}\cdot\frac{2m-1}{2}\cos.^2 \sin.^{2m-2} a \\ & \mp \sin.^{2m} a\end{aligned}$$

$$\begin{aligned}\sin.2ma = {} & \frac{2m}{1}\cos.^{2m-1} a\sin.a \\ & - \frac{2m}{1}\cdots\frac{2m-2}{3}\cos.^{2m-3} a\sin.^3 a \\ & + \frac{2m}{1}\cdots\frac{2m-4}{5}\cos.^{2m-5} a\sin.^5 a \\ & - \frac{2m}{1}\cdots\frac{2m-6}{7}\cos.^{2m-7} a\sin.^7 a \\ & + \;\text{- - - - -} \\ & \;\vdots \\ & \mp \frac{2m}{1}\cdots\frac{2m-2}{3}\cos.^3 a\sin.^{2m-3} a \\ & \pm \frac{2m}{1}\cos.a\sin.^{2m-1} a\end{aligned}$$

cos.

$$\cos.(2m+1)a = \cos.^{2m+1}a$$
$$-\frac{2m+1}{1}.\frac{2m}{2}\cos.^{2m-1}a\sin.^2a$$
$$+\frac{2m+1}{1}\ldots\frac{2m-2}{4}\cos.^{2m-3}a\sin.^4a$$
$$-\frac{2m+1}{1}\ldots\frac{2m-4}{6}\cos.^{2m-5}a\sin.^6a$$
$$+ \; - \; - \; - \; - \; -$$
$$\vdots \qquad\qquad \vdots$$
$$\pm\frac{2m+1}{1}\ldots\frac{2m-1}{3}\cos.^3a\sin.^{2m-2}a$$
$$\mp\frac{2m+1}{1}\cos.a\sin.^{2m}a$$

$$\sin.(2m+1)a = \frac{2m+1}{1}\cos.^{2m}a\sin.a$$
$$-\frac{2m+1}{1}\ldots\frac{2m-1}{3}\cos.^{2m-2}a\sin.^3a$$
$$+\frac{2m+1}{1}\ldots\frac{2m-3}{5}\cos.^{2m-4}a\sin.^5a$$
$$-\frac{2m+1}{1}\ldots\frac{2m-5}{6}\cos.^{2m-6}a\sin.^7a$$
$$+ \; - \; - \; - \; - \; -$$
$$\vdots \qquad\qquad \vdots$$
$$\mp\frac{2m+1}{1}.\frac{2m}{2}\cos.^2a\sin.^{2m-1}a$$
$$\pm \sin.^{2m+1}a.$$

Demonstratur autem modo mathematicis familiari: quod, si hæc lex obtineat pro exponente quocunque proposito, eadem etiam valeat pro exponente unitate majori. Sed lex locum habet pro exponentibus 2, 3 . . . 6: ergo & locum habet pro exponente sequente 7; inde pro exponente 8; & sic deinceps.

§. ab. Termini igitur, quibus expressio cosinus arcus m^{pli} arcus a constat, sunt termini alterni binomii $(\cos.z+\sin.z)^m$; sed alternis vicibus, a primo inde, signis + & — affecti: & termini, quibus constat expressio sinus arcus ejusdem m^{pli} sunt reliqui termini alterni ejusdem binomii, alternis vicibus a secundo inde signis + & — pariter affecti.

Quoniam autem dimidia summa potentiarum ejusdem exponentis m summæ $a+b$ & differentiæ $a-b$ duarum quantitatum realium a & b constat terminis alternis binomii $(a+b)^m$, a primo inde, iisque signo + affectis; dum dimidia differentia harum potestatum continet reliquos terminos alternos etiam positivos: ut expressio cosinuum arcuum multiplorum reduci possit ad summam potestatum similium summæ & differentiæ cosinus & sinus arcus simplicis; sinus hic affici debet coëfficiente tali, ut potentiæ ejus impariter pares signo — afficiantur, potestates vero pariter pares maneant positivæ. Quod fiet, si loco $\cos.z+\sin.z$ substituatur $\cos.z+\sin.z\sqrt{-1}$; unde ad hanc deducimur expressionem $\cos.mz = \frac{(\cos.z+\sin.z\sqrt{-1})^m+(\cos.z-\sin.z\sqrt{-1})^m}{2}$.

Evo-

Evoluta autem duarum poteſtatum $(\text{coſ.}z+\text{ſin.}z\sqrt{-1})^m$, $(\text{coſ.}z-\text{ſin.}z\sqrt{-1})^m$ differentia, omnes ejus termini afficiuntur coëfficiente imaginario $\sqrt{-1}$; unde, ut expreſſio maneat realis, inferimus $\text{ſin.}mz = \frac{(\text{coſ.}z+\text{ſin.}z\sqrt{-1})^m-(\text{coſ.}z-\text{ſin.}z\sqrt{-1})^m}{2\sqrt{-1}}$.

Quodſi enim expreſſiones hæ veræ ſint pro exponente quodam m, eædem etiam valent pro exponente unitate majore $m+1$.

Etenim

$$\text{coſ.}(m+1)z = \text{coſ.}mz\,\text{coſ.}z - \text{ſin.}mz\,\text{ſin.}z$$
$$= \frac{(\text{coſ.}z+\text{ſin.}z\sqrt{-1})^m+(\text{coſ.}z-\text{ſin.}z\sqrt{-1})^m}{2} \times \frac{(\text{coſ.}z+\text{ſin.}z\sqrt{-1})+(\text{coſ.}z-\text{ſin.}z\sqrt{-1})}{2}$$
$$- \frac{(\text{coſ.}z+\text{ſin.}z\sqrt{-1})^m-(\text{coſ.}z-\text{ſin.}z\sqrt{-1})^m}{2\sqrt{-1}} \times \frac{(\text{coſ.}z+\text{ſin.}z\sqrt{-1})-(\text{coſ.}z-\text{ſin.}z\sqrt{-1})}{2\sqrt{-1}}$$

Quæ ſumma reducitur ad $\frac{(\text{coſ.}z+\text{ſin.}z\sqrt{-1})^{m+1}+(\text{coſ.}z-\text{ſin.}z\sqrt{-1})^{m+1}}{2}$, uti propoſitum.

Eodemque modo fit $\text{ſin.}(m+1)z = \frac{(\text{coſ.}z+\text{ſin.}z\sqrt{-1})^{m+1}-(\text{coſ.}z-\text{ſin.}z\sqrt{-1})^{m+1}}{2\sqrt{-1}}$

Obſervatio. Expreſſiones hæ imaginariæ (quæ in calculo inprimis integrali, & in ſerierum ſummatione frequentiſſime uſurpantur) mera ſunt ſigna, majoris calculi facilitatis gratia ab mathematicis introducta. Partes earum $(\text{coſ.}z \pm \text{ſin.}z\sqrt{-1})^m$ ſeorſim ſumtæ omni ſenſu carent; & operationes his ſymbolis juxta regulas calculi litteralis ad eas extenſas inſtitutæ eatenus tantum quantitates reales ultimo exhibent, quatenus termini imaginarii, quos expreſſiones illæ continent, ſeſe mutuo deſtruunt. Scilicet æquatio $2a=(a+d)+(a-d)$ ſemper obtinet, quidquid ſit d; ideoque etiam ſi d involvat ſignum impoſſibilitatis, quod utramque expreſſionem $a+d$, $a-d$ pariter impoſſibilem reddit.

§. *ac.* Quoniam $\text{coſ.}mz = \frac{(\text{coſ.}z+\text{ſin.}z\sqrt{-1})^m+(\text{coſ.}z-\text{ſin.}z\sqrt{-1})^m}{2}$

et $\text{ſin.}mz = \frac{(\text{coſ.}z+\text{ſin.}z\sqrt{-1})^m-(\text{coſ.}z-\text{ſin.}z\sqrt{-1})^m}{2\sqrt{-1}}$

erit $2\,\text{coſ.}mz = (\text{coſ.}z+\text{ſin.}z\sqrt{-1})^m+(\text{coſ.}z-\text{ſin.}z\sqrt{-1})^m$

et $2\,\text{ſin.}mz\sqrt{-1} = (\text{coſ.}z+\text{ſin.}z\sqrt{-1})^m-(\text{coſ.}z-\text{ſin.}z\sqrt{-1})^m$

****

Unde

Unde fit $(\cos. z+\sin. z\sqrt{-1})^m=\cos. mz+\sin. mz\sqrt{-1}$

et $(\cos. z-\sin. z\sqrt{-1})^m=\cos. mz-\sin. mz\sqrt{-1}$

Hinc . $\cos. z+\sin. z\sqrt{-1} =(\cos. mz+\sin. mz\sqrt{-1})^{\frac{1}{m}}$

et $\cos. z-\sin. z\sqrt{-1} =(\cos. mz-\sin. mz\sqrt{-1})^{\frac{1}{m}}$

Unde $$\cos. z =\frac{(\cos. mz+\sin. mz\sqrt{-1})^{\frac{1}{m}}+(\cos. mz-\sin. mz\sqrt{-1})^{\frac{1}{m}}}{2}$$

$$\sin. z =\frac{(\cos. mz+\sin. mz\sqrt{-1})^{\frac{1}{m}}-(\cos. mz-\sin. mz\sqrt{-1})^{\frac{1}{m}}}{2\sqrt{-1}}.$$

Proinde eadem lex, quæ valet pro ſinibus & coſinibus arcuum multiplorum, pariter valet pro ſinibus & coſinibus arcuum ſubmultiplorum.

§. *ad.* *Theorema Coteſianum* (a ſagaciſſimo ejus auctore denominatum) illud eſt, quo demonſtratur, functiones utriusque formæ $a^n \pm b^n$ in factores binomios aut trinomios reſolvi modo ſequenti:

1°. $$a^{2n}+b^{2n}=(aa-2ab\cos.\tfrac{1}{2n}\pi+bb)\times(aa-2ab\cos.\tfrac{3}{2n}\pi+bb)\times(aa-2ab\cos.\tfrac{5}{2n}\pi+bb)\times\ldots\times(aa-2ab\cos.\tfrac{2n-3}{2n}\pi+bb)\times(aa-2ab\cos.\tfrac{2n-1}{2n}\pi+bb)$$

2°. $$a^{2n+1}+b^{2n+1}=(a+b)\times(aa-2ab\cos.\tfrac{2}{2n+1}\pi+bb)\times(aa-2ab\cos.\tfrac{4}{2n+1}\pi+bb)\times(aa-2ab\cos.\tfrac{6}{2n+1}\pi+bb)\times\ldots\times(aa-2ab\cos.\tfrac{2n-2}{2n+1}\pi+bb)\times(aa-2ab\cos.\tfrac{2n}{2n+1}\pi+bb)$$

3°. $$a^{2n}-b^{2n}=(aa-bb)\times(aa-2ab\cos.\tfrac{2}{2n}\pi+bb)\times(aa-2ab\cos.\tfrac{4}{2n}\pi+bb)\times(aa-2ab\cos.\tfrac{6}{2n}\pi+bb)\times\ldots\times(aa-2ab\cos.\tfrac{2n-4}{2n}\pi+bb)\times(aa-2ab\cos.\tfrac{2n-2}{2n}\pi+bb)$$

4°. $$a^{2n+1}-b^{2n+1}=(a-b)\times(aa-2ab\cos.\tfrac{2}{2n+1}\pi+bb)\times(aa-2ab\cos.\tfrac{4}{2n+1}\pi+bb)\times(aa-2ab\cos.\tfrac{6}{2n+1}\pi+bb)\times\ldots\times(aa-2ab\cos.\tfrac{2n-2}{2n+1}\pi+bb)\times(aa-2ab\cos.\tfrac{2n}{2n+1}\pi+bb).$$

Theo-

Theorematі huic demonſtrando (quod a variis auctoribus præſtitum eſt) non immoror; quamvis talis ejus demonſtrationis compotem me eſſe credam, quæ ſimplicitate ſua ſeſe commendat. Pergo ideo ad illlus applicationem, quæ deinceps utilis erit.

§. *ae.* 1°. In prima formula fiat $a = b = 1$:

$$\begin{aligned} \text{fit} \quad 2 = & \; 2(1-\text{cof.}\tfrac{1}{2n}\pi) \\ & \times 2(1-\text{cof.}\tfrac{3}{2n}\pi) \\ & \times 2(1-\text{cof.}\tfrac{5}{2n}\pi) \\ & \quad \vdots \qquad \vdots \\ & \times 2(1-\text{cof.}\tfrac{2n-3}{2n}\pi) \\ & \times 2(1-\text{cof.}\tfrac{2n-1}{2n}\pi) \\ = & \; 4^n \text{fin.}^2 \tfrac{1}{2n}p\, \text{fin.}^2 \tfrac{3}{2n}p\, \text{fin.}^2 \tfrac{5}{2n}p \ldots \text{fin.}^2 \tfrac{2n-3}{2n}p\, \text{fin.}^2 \tfrac{2n-1}{2n}p \end{aligned}$$

Unde $\sqrt{2} = 2^n \text{fin.}\frac{1}{2n}p\,\text{fin.}\frac{3}{2n}p\,\text{fin.}\frac{5}{2n}p \ldots \text{fin.}\frac{2n-3}{2n}p\,\text{fin.}\frac{2n-1}{2n}p.$

2°. Ex ſecunda formula fit pariter

$$\sqrt{2} = 2^n \text{fin.}\frac{2}{2n+1}p\,\text{fin.}\frac{4}{2n+1}p\,\text{fin.}\frac{6}{2n+1}p \ldots \text{fin.}\frac{2n-2}{2n+1}p\,\text{fin.}\frac{2n}{2n+1}p.$$

3°. $a^{2n}-b^{2n} = (aa-bb)(a^{2n-2}+a^{2n-4}bb+a^{2n-6}b^4+\ldots+a^2b^{2n-4}+b^{2n-2})$

Hinc

$$\begin{aligned} a^{2n-2}+a^{2n-4}bb+a^{2n-6}b^4+\ldots+a^2b^{2n-4}+b^{2n-2} = & \; (aa-2ab\,\text{cof.}\tfrac{1}{n}\pi+bb) \\ & \times(aa-2ab\,\text{cof.}\tfrac{2}{n}\pi+bb) \\ & \times(aa-2ab\,\text{cof.}\tfrac{3}{n}\pi+bb) \\ & \quad \vdots \qquad \vdots \\ & \times(aa-2ab\,\text{cof.}\tfrac{n-2}{n}\pi+bb) \\ & \times(aa-2ab\,\text{cof.}\tfrac{n-1}{n}\pi+bb). \end{aligned}$$

Hinc factis $a=b=1$: $n = 4^{n-1}\text{fin.}^2\frac{1}{n}p\,\text{fin.}^2\frac{2}{n}p\,\text{fin.}^2\frac{3}{n}p \ldots \text{fin.}^2\frac{n-2}{n}p\,\text{fin.}^2\frac{n-1}{n}p$

et $\sqrt{n} = 2^{n-1}\,\text{fin.}\frac{1}{n}p\,\text{fin.}\frac{2}{n}p\,\text{fin.}\frac{3}{n}p \ldots \text{fin.}\frac{n-2}{n}p\,\text{fin.}\frac{n-1}{n}p.$

4°. Ex

4°. Ex quarta formula fit pariter

$$\sqrt{(2n+1)} = 2^n \sin.\frac{2}{2n+1}p \sin.\frac{4}{2n+1}p \sin.\frac{6}{2n+1}p \ldots \sin.\frac{2n-2}{2n+1}p \sin.\frac{2n}{2n+1}p.$$

§. af. Post *Cotesium* theorema ipsius fuit amplificatum, & formula quoque $a^{2n} \pm 2a^n b^n \cos.\varphi + b^{2n}$ resoluta in factores, ut sequitur, prouti n numerus est par aut impar:

1°. $a^{4n} - 2a^{2n}b^{2n}\cos.2\varphi + b^{4n}$

$$= (aa - 2ab\cos.\frac{2\varphi}{2n} + bb)$$
$$\times (aa - 2ab\cos.\frac{2\pi - 2\varphi}{2n} + bb)$$
$$\times (aa - 2ab\cos.\frac{2\pi + 2\varphi}{2n} + bb)$$
$$\times (aa - 2ab\cos.\frac{4\pi - 2\varphi}{2n} + bb)$$
$$\times (aa - 2ab\cos.\frac{4\pi + 2\varphi}{2n} + bb)$$
$$\vdots$$
$$\times (aa - 2ab\cos.\frac{(2n-2)\pi - 2\varphi}{2n} + bb)$$
$$\times (aa - 2ab\cos.\frac{(2n-2)\pi + 2\varphi}{2n} + bb)$$
$$\times (aa - 2ab\cos.\frac{2n\pi - 2\varphi}{2n} + bb)$$

2°. $a^{4n+2} - 2a^{2n+1}b^{2n+1}\cos.2\varphi + b^{4n+2}$

$$= (aa - 2ab\cos.\frac{2\varphi}{2n+1} + bb)$$
$$\times (aa - 2ab\cos.\frac{2\pi - 2\varphi}{2n+1} + bb)$$
$$\times (aa - 2ab\cos.\frac{2\pi + 2\varphi}{2n+1} + bb)$$
$$\times (aa - 2ab\cos.\frac{4\pi - 2\varphi}{2n+1} + bb)$$
$$\times (aa - 2ab\cos.\frac{4\pi + 2\varphi}{2n+1} + bb)$$
$$\vdots$$
$$\times (aa - 2ab\cos.\frac{(2n-2)\pi + 2\varphi}{2n+1} + bb)$$
$$\times (aa - 2ab\cos.\frac{2n\pi - 2\varphi}{2n+1} + bb)$$
$$\times (aa - 2ab\cos.\frac{2n\pi + 2\varphi}{2n+1} + bb).$$

§. ag. Hinc factis $a = b = 1$, erit

1°. $$4\sin.^2\varphi = 4^{2n}\sin.^2\frac{\varphi}{2n}\sin.^2\frac{\pi-\varphi}{2n}\sin.^2\frac{\pi+\varphi}{2n}\sin.^2\frac{2\pi-\varphi}{2n}\sin.^2\frac{2\pi+\varphi}{2n}\ldots\sin.^2\frac{(n-1)\pi-\varphi}{2n}\sin.^2\frac{(n-1)\pi+\varphi}{2n}\sin.^2\frac{n\pi-\varphi}{2n}$$

et $$\sin.\varphi = 2^{2n-1}\sin.\frac{\varphi}{2n}\sin.\frac{\pi-\varphi}{2n}\sin.\frac{\pi+\varphi}{2n}\sin.\frac{2\pi-\varphi}{2n}\sin.\frac{2\pi+\varphi}{2n}\ldots\sin.\frac{(n-1)\pi-\varphi}{2n}\sin.\frac{(n-1)\pi+\varphi}{2n}\sin.\frac{n\pi-\varphi}{2n}$$

2°. $$4\sin.^2\varphi = 4^{2n+1}\sin.^2\frac{\varphi}{2n+1}\sin.^2\frac{\pi-\varphi}{2n+1}\sin.^2\frac{\pi+\varphi}{2n+1}\sin.^2\frac{2\pi-\varphi}{2n+1}\sin.^2\frac{2\pi+\varphi}{2n+1}\ldots\sin.^2\frac{n\pi-\varphi}{2n+1}\sin.^2\frac{n\pi+\varphi}{2n+1}$$

et $$\sin.\varphi = 2^{2n}\sin.\frac{\varphi}{2n+1}\sin.\frac{\pi-\varphi}{2n+1}\sin.\frac{\pi+\varphi}{2n+1}\sin.\frac{2\pi-\varphi}{2n+1}\sin.\frac{2\pi+\varphi}{2n+1}\ldots\sin.\frac{n\pi-\varphi}{2n+1}\sin.\frac{n\pi+\varphi}{2n+1}.$$

CAPUT I.

Caput Primum.

De Limitibus Quantitatum et Rationum, seu de Methodo Exhaustionis.

§. 1.

Definitiones. 1°. Si quantitas mutabilis semper minor fuerit quantitate datâ (mutabili scilicet homogenea, quod deinceps semper subintelligatur); sed ita augeri poterit, ut major fiat quacunque quantitate datâ, quæ minor est prima quantitate datâ (& tam cum hac quam cum mutabili homogenea, quod pariter deinceps semper subintelligatur): hæc prima quantitas data, dicitur *Limes Quantitatis mutabilis crescentis.*

2°. Et si quantitas mutabilis semper major fuerit quantitate datâ, sed ita minui poterit, ut minor fiat quacunque quantitate datâ, quæ major est prima quantitate datâ: hæc prima quantitas data dicitur *Limes Quantitatis mutabilis decrescentis.*

3°. Si ratio mutabilis semper minor fuerit quam ratio data; sed ita augeri poterit, ut major fiat ratione quacunque datâ, quæ minor est ratione prima datâ: hæc prima ratio data dicitur *Limes Rationis mutabilis crescentis.*

4°. Si ratio mutabilis semper major fuerit quam ratio data; sed ita minui poterit, ut minor fiat ratione quacunque datâ, quæ major est ratione prima datâ: hæc prima ratio data dicitur *Limes Rationis mutabilis decrescentis.* (*a*)

E re esse censeo definitiones has nonnullis exemplis illustrare.

Sit progressio geometrica decrescens: $1, p, p^2, p^3, p^4 \ldots\ldots p^{n-1}$; in qua proinde p est minor unitate. Summa n primorum terminorum hujus progres-

(*a*) Definitiones hæ, pariter ac propositiones nonnullæ hoc capite expositæ, desumptæ sunt ex Opusculo R. Simson, inscripto: *De Limitibus Quantitatum & Rationum Fragmentum;* quod reperitur inter Opera posthuma insignis illius Geometræ, impensis Viri eruditissimi, & scientiarum mathematicarum fautoris generosissimi Comitis Stanhope in lucem edita: Glasguæ 1776.

greſſionis eſt: $\frac{1-p^n}{1-p} = \frac{1}{1-p} - \frac{p^n}{1-p}$. Ideo ſumma quotcunque terminorum hujus progreſſionis minor eſt quantitate $\frac{1}{1-p}$. Atqui: auctô n, quantitas p^n, & ideo etiam quantitas $\frac{p^n}{1-p}$, poteſt fieri minor quacunque quantitate datâ; ideo quantitas $\frac{1}{1-p} - \frac{p^n}{1-p}$ poteſt fieri major quacunque quantitate datâ, quæ minor eſt quantitate datâ $\frac{1}{1-p}$. Hinc quantitas $\frac{1}{1-p}$ dicitur limes valoris cum numero terminorum continuè creſcentis, quem ſumma progreſſionis illius geometricæ obtinet.

Idem valet de ſummis nonnullarum aliarum progreſſionum, quales ſunt ſeries numerorum figuratorum reciprocorum, aut æquimultiplorum ſive ſubmultiplorum eorundem.

Sit v. gr. $S = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} + \frac{1}{6.7} \ldots\ldots + \frac{1}{n-1.n} + \frac{1}{n.n+1}$.

$$
\begin{aligned}
\text{Erit } S = {} & \tfrac{1}{1} - \tfrac{1}{2} \\
& + \tfrac{1}{2} - \tfrac{1}{3} \\
& + \tfrac{1}{3} - \tfrac{1}{4} \\
& + \tfrac{1}{4} - \tfrac{1}{5} \\
& + \tfrac{1}{5} - \tfrac{1}{6} \\
& + \tfrac{1}{6} \ldots\ldots\ldots \\
& \quad \vdots \\
& - \tfrac{1}{n-1} \\
& + \tfrac{1}{n-1} - \tfrac{1}{n} \\
& + \tfrac{1}{n} - \tfrac{1}{n+1}. \\
= {} & 1 \ldots\ldots\ldots\ldots\ldots\ldots - \tfrac{1}{n+1}.
\end{aligned}
$$

Auctò n; quantitas $\frac{1}{n+1}$ poteſt reddi minor quacunque quantitate propoſita. Ideo 1 limes eſt ſummæ creſcentis progreſſionis illius.

Diſtinctio inter valorem & limitem quantitatis per incrementa continuè minora mutabilis apte illuſtratur exemplo ſpatiorum aſymptoticorum; quorum magnitudo ſæpe limitatur, quamvis aliqua dimenſionum eorundem ſine fine creſcere poſſit. Sic ſpatium inter duas lineas ordinatim applicatas curvæ loga-

logarithmicæ (ad Aſymptotum tanquam Axem relatæ) comprehenſum, accurate æquale eſt rectangulo facto ex ſubtangente & differentia illarum applicatarum. Atqui minor harum ordinatarum poteſt reddi minor lineâ quacunque magnitudine datâ; proinde major harum ordinatarum limes eſt differentiæ creſcentis earundem. Et rectangulum factum ex ſubtangente & majori ordinata, eſt limes ſpatii logarithmici creſcentis, quem ſpatium hoc nunquam attingit, ſed ad quem accedere poteſt propius quam ſpatium quodcunque dato prædicto rectangulo minus.

Exemplum 2dum. Demonſtrante ARCHIMEDE (*de Sphæra & Cylindro*) licet circulo inſcribere & circumſcribere polygona ordinata cognomina ſeu ſimilia, quorum ambitus & ſuperficies ab ambitu & ſuperficie circuli differunt minus quam ullâ lineâ aut ullâ ſuperficie aſſignatâ. Jam vero ambitus & ſuperficies circuli majores ſunt perimetris & ſuperficiebus polygonorum inſcriptorum; minores vero perimetris & ſuperficiebus polygonorum circumſcriptorum. Ideo ambitus & ſuperficies circuli ſunt reſpectivè limites perimetrorum & ſuperficierum (creſcentium aut decreſcentium) polygonorum ordinatorum, quæ circulo inſcribuntur, aut eidem circumſcribuntur. Eodem modo ſe habent ſuperficies & capacitates cylindrorum, conorum, ſphærarum, ad ſuperficies ac ſolida priſmatum, pyramidum, polyhedrorum, illis inſcriptorum aut circumſcriptorum.

Item: Ratio æqualitatis limes eſt rationum creſcentium aut decreſcentium, quas perimetri aut ſuperficies polygonorum regularium circulo inſcriptorum aut circumſcriptorum habent ad ambitum & ſuperficiem circuli. Idem valet de rationibus, quas cylindri, coni, ſphæræ habent ad priſmata, pyramides, polyhedra, ipſis inſcripta aut circumſcripta.

Exemplum 3tium. Sit curva quæcunque ad axem aliquem relata. Rectæ huic axi ordinatim applicatæ continuo creſcant; a zero usque ad maximam ipſarum, quæ ſumatur pro baſi. Axis dividatur in partes quotcunque inter ſe æquales. Per omnia puncta diviſionis ducantur ad curvam rectæ axi ordinatim applicatæ. Super baſi & omnibus rectis ordinatim applicatis conſtruantur verſus verticem parallelogramma, quorum altera latera ſint partes æqua-

 les

les axis. Hæc parallelogramma dicantur curvæ circumſcripta. Pariter ſuper omnibus rectis ordinatim applicatis baſi parallelis conſtituantur parallelogramma ad partes baſis, quorum altera latera ſint partes æquales axis. Hæc parallelogramma dicantur curvæ inſcripta. (Vide Fig. 1am.)

Exceſſus quô ſumma parallogrammorum circumſcriptorum ſuperat ſummam parallelogrammorum inſcriptorum; æqualis eſt parallelogrammo circumſcripto omnium maximo. Atqui baſis hujus parallelogrammi datur magnitudine: ergo, imminutâ altitudine, parallelogrammum hoc poteſt fieri minus quocunque ſpatio dato. Ideo differentia prædictarum ſummarum poteſt fieri minor quocunque ſpatio dato. Jam vero ſuperficies curvæ major eſt ſummâ parallelogrammorum inſcriptorum, ſed eadem minor eſt ſummâ parallelogrammorum circumſcriptorum; & quantitas, quâ ab alterutra harum ſummarum differt, minor eſt differentiâ mutuâ prædictarum ſummarum. Ergo figura curvilinea limes eſt, tam ſummæ creſcentis parallelogrammorum ipſi inſcriptorum, quam ſummæ decreſcentis parallelogrammorum eidem circumſcriptorum.

Rectæ axi ordinatim applicatæ ponantur eidem perpendiculares. Solidum rotatione prædictæ figuræ circa axem genitum limes erit tam ſummæ decreſcentis cylindrorum ſimul genitorum rotatione rectangulorum curvæ circumſcriptorum, quam ſummæ creſcentis cylindrorum ſimul genitorum rotatione rectangulorum curvæ inſcriptorum.

§. 2. *Scholium.* Sit quantitas aliqua data limes quantitatis mutabilis creſcentis vel decreſcentis. Utrique illarum addatur, aut ab utraque ſubtrahatur eadem aliqua quantitas. Prior ſumma, aut prior differentia, limes etiam erit poſterioris ſummæ, aut differentiæ, creſcentis vel decreſcentis. Si vero utraque quantitas ab aliqua eadem quantitate ſubtrahatur: prius reſiduum limes erit poſterioris decreſcentis aut creſcentis.

Fig. 2. Nempe denotet AB quantitatem datam, quæ limes ſit quantitatis mutabilis v. gr. creſcentis AX. Utrique illarum addatur, aut ab utraque ſubtrahatur eadem quantitas AC aut AC': prior ſumma CB, aut prior differentia $C'B$, limes quoque eſt creſcentis poſterioris ſummæ CX, aut poſterioris differen-

ferentiæ $C'X$. Si vero utraque illarum ſubtrahatur ab eadem quantitate AC'': prius reſiduum $C''B$ limes eſt poſterioris reſidui decreſcentis $C''X$. Quod per ſe patet.

§. 3. *Theorema.* Sit quantitas data limes quantitatis mutabilis creſcentis vel decreſcentis. Dico rationem æqualitatis limitem eſſe rationis decreſcentis vel creſcentis, prioris quantitatis ad poſteriorem.

Nempe. Sit quantitas data AB limes quantitatis creſcentis v. gr. AX. Dico rationem $AB : AX$ poſſe accedere ad rationem æqualitatis, propius quam ad eam accedit data quæcunque ratio majoris ad minorem.

Demonſtratio. Quæcunque detur ratio majoris ad minorem; fiat ipſi æqualis ratio AB ad AD, quæ minor erit quam AB. Et fiat $AX > AD$ (quod poſſibile eſt per hyp.) Erit $AB : AX < AB : AD$. (*a*)

Eodem modo demonſtratur de altero Limite.

Viciſſim ſi ratio æqualitatis fuerit limes rationis decreſcentis aut creſcentis alicujus quantitatis datæ ad quantitatem mutabilem: Dico quantitatem datam limitem eſſe quantitatis mutabilis creſcentis aut decreſcentis.

Sit (ex. gr.) ratio æqualitatis limes rationis decreſcentis quantitatis datæ AB ad quantitatem mutabilem AX. Dico: quantitatem mutabilem AX poſſe fieri majorem quacunque quantitate datâ AD, quæ minor eſt quam AB.

Etenim per hyp. ratio $AB : AX$ poteſt fieri minor ratione data $AB : AD$ majoris ad minorem. Factum ſit: et cum $AB : AX < AB : AD$

erit $AX > AD$.

Eodem modo demonſtratur de altero limite.

Exempla. Cum perimeter & ſuperficies circuli limites ſint perimetrorum & ſuperficierum creſcentium aut decreſcentium polygonorum ordinatorum, quæ circulo inſcribuntur & circumſcribuntur: ratio æqualitatis eſt limes rationum

 decre-

(*a*) Quoniam demonſtrationes plerarumque propoſitionum hoc Capite evolutarum inæqualium rationum proprietatibus nituntur: ſi cui non fuerint ſatis notæ; ille adeat eximiam Diſſertationem inſcriptam: *Propoſitionum de Rationibus inter ſe diverſis Demonſtrationes, ex ſolis Libri V. Element. definitionibus & propoſitionibus deductæ;* quam Præſide celeb. Prof. Pfleiderer publice propoſuit & defendit ornat. Candid. Hauber. *Tubing.* 1793.

decreſcentium aut creſcentium perimetri & ſuperficiei circuli, ad perimetros & ſuperficies polygonorum regularium ipſi inſcriptorum aut circumſcriptorum. Idem valet de ſuperficiebus & capacitatibus cylindrorum, conorum, ſphærarum, reſpectu priſmatum, pyramidum, polyhedrorum, quæ ipſis inſcribuntur & circumſcribuntur.

Item: cum area figuræ §. 1. Ex. 3. Fig. 1. limes ſit ſummarum creſcentium aut decreſcentium parallelogrammorum, quæ ipſi inſcribuntur aut circumſcribuntur: ratio æqualitatis eſt limes rationis decreſcentis aut creſcentis ejusdem figuræ ad ſummas prædictas. Idemque valet de ſolidis ibidem commemoratis; & de ſummis cylindrorum, qui ipſis inſcribuntur & circumſcribuntur.

§. 4. *Theorema.* Sint duæ quantitates homogeneæ limitum capaces; ad quos vel utraque quantitas creſcendo, vel utraque quantitas decreſcendo, continue propius accedit. Sitque ratio harum quantitatum mutabilium ſemper eadem; ſeu æqualis rationi datæ: dico rationem limitum ipſarum æqualem eſſe eidem rationi datæ.

Fig. 3. Sint AB, CD, limites duarum quantitatum mutabilium AX, $C\Upsilon$. Sit ratio $AX : C\Upsilon$ ſemper æqualis rationi datæ $m : n$: dico rationem limitum AB, CD æqualem eſſe eidem rationi datæ $m : n$.

1°. 1°. Sint AB, CD limites quantitatum mutabilium creſcentium AX, $C\Upsilon$.

Si ratio $AB : CD$ non ſit æqualis rationi datæ $m : n$; erit ea major aut minor. Tum vero, priore quidem caſu, ratio $m : n$ æqualis erit rationi alicujus quantitatis minoris ipſâ AB ad CD; altero autem caſu, ratio $m : n$ æqualis erit rationi AB ad aliquam quantitatem minorem quam CD. Proinde utroque caſu unus ex limitibus minui deberet, ut ratio limitis ita imminuti ad alterum limitem fieret æqualis rationi $m : n$.

Sit itaque, ſi fieri poſſit: $m : n = AB : CD' \,(< CD)$. Sumatur $C\Upsilon > CD'$ (quod fieri poteſt per hyp.)

$$\begin{array}{llcll}
\text{Tum fiat} & n : m & = & C\Upsilon : AX & \\
\text{Atqui} & n : m & = & CD' : AB & \\
\text{Ergo} & C\Upsilon : AX & = & CD' : AB & \\
\text{Sed} & C\Upsilon & > & CD' & (\textit{conſtr.}) \\
\text{Ergo} & AX & > & AB &
\end{array}$$

contra hypotheſin; cum AB ſit limes quantitatis creſcentis AX.

2°.

2°. Sint AB, CD limites quantitatum mutabilium decrescentium AX, CY. Fig. 3. 2°.

Si ratio $AB : CD$ non sit æqualis rationi datæ $m : n$; rursus erit ea major aut minor. Tum priore casu ratio $m : n$ æqualis erit rationi ipsius AB ad aliquam quantitatem majorem ipsâ CD; altero casu ratio $m : n$ æqualis erit rationi alicujus quantitatis majoris ipsâ AB ad CD. Ideo utroque casu unus ex limitibus augeri deberet, ut ratio hujus limitis ita aucti ad alterum limitem fieret æqualis rationi $m : n$.

Sit itaque $m : n = AB : CD'$ ($> CD$). Sumatur $CY < CD'$ (quod fieri potest per hyp.): Et sit $n : m = CY : AX$

Atqui $n : m = CD' : AB$

Ergo $CY : AX = CD' : AB$

Sed $CY < CD'$

Ergo $AX < AB$, contra hypothesin; cum AB sit limes quantitatis decrescentis AX.

Ratio igitur $m : n$ neque major est neque minor ratione limitum. Proinde ratio limitum æqualis est rationi $m : n$.

Observatio. Hæc propositio unum est ex præcipuis fundamentis methodi exhaustionis, qualis sedulo fuit ab Antiquis exculta, atque ab Euclide & Archimede nobis transmissa.

En aliquot exempla illam illustrantia.

Perimetri polygonorum ordinatorum similium, circulis diversis inscriptorum, sunt in ratione data radiorum circulorum, intra vel circa quos describuntur. Sed perimetri circulorum sunt limites perimetrorum crescentium polygonorum ipsis inscriptorum, & limites perimetrorum decrescentium polygonorum circumscriptorum. Itaque perimetri circulorum sunt inter se in eadem ratione data; scilicet in ratione horum radiorum.

Item: superficies circuli limes est superficierum crescentium aut decrescentium polygonorum ordinatorum, quæ ipsi inscribuntur aut circumscribuntur. Sed superficies polygonorum similium diversis circulis inscriptorum aut circumscriptorum sunt in ratione duplicata radiorum prædictorum circulorum. Proinde superficies circulorum sunt etiam in ratione duplicata radiorum suorum.

Sint

Sint duæ pyramides æque altæ, quarum bafes in eodem plano, & quæ fitæ fint ad easdem partes hujus plani. Altitudo communis harum pyramidum fecetur in quotcunque partes æquales: per puncta divifionis agantur plana bafi parallela; & utrique pyramidi infcribantur & circumfcribantur prismata, ad normam eorum, quæ de parallelogrammis infcriptis & circumfcriptis §. 1. Ex. 3. dicta fuerunt. Utraque pyramis limes eft fummarum crefcentium aut decrefcentium prismatum, quæ ipfi infcribuntur & circumfcribuntur. Proinde demonftrato, prædictas prismatum correfpondentium fummas inter fe effe in ratione data bafium pyramidum; licebit concludere: rationem limitum harum fummarum, feu ipfarum pyramidum, æqualem effe eidem rationi datæ bafium.

Eodem modo demonftratur: tam cylindros quam conos æquè-altos effe inter fe uti bafes: cum folida hæc limites fint prismatum aut pyramidum, quæ ipfis infcribuntur & circumfcribuntur.

Hinc etiam fi cylinder & conus æquè alti eidem bafi infiftant: cylinder eft triplus coni.

Hinc rurfus fuperficies tam cylindrorum quam conorum fimilium funt in duplicata ratione dimenfionum fuarum homologarum; foliditates autem in eadem ratione triplicata.

Hinc tandem fluunt nunquam fatis laudata Archimedis inventa de fuperficie & foliditate fphærarum.

Super diametro circuli tanquam axe defcribatur ellipfis quæcunque. Diameter hæc fecetur in partes quotcunque æquales; & utrique figuræ infcribantur & circumfcribantur rectangula ad normam eorum, quæ §. 1. Ex. 3. conftructa fuerunt. Circulus & ellipfis limites refpectivè funt tam fummarum crefcentium parallelogrammorum, quæ ipfis infcribuntur, quam fummarum decrefcentium parallelogrammorum, quæ ipfis circumfcribuntur. Sed fummæ rectangulorum circulo & ellipfi infcriptorum aut circumfcriptorum funt inter fe in ratione conftanti prioris axis ellipfis ad axem ei conjugatum. Ergo etiam circulus & ellipfis funt inter fe in ratione data horum axium.

Similiter comparatio inftituitur fphæræ & ellipfoïdis, rotatione circuli & ellipfis

ellipſis circa eundem axem genitarum. Nempe ſolida hæc reſpectivè limites ſunt ſummarum cylindrorum, qui ipſis inſcribuntur & circumſcribuntur. Atqui ſummæ iſtæ ſunt inter ſe in ratione conſtanti, quæ eſt duplicata rationis hujus axis ad axem ipſi conjugatum: ergo etiam ſphæra & ellipſoïs ſunt inter ſe in eadem ratione duplicata.

Utilitas præſentis propoſitionis plurimis aliis applicationibus illuſtrari poſſet. Sed hæc abunde ſufficiant.

§. 5. *Theorema.* Duæ quantitates mutabiles heterogeneæ, ſive creſcentes ambæ, ſive ambæ decreſcentes, ſint limitum capaces: et rationes harum quantitatum ad duas quantitates datas ſint ſemper inter ſe æquales. Dico: rationes limitum harum quantitatum mutabilium, ad eaſdem quantitates datas, eſſe etiam inter ſe æquales.

Sint Q & Q' duæ quantitates mutabiles. Sint L & L' limites harum quantitatum ſimul creſcentium, aut ſimul decreſcentium. Sint C & C' duæ quantitates conſtantes. Sit ſemper $Q : C = Q' : C'$
Dico eſſe $L : C = L' : C'$.

Primus Caſus. Uterque limes L & L' ſit limes quantitatum Q & Q' creſcentium.

Si non eſt $L : C = L' : C'$; alterutra rationum $L : C$, $L' : C'$; major erit altera; & proinde minuendus erit antecedens majoris rationis, ut alteri fiat æqualis. Sit igitur, ſi fieri poſſit: $L - q : C = L' : C'$.

Fiat $Q > L - q$ (quod fieri poteſt per hyp.)
Erit $Q : C > L - q : C$
Sed $Q : C = Q' : C'$ (hyp.)
Et $L - q : C = L' : C'$.
Ergo $Q' : C' > L' : C'$
ideo $Q' > L'$ (contra hyp.)

Secundus Caſus. Uterque limes L & L' ſit limes quantitatum Q & Q' decreſcentium.

Si non eſt $L : C = L' : C'$: alterutra rationum $L : C$, $L' : C'$; minor erit alterâ; & proinde augendus erit antecedens minoris rationis, ut alteri fiat æqualis. Sit ideo, ſi fieri poſſit: $L + q : C = L' : C'$.

Fiat	$Q \prec L+q$ (quod fieri poteſt per hyp.)	
Erit	$Q : C \prec L+q : C$	
Sed	$Q : C = Q' : C'$	
Et	$L+q : C = L' : C'$	
Ergo	$Q' : C' \prec L' : C'$	
et ideo	$Q' \prec L'$ (contra hyp.)	

Neutra ideo rationum $L : C$, $L' : C'$, major aut minor eſt alterâ. Proinde hæ duæ rationes ſunt inter ſe æquales; ſeu $L : C = L' : C'$.

Obſervatio. Hæc propoſitio pariter eſt unum ex fundamentis methodi exhauſtionis Veterum. Illius ope menſuram obtinere licet plurimorum extenſorum, datâ menſura aliarum quarundam magnitudinum; uti paucis oſtendam exemplis, deſumptis a quadratura parabolæ conicæ, & cubatura paraboloïdis rotatione parabolæ hujus circa axem genitæ.

Fig. 4. Sit nempe $SMAB$ dimidium ſegmentum parabolæ conicæ, cujus vertex S, abſciſſa axis SB, & baſis axi ordinatim applicata AB. Et quæratur area hujus ſpatii.

Per verticem S, ductâ tangente SD, compleatur rectangulum $ABSD$, & jungatur SA recta. Dividatur tangens SD in partes quotcunque inter ſe æquales, quarum una ſit PP'. Spatio parabolico exteriori $SDAM$, & triangulo SDA circumſcribantur (aut inſcribantur) rectangula æquè-alta (ad normam eorum, quæ §. 1. Ex. 3. ſunt deſcripta). Sint $PMmP'$, $PQqP'$, duo rectangula ſibi mutuò reſpondentia, ſpatio illi & triangulo v. gr. circumſcripta. Ordinatæ PM, $P'M'$ occurrant baſi AB in punctis R & R'.

Concipiatur triangulum SAD, & rectangulum $SBAD$; gyrari circa tangentem SD tanquam axem. Triangulum SAD gignet conum, cui circumſcribentur cylindri rectangulis Pq geniti; & rectangulum $SBAD$ gignet cylindrum.

Per naturam parabolæ
$$MP : AD = SP^2 : SD^2$$
$$= PQ^2 : AD^2$$
$$= PQ^2 : PR^2$$
$$= \text{cyl.}\, PQqP' : \text{cyl.}\, PRR'P'$$
Sed
$$MP : AD = MP : PR$$
$$= \text{rect.}\, PMmP' : \text{rect.}\, PRR'P'.$$
Ergo
$$\text{rect.}\, PMmP' : \text{rect.}\, PRR'P' = \text{cyl.}\, PQqP' : \text{cyl.}\, PRR'P'.$$

Alti-

Altitudine PP' manente eâdem; termini consequentes omnium similium proportionum sunt etiam constantes.

$$
\begin{array}{llcl}
\text{Hinc} & \text{s. rect. } PMmP' : \text{s. rect. } PRR'P' & = & \text{s. cyl. } PQqP' : \text{s. cyl. } PRR'P' \\
\text{seu} & \text{s. rect. } PMmP' : ABSD & = & \text{s. cyl. } PQqP' : \text{cyl. } ABSD. \\
\text{Hinc (§. praes. 1)} & \text{lim. s. rect. } PMmP' : ABSD & = & \text{lim. s. cyl. } PQqP' : \text{cyl. } ABSD. \\
\text{Sed (§. 1.)} & \text{lim. s. rect. } PMmP' & = & \text{spatio parabolico } AMSD \\
\text{et} & \text{lim. s. cyl. } PQqP' & = & \text{cono gyratione trianguli } SAD \text{ genito.} \\
\text{Ergo} & AMSD : ABSD & = & \text{con. } ASD : \text{cyl. } ABSD \\
\text{Atqui (Archim. de cono \& cyl.)} & & & \text{con. } ASD = \tfrac{1}{3} \text{cyl. } ABSD \\
\text{Ergo} & AMSD = \tfrac{1}{3} ABSD & & \\
\text{Et} & AMSB = \tfrac{2}{3} ABSD. & &
\end{array}
$$

Hoc est: dimidium segmentum parabolæ, abscissa axis, recta axi ordinatim applicata, & arcu curvæ contentum, æquale est duobus trientibus rectanguli circumscripti.

Porro: sit SAB dimidium segmentum parabolicum, quod gyratione sua circa axem SB gignat segmentum paraboloïdicum. Fig. 5.

Jungatur SA recta. Ductâ per verticem S, tangente SD, compleatur rectangulum $ABSD$. Dividatur axis SB in partes quotcunque æquales, quarum una sit PP'. Spatio parabolico & triangulo SAB circumscribantur (vel inscribantur) rectangula æquè alta (ad normam §. 1. Ex. 3.). Sint $PMmP'$, $PQqP'$ duo rectangula sibi invicem respondentia, segmento parabolico & triangulo circumscripta, & rectæ PM, $P'M'$, axi ordinatim applicatæ, occurrant rectæ AD in punctis R & R'.

$$
\begin{array}{lrcl}
\text{Per naturam parabolæ est} & SP : SB & = & PM^2 : AB^2 \\
 & & = & PM^2 : PR^2 \\
 & & = & \text{cyl. } PMmP' : \text{cyl. } PRR'P' \\
\text{Atqui} & SP : SB & = & PQ : AB \\
 & & = & PQ : PR \\
 & & = & \text{rect. } PQqP' : \text{rect. } PRR'P'
\end{array}
$$

$$\text{Ergo rect. } PQqP' : \text{rect. } PRR'P' = \text{cyl. } PMmP' : \text{cyl. } PRR'P'.$$

Altitudine PP' manente eadem: termini consequentes omnium similium proportionum sunt etiam constantes.

Ergo $\quad$ f. rect. $PQqP'$: f. rect. $PRR'P'$ = f. cyl. $PMmP'$: f. cyl. $PRR'P'$

feu $\quad$ f. rect. $PQqP'$: rect. $ABSD$ = f. cyl. $PMmP'$: cyl. $ABSD$

Ergo $\quad$ lim. f. rect. $PQqP'$: rect. $ABSD$ = lim. f. cyl. $PMmP'$: cyl. $ABSD$

Sed (§. 1. Ex. 3.) $\quad$ lim. f. rect. $PQqP'$ = triang. ASB

$\quad$ lim. f. cyl. $PMmP'$ = paraboloïdi $AMSB$.

Ergo $\quad$ triang. ASB : rect. $ABSD$ = parab. $AMSB$: cyl. $ABSD$

Sed $\quad$ triang. $ASB = \frac{1}{2}$ rect. $ABSD$

Ergo - - - - - - - - - parab. $AMSB = \frac{1}{2}$ cyl. $ABSD$.

Hoc eft: paraboloïdes gyratione dimidii fegmenti parabolici circa axem fuum genita eft fubdupla cylindri circumfcripti.

Scholium. Exempla hæc declarant, quomodo cognita ratione duorum extenforum fæpius determinari poffit ratio, quam invicem habent duo alia extenfa prioribus heterogenea. In priori exemplo, ratio data, quam mutuò habent conus & cylindrus æquè alti fuper eadem bafi, deduxit ad rationem, quæ intercedit inter dimidium fegmentum parabolæ & rectangulum circumfcriptum; in pofteriori, ratio data trianguli & rectanguli æquè altorum fuper eadem bafi idem præftitit pro paraboloïde & cylindro huic circumfcripto.

§. 6. *Theorema.* Sint duæ aut plures quantitates mutabiles, quæ fimul fieri poffint minores quacunque quantitate propofita: dico earum fummam poffe etiam quacunque data quantitate fieri minorem.

Sint x, y, z, v . . . . . quantitates mutabiles, quarum numerus n, quæ fimul fieri poffint quacunque quantitate propofita minores: dico, fummam harum quantitatum mutabilium poffe fieri quacunque quantitate data a minorem.

Etenim dividatur quantitas a in tot partes æquales, quot funt quantitates mutabiles. Fiant fimul (quod poffibile per hyp.) fingulæ quantitates mutabiles unâ illarum partium minores. Proinde fumma omnium quantitatum mutabilium minor erit prædicta parte, toties fumptâ quot funt quantitates mutabiles, feu minor quantitate propofita.

Corollarium 1. Idem a fortiori valet de differentia duarum ejusmodi quantitatum mutabilium, & de exceffu, quo fumma aliquot ejusmodi quantitatum mutabilium fuperat fummam reliquarum.

Corol-

Corollarium 2. Sit quantitas mutabilis, quæ minor fieri poteſt quacunque quantitate data: & ea quantitas multiplicetur per numerum quemcunque n (poſitivum). Dico, produćtum inde ortum poſſe etiam minus fieri quacunque quantitate propoſita.

Sit n numerus integer: res patet per Propoſitionem.

Si vero n non ſit numerus integer: fiat produćtum ex quantitate mutabili per numerum integrum, ipſo n proxime majorem, minus quantitate data: erit a fortiori produćtum ex quantitate mutabili per numerum n minus eadem quantitate data.

§. 7. *Theoremata varia.*

1°. Si rationis mutabilis creſcentis v. gr. (*a*) $X : Y$ limes ſit ratio data $A : B$. Erit (invertendo) rationis decreſcentis $Y : X$ limes ratio $B : A$.

Quoniam (per hyp.) $X : Y \lessdot A : B$

eſt $Y : X \gtrdot B : A$. Sit $B' \gtrdot B$, & proinde $B' : A \gtrdot B : A$;

dico fieri poſſe $Y : X \lessdot B' : A$.

Etenim per hyp. fieri poteſt $X : Y \gtrdot A : B'$,

fiat; & erit $Y : X \lessdot B' : A$.

2°. Si rationis mutabilis creſcentis v. gr. $X : Y$ limes ſit ratio data $A : B$: erit (convertendo) rationis mutabilis decreſcentis $X : X - Y$, limes ratio data $A : A - B$.

Quoniam (per hyp.) $X : Y \lessdot A : B$

eſt $X : X - Y \gtrdot A : A - B$. Sit autem $B' \gtrdot B$, & proinde

$A - B' \lessdot A - B$

& $A : A - B' \gtrdot A : A - B$

Dico fieri poſſe $X : X - Y \lessdot A : A - B'$.

Etenim quoniam $B' \gtrdot B$; eſt $A : B' \lessdot A : B$;

Sed (per hyp.) fieri poteſt $X : Y \gtrdot A : B'$,

Fiat; & erit - - - $X : X - Y \lessdot A : A - B'$.

3°. Si rationis mutabilis creſcentis v. gr. $X : Y$, limes ſit ratio data $A : B$; erit $\begin{smallmatrix}\text{componendo}\\ \text{dividendo}\end{smallmatrix}$ rationis mutabilis creſcentis $X \pm Y : Y$ limes, ratio data $A \pm B : B$.

(*a*) Quæcunque dicam de ratione creſcente, dićta intelligantur de ratione decreſcente, mutatis mutandis.

Quoniam (per hyp.) $X : Y < A : B$

$X \pm Y : Y < A \pm B : B.$ Sit autem $A' < A$; & proinde

$A' + B < A + B$

& $A' \pm B : B < A \pm B : B$

Dico fieri posse $X \pm Y : Y > A' \pm B : B$

Etenim quoniam $A' < A$

$A' : B < A : B$

Sed (per hyp.) fieri potest $X : Y > A' : B$

Fiat; & erit $X \pm Y : Y > A' \pm B : B.$

§. 8. *Theorema.* Duæ quantitates mutabiles semper inter se sint in ratione data. Et quantitas data sit limes unius harum quantitatum mutabilium crescentis vel decrescentis: dico, dari etiam quantitatem, quæ sit limes alterius quantitatis, crescentis vel decrescentis, & quantitatem hanc eam esse, ad quam prior data quantitas habet eandem rationem datam.

Fig. 3. 1°. Sint AX, CY duæ quantitates mutabiles datam inter se habentes rationem $a : c$. Sit AB quantitas data, quæ sit limes quantitatis mutabilis AX crescentis v. gr: dico, dari alteram quantitatem CD, quæ erit limes alterius quantitatis mutabilis crescentis CY; & quantitatem hanc CD eam esse, ad quam quantitas data AB habet rationem datam $a : c$.

Etenim fiat $a : c = AB : CD$

per hyp. $AX : CY = a : c$

Ergo $AX : CY = AB : CD$; jam vero $AB > AX$ (hyp.); ergo $CD > CY$

hinc $AB - AX : CD - CY = AB : CD$

seu $XB : DY = AB : CD = a : c$; & $DY = BX \times \frac{c}{a}$.

Sed (per hyp.) BX potest fieri minor qualibet quantitate proposita; ergo (§. 6.) & DY simul minor reddi potest quacunque quantitate proposita. Et proinde quantitas CD, semper major mutabili CY, limes est magnitudinis hujus quantitatis CY.

Eodemque modo propositum ostenditur de limite decrementi.

§. 9. *Theorema.* Duæ quantitates mutabiles eandem semper habeant rationem ad duas quantitates datas A & A'. Detur quantitas C, quæ limes sit unius

unius illarum quantitatum five crefcentis five decrefcentis. Dico, dari etiam aliquam quantitatem C', quæ fit limes pofterioris quantitatis mutabilis pariter crefcentis aut decrefcentis; & quantitatem hanc C' eam effe, quâ fiat $C : A = C' : A'$. Demonftratur eodem modo quo Theorema præcedens.

§. 10. *Theorema.* Duæ rationes mutabiles fint femper inter fe æquales; & fit ratio aliqua data limes unius harum rationum mutabilium crefcentis vel decrefcentis: dico, eandem rationem datam effe etiam limitem pofterioris rationis crefcentis vel decrefcentis.

Sit femper $X : X' = Y : Y'$.

Et femper fit (v. gr.) $X : X' < A : C$; fed lim. $X : X' = A : C$

Dico etiam - - - - - - lim. $Y : Y' = A : C$.

Etenim (per hyp.) $X : X' = Y : Y'$

Sed $X : X' < A : C$

Ergo $Y : Y' < A : C$. Sit $C' > C$; & proinde $A : C > A : C'$.

Fieri poteft (per hyp.) $X : X' > A : C'$;

Fiat; & erit $Y : Y' > A : C'$. Proinde ratio $A : C$ etiam eft limes incrementi rationis $Y : Y'$.

Corollarium. Si ratio aliqua data limes eft rationis duarum quantitatum mutabilium: eadem ratio data limes quoque eft rationis quantitatum, quæ funt æquè-multiplæ vel æquè-fubmultiplæ pofteriorum.

§. 11. *Theorema.* Sint duæ quantitates datæ, quarum una fit limes alicujus quantitatis mutabilis decrefcentis; altera vero limes fit alterius quantitatis mutabilis crefcentis; ita ut magnitudines, quibus duæ quantitates mutabiles a limitibus fuis refpectivè differunt, fimul fieri poffint minores quibuscunque quantitatibus propofitis. Dico: rationem, quam prior quantitas data habet ad pofteriorem, effe limitem rationis decrefcentis, quam prior quantitas mutabilis habet ad pofteriorem.

Sit AB limes quantitatis mutabilis decrefcentis AX; & CD limes quantitatis mutabilis crefcentis CY. Quoniam $\frac{AX > AB}{CY < CD}$, eft $AX : CY > AB : CD$. Fig. 6.
Dico: rationem $AX : CY$ poffe reddi minorem quacunque ratione propofita, quæ major fit ratione $AB : CD$.

Sit

Sit enim propoſita ratio quæcunque $AB : CE$ major ratione $AB : CD$. Erit $CE < CD$. Porro, ſit quantitas $CF \begin{smallmatrix} > CE \\ < CD \end{smallmatrix}$; & fiat, uti $AB : CE$, ita $AG : CF$. Quoniam $CF > CE$, erit $AG > AB$. Fiat ſimul $\begin{smallmatrix} AX < AG \\ CY > AF \end{smallmatrix}$ (quod fieri poteſt per hyp.)

Dico, fore $AX : CY < AB : CE$

Quoniam $AX < AG$; . . . $AX : CF < AG : CF$

$< AB : CE$

Atqui $CF < CY$; ergo $AX : CY < AX : CF$

Ergo a fortiori . . . . $AX : CY < AB : CE$.

Obſervatio. Hinc ratio poſterioris quantitatis datæ ad priorem eſt limes rationis creſcentis poſterioris quantitatis mutabilis ad priorem (§. 1.)

§. 12. *Theorema.* Sint duæ quantitates datæ, quæ ambæ limites ſunt duarum quantitatum mutabilium ſimul decreſcentium; ita ut exceſſus, quibus limites ſuos ſuperant, poſſint ſimul fieri minores quibuslibet magnitudinibus datis. Dico, rationem duarum quantitatum mutabilium ſimul fieri poſſe majorem quacunque ratione datâ, quæ ſit minor ratione prioris limitis ad poſteriorem, minorem vero quacunque ratione datâ, quæ major ſit ratione prioris limitis ad poſteriorem.

Fig. 7. Sint AB, CD limites duarum quantitatum mutabilium decreſcentium AX, CY. Dico, rationem $AX : CY$ poſſe fieri ſimul majorem quacunque ratione datâ $AB : CE$, quæ ſit minor ratione $AB : CD$; & minorem ratione datâ $AF : CD$, quæ ſit major ratione $AB : CD$.

Demonſtratio. Etenim, fiant ſimul $\begin{smallmatrix} AX < AF \\ CY < CE \end{smallmatrix}$ (quod fieri poteſt per hyp.)

1°. Quoniam $AX > AB$, eſt $AX : CE > AB : CE$

Atqui $CY < CE$; ergo $AX : CY > AX : CE$

Ergo a fortiori . . . . . . $AX : CY > AB : CE$

2°. Quoniam $CY > CD$, eſt $AF : CY < AF : CD$

Atqui $AX < AF$; ergo $AX : CY < AF : CY$

Ergo a fortiori . . . . . . $AX : CY < AF : CD$.

NB. Idem obtinebit, ſi quantitates datæ fuerint limites duarum quantitatum mutabilium creſcentium.

Obſer-

Observatio. Quamvis hisce casibus ratio limitum AB & CD non possit dici semper esse major aut semper minor ratione quantitatum mutabilium AX, $C\Upsilon$, quod stricto sensu juxta definitiones (§. 1.) requiritur, ut ratio data $AB : CD$ dici possit limes rationis mutabilis $AX : C\Upsilon$; attamen, quoniam posterior ratio, sive sit major priore, sive eâ minor, ad illam ita accedere potest, ut minus ab ea differat, quam duæ rationes propositæ quæcunque, quarum una major altera minor est ratione limitum, ita ut nullus sit exiguitatis quantitatum BF, DE limes, quo rationum $AB : CE$, $AF : CD$ ad rationem $AB : CD$ accessus cohiberetur; liquet tanto magis, rationis $AX : C\Upsilon$ a ratione $AB : CD$ discrepantiam intra limites continue arctiores posse contineri. Et proinde ratio data $AB : CD$ dici quoque potest ea, ad quam ratio $AX : C\Upsilon$ propius propiusque accedit, quamvis fluxu non æquè continuo & regulari, uti in omnibus exemplis antea propositis: cum ratio mutabilis $AX : C\Upsilon$ alternis vicibus possit major aut minor fieri ratione data $AB : CD$.

Exemplo hoc ducimur ad alteram speciem limitum rationum, paululum ab ea, quam §. 1. definivimus, diversam.

§. 13. *Definitio.* Ratio mutabilis possit alternis vicibus fieri major aut minor ratione datâ; ita tamen ut ad rationem datam propius accedere possit, quam ad eandem accedit quæcunque alia ratio proposita, sive major priori ratione datâ, sive minor: prior ratio data dicitur etiam limes rationis mutabilis; & heic quidem obtinet limes rationis tam crescentis quam decrescentis.

Idem discrimen locum etiam habere potest in limitibus quantitatum, quod sequenti exemplo prorsus familiari declaratur.

Sit p minor unitate, & sit progressio $1-p+p^2-p^3+p^4-p^5 \ldots\ldots \mp p^{n-2} \pm p^{n-1}$. Vera summa hujus progressionis est $\frac{1}{1+p} \pm \frac{p^n}{1+p}$; prouti n est numerus $\begin{matrix}\text{impar}\\ \text{par}\end{matrix}$. Proinde numero n alterne sumto impari & pari, summa illa etiam alterne fit major & minor quantitate $\frac{1}{1+p}$. Quare sensu stricto definitionis (§. 1.) quantitas $\frac{1}{1+p}$ dici nequit limes esse summæ seriei propositæ; cum hæc summa neque semper major sit, neque semper minor quantitate $\frac{1}{1+p}$. Cum vero tam

exces-

excessus, quam defectus, quibus summa progressionis differt a quantitate $\frac{1}{1+p}$, possint quacunque quantitate proposita fieri minores; quantitas $\frac{1}{1+p}$ jure etiam nunc dicitur limes summæ, sive crescentis, sive decrescentis, propositæ progressionis.

Idemque dicatur de plurimis progressionibus decrescentibus, quæ alternè excessu & defectu differunt ab aliqua quantitate data; ita tamen ut tam excessus quam defectus possint fieri minores quacunque quantitate proposita. Ex. gr. sit p quadrans circumferentiæ, cujus radius $= 1$:

$$\text{fit } \tfrac{1}{2}p = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} \ldots\ldots$$

Et quamvis summa hujus seriei alternè sit major & minor eadem quantitate data $\frac{1}{2}p$; hæc tamen absque dubio limes illius esse censetur. Sic & log. hyp. 2 dicitur limes summæ $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots\ldots$

Conjuncta utraque limitum notione, duo postrema theoremata brevius sic enunciantur: *Ratio limitum duarum quantitatum mutabilium limes est rationis ipsarum quantitatum mutabilium.*

Exempla. Rationes tam perimetrorum quam superficierum duorum circulorum limites sunt rationum perimetrorum & superficierum polygonorum regularium sive similium sive dissimilium, quæ circulis circumscribuntur & inscribuntur; sive polygona circulis simul circumscribantur, sive ipsis simul inscribantur; sive tandem polygona sint uni circulo circumscripta, alteri inscripta.

Pariter si absque limite imminui possunt quantitates x & y: ratio quantitatum a & b limes est rationum $a \pm x : b \pm y$.

§. 14. *Theorema.* Ratio composita ex rationibus, quæ limites sunt duarum pluriumve rationum mutabilium, limes est rationis compositæ ex iisdem rationibus mutabilibus.

1°. Sint duæ rationes mutabiles semper respectivè $\begin{matrix}\text{majores}\\ \text{minores}\end{matrix}$ duabus rationibus datis; sed quæ fieri possint *simul* respectivè $\begin{matrix}\text{minores}\\ \text{majores}\end{matrix}$ duabus rationibus quibuscunque $\begin{matrix}\text{majoribus}\\ \text{minoribus}\end{matrix}$ quam prædictæ rationes datæ. Dico: rationem compositam ex duabus rationibus mutabilibus (quæ semper $\begin{matrix}\text{major}\\ \text{minor}\end{matrix}$ est ratione composita

ſita ex duabus rationibus datis), poſſe fieri $\begin{matrix}\text{minorem}\\ \text{majorem}\end{matrix}$ quacunque ratione data, quæ $\begin{matrix}\text{major}\\ \text{minor}\end{matrix}$ ſit ratione compoſita ex rationibus datis.

Cum modus demonſtrandi idem ſit pro utraque inæqualitatis ſpecie: ſufficiat (brevitatis cauſa) de altera illarum propoſitum evincere.

Sint igitur duæ rationes datæ $\begin{matrix}AB : BC\\ BC : CD\end{matrix}$, & ſint duæ rationes mutabiles $\begin{matrix}X : Y\\ Y : Z\end{matrix}$. 1°. Sit ſemper $\begin{matrix}X : Y > AB : BC\\ Y : Z > BC : CD\end{matrix}$; ſed duæ rationes $\begin{matrix}X : Y\\ Y : Z\end{matrix}$ ſimul fieri poſſint minores quibuscunque rationibus datis, quæ ſint rationibus $\begin{matrix}AB : BC\\ BC : CD\end{matrix}$ reſpectivè majores. Dico, rationem $X : Z$ (quæ componitur ex rationibus $\begin{matrix}X : Y\\ Y : Z\end{matrix}$) pariter minorem fieri poſſe quacunque ratione propoſita, quæ major eſt ratione $AB : CD$ (quæ componitur ex rationibus $\begin{matrix}AB : BC\\ BC : CD\end{matrix}$). Fig. 8.

Et primo quidem, quoniam $\begin{matrix}X : Y > AB : BC\\ Y : Z > BC : CD\end{matrix}$

erit ſemper $X : Z > AB : CD$.

Jam proponatur quælibet ratio $AB : CE$ major ratione $AB : CD$ (& proinde ſit $CE < CD$). Probandum eſt fieri poſſe $X : Y < AB : CE$.

Fiat, uti $BC : CD$, ita $BF : CE$. Et quoniam $CD > CE$; erit $BC > BF$. Fiat etiam (quod poſſibile per hyp.) $X : Y (< AB : BF) = AB : BG (BG > BF)$; & fiat ſimul (quod poſſibile per hyp.) $Y : Z \left(\begin{matrix}> BC : CD\\ \text{ſeu } BF : CE\end{matrix}\right) < BG : CE$

Erit ideo $X : Z \;.\;.\;.\;.\;.\;.\; < AB : CE$.

Nempe ratio $X : Z$ potuit fieri minor ratione quacunque propoſita $AB : CE$, quæ major eſt ratione data $AB : CD$.

2°. Sint duæ rationes mutabiles, quarum una ſemper major ſit aliqua ratione data, altera vero ſemper ſit minor aliqua altera ratione data: ita tamen ut prior ratio mutabilis poſſit fieri minor quacunque ratione propoſita, qu major ſit priore ratione data; & *ſimul* poſterior ratio mutabilis poſſit fieri major quacunque ratione propoſita, quæ minor eſt poſteriori ratione data. Dico: rationem compoſitam ex duabus rationibus mutabilibus poſſe fieri tum majorem quacunque ratione propoſita, quæ minor fuerit ratione compoſita ex duabus prioribus rationibus datis; tum & minorem quacunque ratione propoſita, quæ major fuerit ratione compoſita ex prædictis rationibus datis.

Sint $\begin{matrix} X : Y \\ Y : Z \end{matrix}$ duæ rationes mutabiles, & ſint $\begin{matrix} AB : BC \\ BC : CD \end{matrix}$ duæ rationes datæ; ita ut ſemper $\begin{matrix} X : Y > AB : BC \\ Y : Z < BC : CD \end{matrix}$. Liquet nihil exinde concludi poſſe, quod ad inæqualitatem (imo & quod ad æqualitatem) rationum, quæ ex rationibus mutabilibus, & ex rationibus datis componuntur.

Sit autem ratio $AB : BC$ limes rationis decreſcentis $X : Y$; & ſimul ſit ratio $BC : CD$ limes rationis creſcentis $Y : Z$.

Dico, rationem - - - - - - - - $X : Z$ poſſe fieri $\begin{matrix} \text{minorem} \\ \text{majorem} \end{matrix}$ quacunque ratione propoſita, quæ $\begin{matrix} \text{major} \\ \text{minor} \end{matrix}$ ſit ratione data $AB : CD$.

Fig. 9. 1°. Sit ratio propoſita $AE : CD$ major ratione $AB : CD$ (& proinde ſit $AE > AB$).

$$\begin{array}{rl} \text{Fiat (quod poſſibile per hyp.)} & X : Y < AE : BC \\ \text{Sed eſt ſemper (per hyp.)} & \underline{Y : Z < BC : CD} \\ \text{Ergo erit} & X : Z < AE : CD. \end{array}$$

2°. Sit ratio propoſita $AB : CF$, minor ratione $AB : CD$ (& proinde $CF > CD$).

$$\begin{array}{rl} \text{Fiat (quod poſſibile per hyp.)} & Y : Z > BC : CF \\ \text{Quoniam eſt ſemper (per hyp.)} & X : Y > AB : BC \\ \text{\& factum fuit} & \underline{Y : Z > BC : CF} \\ \text{Erit} & X : Z > AB : CF. \end{array}$$

Proinde, propoſitis duabus rationibus, $AE : BC$, & $BC : CF$, quarum prior major ſit ratione data $AB : BC$, & poſterior minor ſit ratione data $BC : CD$; fiat ſimul $\begin{matrix} X : Y < AE : BC \\ Y : Z > BC : CF \end{matrix}$; erit ſimul $\begin{matrix} X : Z < AE : CD \\ X : Z > AB : CF \end{matrix}$.

Quoniam autem rationes propoſitæ $\begin{matrix} AE : CD \\ AB : CF \end{matrix}$ poſſunt eſſe rationes quæcunque $\begin{matrix} \text{majores} \\ \text{minores} \end{matrix}$ ratione $AB : CD$, utcunque parum ab ea diverſæ; tanto magis ratio $X : Z$, ſive major fuerit ratione $AB : CD$, ſive eadem minor, poteſt ad illam accedere, ut ab ea minus diſcrepet, quam alia ratio quæcunque, ſive major ſive minor ratione $AB : CD$. Hinc ratio $AB : CD$ eſt limes (juxta §. 13.) rationis mutabilis $X : Z$.

3°. Sint plures rationes mutabiles ſemper reſpectivè majores totidem rationibus datis; ita ut priores rationes poſſint fieri ſimul minores totidem rationibus

nibus propositis, quæ fuerint majores rationibus datis respectivè: dico, rationem compositam ex rationibus datis esse limitem rationis decrescentis compositæ ex rationibus mutabilibus.

$$\begin{array}{rl}
\text{Scilicet, sit v. gr.} & A:B < X:Y = \text{lim. decr. } X:Y \\
 & B:C < Y:Z = \text{lim. decr. } Y:Z \\
 & C:D < Z:V = \text{lim. decr. } Z:V \\
\text{Erit} & A:C < X:Z = \text{lim. decr. } X:Z \\
\text{Atqui} & C:D < Z:V = \text{lim. decr. } Z:V \\
\text{Ergo} & A:D < X:V = \text{lim. decr. } X:V.
\end{array}$$

Et si proposita fuerit quarta ratio tam mutabilis, quam limes hujus rationis decrescentis; eodem modo demonstrabitur, rationem ex quatuor rationibus datis compositam limitem esse rationis decrescentis compositæ ex quatuor rationibus mutabilibus: & sic deinceps, quicunque sit numerus rationum propositarum.

4°. Sit ratio $A:B$ limes rationis crescentis $X:Y$
$B:C$ limes rationis decrescentis $Y:Z$
$C:D$ limes rationis crescentis $Z:V$.

Dico: rationem $X:V$ (quæ componitur ex rationibus $\left.\begin{matrix}X:Y\\Y:Z\\Z:V\end{matrix}\right]$, posse fieri $\genfrac{}{}{0pt}{}{\text{majorem}}{\text{minorem}}$, quacunque ratione proposita, quæ $\genfrac{}{}{0pt}{}{\text{minor}}{\text{major}}$ sit ratione $A:D$ (quæ componitur ex rationibus $\left.\begin{matrix}A:B\\B:C\\C:D\end{matrix}\right]$; seu rationem $C:D$ esse limitem rationis $X:V$.

Sint AB, BC, CD, DE ipsis Fig. 10.
A, B, C, D, respectivè æquales.

1°. Sit ratio proposita $Ab:CD$ minor ratione $AB:CD$; & proinde sit $Ab < AB$. Sumatur $Ab' \genfrac{}{}{0pt}{}{> Ab}{< AB}$; & sit, uti $Ab:DE$, ita $Ab':De'$; & proinde sit $De' > DE$.

$$\begin{array}{rl}
\text{Fiat (quod possibile per hyp.)} & X:Y > Ab':BC \\
\text{Est semper} & Y:Z > BC:CD \\
\text{Fiat (quod possibile per hyp.)} & Z:V > CD:De' \\
\text{Erit} & X:V > Ab':De' \\
 & > Ab:DE.
\end{array}$$

2°. Sit ratio proposita $AB:De$ major ratione $AB:DE$ (& proinde $De < DE$. Fiat, uti $CD:DE$, ita $Cd:De$; & proinde sit $Cd < CD$.

 Est

Eſt ſemper (hyp.) $X : Y < AB : BC$
Fiat (quod poſſibile per hyp.) $Y : Z < BC : Cd$
Et eſt ſemper (hyp.) $Z : V < Cd : De$
Ergo $X : V < AB : De.$

Proinde ratio $X : V$ fieri poteſt $\begin{matrix}\text{major}\\ \text{minor}\end{matrix}$ quacunque ratione propoſita $\begin{matrix}\text{minori}\\ \text{majori}\end{matrix}$, quam ratio $AB : DE$. Proinde (§. 13.) ratio $AB : DE$ eſt limes rationis $X : V$.

Methodum demonſtrandi reliquos, qui heic obtinere poſſunt caſus, abunde hoc exemplo liquere exiſtimo.

Obſervatio. Caſus, ubi una aut plures rationes conſtantes occurrunt componendæ cum rationibus mutabilibus (limitum capacibus), ita facilis & evidens eſt ex præcedentibus, ut uno exemplo eum ſufficiat illuſtrare.

Sit ratio $A : B$, limes rationis mutabilis decreſcentis $X : Y$.
Sit ratio $B : C$, æqualis ſemper rationi $Y : Z$.
Dico, rationem $A : C$ eſſe limitem rationis decreſcentis $X : Z$.

Etenim quoniam $X : Y > A : B$
et $Y : Z = B : C$
Erit ſemper $X : Z > A : C$

Sit ratio data $A' : C$ major ratione data $A : C\,(A' > A)$.

Fiat (quod poſſibile per hyp.) $X : Y < A' : B$
Eſt ſemper $Y : Z = B : C$
Ergo $X : Z < A' : C.$

Proinde ratio $A : C$ compoſita ex rationibus $\begin{matrix}A : B\\ B : C\end{matrix}$ limes eſt rationis decreſcentis $X : Z$.

Corollarium 1. Nominatim, ſit aliqua ratio data limes rationis mutabilis, creſcentis vel decreſcentis; ratio duplicata, triplicata, quadruplicata, - - - - - prioris pariter erit limes rationis creſcentis aut decreſcentis, quæ eſt duplicata, triplicata, quadruplicata - - - - - rationis mutabilis.

Hinc ſpeciatim, ſi ratio quæpiam data limes eſt rationis mutabilis creſcentis vel decreſcentis dimenſionum homologarum duarum figurarum ſimilium, ſuperficialium vel ſolidarum: prior ratio duplicata limes eſt rationis ſuperficierum figurarum; eademque ratio triplicata eſt limes rationis capacitatum figurarum ſolidarum.

Corol-

Corollarium 2. Si rationes, quæ sunt limites duarum pluriumve rationum mutabilium, respectivè æquales sint rationibus, quæ sunt limites totidem aliarum rationum mutabilium; dico: rationem, quæ est limes rationis ex prioribus rationibus compositæ, æqualem esse rationi, quæ est limes rationis ex posterioribus compositæ.

Exemplum. Sit $\lim. X : Y (= a : b) = \lim. X' : Y'$
$\lim. Y : Z (= b : c) = \lim. Y' : Z'$
$\lim. Z : V (= c : d) = \lim. Z' : V'$, & sic deinceps
erit $\lim. X : V (= a : d) = \lim. X' : V'$.

Pariter sit $\lim. X : Y (= a : b) = \lim. X' : Y'$
$Y : Z (= b : c) = \lim. Y' : Z'$
$\lim. Z : V (= c : d) = Z' : V'$
Erit $\lim. X : Y (= a : d) = \lim. X' : Z'$; & sic deinceps, quicunque sit rationum numerus, & quomodocunque rationes limites & rationes constantes in similibus proportionibus inter se combinentur.

Corollarium 3. Si rationis $X : Y$ limes sit ratio $A : B$
& rationis $Z : V$ limes sit ratio $C : D$
sit etiam rationis $XZ : YV$ limes ratio $AC : BD$.

Etenim quoniam $A : B = \lim. X : Y$
$AC : BC = \lim. XZ : YZ$
Pariter $BC : CD = \lim. YZ : YV$
Hinc $AC : CD = \lim. XZ : YV$; quod verum, quicunque sit rationum numerus.

Corollarium 4. Rationis $A : A \pm x$ limes sit ratio æqualitatis. Rationis $A \pm x : Y$ limes sit ratio data $a : b$. Dico, rationis $A \pm x : Y \mp x$ limitem esse rationem $a : b$.

Dem. Quoniam $\lim. A : A \pm x = 1 : 1$
et $\lim. A \pm x : Y = a : b$
Erit $\lim. A : Y = a : b$ (§. 14.)
Hinc $\lim. A : A + Y = a : a + b$ (§. 7.)
Sed $\lim. A \pm x : A = 1 : 1$
Ergo $\lim. A \pm x : A + Y = a : a + b$ (§. 14.)
Hinc $\lim. A \pm x : Y \mp x = a : b$ (§. 7.)

§. 16.

§. 15. *Theorema.* Rationis $X : Y$ limes sit ratio data $a : b$
& rationis $X' : Y'$ limes sit eadem ratio $a : b$.
Dico, rationis $X+X' : Y+Y'$ limitem esse rationem $a : b$.

1°. Ratio data $a : b$, limes sit utriusque rationis decrescentis $X : Y$, $X' : Y'$.

Est ideo semper $X : Y > a : b$
$X' : Y' > a : b$
Proinde $X+X' : Y+Y' > a : b$

Sit ratio proposita quæcunque $a+\alpha : b$ major ratione $a : b$. Dico, fieri posse $X+X' : Y+Y' < a+\alpha : b$.

Etenim, fiat simul $X : Y < a+\alpha : b$
$X' : Y' < a+\alpha : b$ (quod fieri potest per hyp.)
et erit $X+X' : Y+Y' < a+\alpha : b$.

2°. Demonstratio eadem est, si ratio data $a : b$ limes sit utriusque rationis crescentis $X : Y$, $X' : Y'$.

3°. Ratio data $a : b$ limes sit rationis decrescentis $X : Y$
limes vero rationis crescentis $X' : Y'$.

Dico, rationem $X+X' : Y+Y'$ posse fieri $\begin{matrix}\text{minorem}\\ \text{majorem}\end{matrix}$ quacunque ratione data $\begin{matrix} a+\alpha : b & \text{majore} \\ a-\alpha : b & \text{minore}\end{matrix}$ quam ratio $a : b$.

Etenim per supp. fieri potest $X : Y < a+\alpha : b$
Fiat, & est semper $X' : Y < a+\alpha : b$
Ergo erit $X+X' : Y+Y' < a+\alpha : b$.
Item per supp. fieri potest $X' : Y' > a-\alpha : b$
Fiat, & est semper $X : Y > a-\alpha : b$
Ergo erit $X+X' : Y+Y' > a-\alpha : b$.

Corollarium. Si rationum quotcunque $X : Y$, $X' : Y'$, $X'' : Y''$, $X''' : Y'''$ ..... limes sit ratio data $a : b$; erit etiam rationis $X+X'+X''+X'''$ .....: $Y+Y'+Y''+Y'''$ ..... limes ratio $a : b$.

§. 16. *Theorema.* Sit x quantitas mutabilis, quæ potest reddi minor quacunque quantitate proposita. Sint $A, B, C, D \ldots\ldots L, M, N$ coëfficientes magnitudine dati. Sint etiam $a, b, c, d \ldots\ldots l, m, n$ exponentes positivi dati non minores unitate, & successive crescentes. Sit Q functio quantitatis

titatis mutabilis x, qualis ſequitur:

$$Q = Ax^a + Bx^b + Cx^c + Dx^d + \ldots . Lx^l + Mx^m + Nx^n.$$

Dico, functionem Q poſſe fieri minorem quacunque quantitate propoſita.

Conſtructio. Determinetur quantitas x, ſic, ut quilibet terminus ſeriei propoſitæ inde a primo Ax^a major ſit termino qui illum immediate ſequitur bis ſumtô. Scilicet, ſi duo termini quicunque contigui, ſint Lx^l, Mx^m; fiat $Lx^l > 2Mx^m$; ſeu $x^{m-l} < \frac{L}{2M}$; & fiat x minor quolibet horum valorum ita determinatorum.

Primus Caſus. Omnes coëfficientes A, B, C, $D \ldots L$, M, N ſint poſitivi.

Quoniam unusquisque terminus major eſt termino qui illum immediate ſequitur bis ſumtô, quilibet terminus major eſt ſummâ omnium terminorum ſubſequentium, & nominatim, primus terminus Ax^a major eſt ſummâ omnium reliquorum. Proinde $Q < 2Ax^a$. Atqui cum x fieri poſſit minor quacunque quantitate data, a fortiori x^a poteſt reddi minor quacunque quantitate data, & etiam (§. 6.) $2Ax^a$ poteſt reddi minor quacunque quantitate propoſita; & proinde Q poteſt reddi minor quacunque quantitate propoſita.

Secundus Caſus. Coëfficientes dati B, C, D, E . . . . L, M, N, qui primum ſequuntur, non ſint omnes poſitivi.

Omnibus ut ante factis, tanto magis hoc caſu erit $Q < 2Ax^a$; unde Q poteſt reddi minor quacunque quantitate propoſita.

Quod ſi A ſit quantitas negativa: omnia pariter vera erunt, ſignis mutatis.

Nominatim. Exponentes a, b, c, d . . . . l, m, n, ſequantur progreſſionem arithmeticam numerorum naturalium; ita ut ſit

$$Q = Ax + Bx^2 + Cx^3 + Dx^4 + \ldots Lx^{n-2} + Mx^{n-1} + Nx^n.$$

Facto $Q < 2Ax$, poterit Q eſſe minor quacunque quantitate propoſita. Et quoniam hic caſus eſt omnium frequentiſſimus: ad illum obſervationes ſequentes earumque demonſtrationes applicare ſufficiet.

Obſervatio 1. Quæcunque dicta ſunt de ratione dupla, potuiſſent etiam dici de quavis alia ratione majoris ad minorem. Scilicet, p exiſtente numero majore unitate, ſi ratio cujuslibet termini ſeriei Q ad terminum, qui illum immediate ſequitur, facta fuerit major ratione numeri p ad unitatem, erit

 $Q <$

$Q < \frac{p}{p-1} Ax$; & proinde (per hyp. & §. 6.) Q poteſt reddi minor quacunque quantitate propoſita.

Obſervatio 2. Quæ dicta fuerunt de caſu, quo numerus terminorum ſeriei Q datus eſt, applicari etiam, ſub datis conditionibus, poſſunt ad caſum, quo numerus terminorum eſt illimitatus. Quod accurate evolvi meretur, & in Diſſertatione jam memorata *Expoſition elementaire des Principes des Calculs ſuperieurs* haud congrue fuerat omiſſum.

1°. Coëfficientes $A, B, C, D \ldots\ldots$ ſequantur legem aliquam decreſcentem. Fiat $x < \frac{1}{2}$. Erit $x^l > 2x^{l+1}$.

Sed ponitur $L > M$

Ergo a fortiori $Lx^l > 2Mx^{l+1}$. Proinde facto $x < \frac{1}{2}$, omnes termini functionis Q ſequuntur progreſſionem decreſcentem, & quidem velocius quam progreſſio geometrica, cujus quilibet terminus eſt ſubduplus termini præcedentis. Proinde $2Ax$ major eſt ſumma progreſſionis, quicunque ſit numerus terminorum, & Q poteſt reddi minor quacunque quantitate data.

2°. Coëfficientes $A, B, C, D \ldots\ldots$ creſcant juxta progreſſionem geometricam quamcunque. Sit ideo $M = pL$

facto $x < \frac{1}{2p}$

erit $x^{l+1} < \frac{1}{2p} x^l$

et $Mx^{l+1} < \frac{1}{2} Lx^l$; ſeu $Lx^l > 2Mx^{l+1}$. Ideo terminus quilibet minor erit duplo præcedentis: unde ut prius concludetur.

3°. Coëfficientes $A, B, C, D \ldots\ldots$ creſcant quidem, ſed lentius quam juxta aliquam progreſſionem geometricam.

Ita omnes termini, qui priores duos ſubſequuntur, minores erunt, quam ſi coëfficientes dati creſcerent juxta progreſſionem geometricam, cujus exponens eſſet exponens rationis ſecundi termini ad primum. Unde præcedens concluſio valet a fortiori.

4°. Coëfficientes $A, B, C, D \ldots\ldots$ creſcant quidem a primo inde usque ad coëfficientem datum L, velocius quam in progreſſione geometrica, ſed ab hoc inde (ſi creſcere pergant) lentius quam in progreſſione geometrica creſcant.

Fiat

Fiat (uti 3°.) terminus, cujus coëfficiens L major summa omnium terminorum sequentium. Tum per constructionem hujus theorematis fiat primus terminus major summa omnium terminorum usque ad terminum, cujus coëfficiens L. Erit, a fortiori, primus terminus major summa omnium terminorum sequentium.

Quod attinet ad casus, quibus coëfficientes non sunt omnes positivi, aut quibus alterne crescunt & decrescunt juxta legem datam, & quod ad incrementa illorum attinet, modo conditionibus præcedentibus consentaneo: conclusiones illis casibus a fortiori nectuntur.

Ceteris vero casibus, quibus coëfficientes ultra omnem limitem crescentes aliquam legum incrementi præcedentium sequi non constat: hactenus dubito, theorema posse seriebus, quarum numerus terminorum est indeterminatus, applicari.

§. 17. Ut necessitas præcedentium distinctionum luculentius pateat, exemplum evolvam, quod in sequentibus magni erit momenti.

Exemplum. Sit x^n potentia quæcunque quantitatis variabilis x, quæ, mutatâ x in $x \pm \Delta x$; fit

$$x^n \pm nx^{n-1}\Delta x + \frac{n}{1}.\frac{n-1}{2}x^{n-2}\Delta x^2 \pm \frac{n}{1}\ldots\frac{n-2}{3}x^{n-3}\Delta x^3 + \frac{n}{1}\ldots\frac{n-3}{4}x^{n-4}\Delta x^4 \pm \ldots\ldots$$

Proinde potentia x^n, mutatâ x in $x \pm \Delta x$, accipit mutationem,

$$\pm\left(nx^{n-1}\Delta x \pm \frac{n}{1}.\frac{n-1}{2}x^{n-2}\Delta x^2 + \frac{n}{1}\ldots\frac{n-2}{3}x^{n-3}\Delta x^3 \pm \frac{n}{1}\ldots\frac{n-3}{4}x^{n-4}\Delta x^4 + \ldots\ldots\right)$$

Jam vero ponatur, mutationem Δx posse fieri minorem quacunque quantitate proposita: dico, mutationem simul factam potentiæ x^n etiam quacunque quantitate proposita fieri posse minorem.

1°. Sit n numerus integer positivus. Prior series abrumpitur, & proinde assertum verum est per §. 16.

2°. Sit n numerus positivus non-integer major unitate. Series præcedens non abrumpitur quidem, sed facile demonstratur: coëfficientes terminorum, in quibus exponens mutationis Δx sit major quam $n+1$, continue decrescere.

Etenim sit m numerus integer positivus immediate minor quam $n+1$. Coëfficiens termini $x^{n-m}\Delta x^m$, erit $\frac{n}{1}.\frac{n-1}{2}\ldots\ldots\frac{n-(m-1)}{m}$. Coëfficiens alius

cujus-

cujuscunque ſequentis termini $n^{n-(m+r)} \Delta x^{m+r}$ eſt

$$\frac{n}{1} \cdot \frac{n-1}{2} \cdot \ldots \ldots \frac{n-m}{m+1} \cdot \frac{n-(m+1)}{m+2} \cdot \ldots \ldots \frac{n-(m+r)}{m+r+1};$$

cujus cum certe unusquisque factor, inde a factore $\frac{n-m}{m+1}$ ſit fractio vera, prædicti coëfficientes continue decreſcunt; præterea iidem alterne fiunt poſitivi & negativi. Proinde (§. 16.) ſeries propoſita poteſt fieri minor quacunque quantitate propoſita.

3°. Sit n numerus negativus quicunque major unitate.

Series præcedens non abrumpitur quidem, ſed exponens rationis duorum coëfficientium ſeſe proxime ſequentium continue decreſcit. Etenim coëfficientes trium terminorum ſeſe proxime ſequentium, ſint

$$\frac{n}{1} \cdot \frac{n+1}{2} \cdot \ldots \cdot \frac{n+m}{m+1}$$

$$\frac{n}{1} \cdot \frac{n+1}{2} \cdot \ldots \ldots \ldots \frac{n+m+1}{m+2}$$

$$\frac{n}{1} \cdot \frac{n+1}{2} \cdot \ldots \ldots \ldots \ldots \ldots \frac{n+m+2}{m+3}.$$

Exponens rationis ſecundi horum coëfficentium ad primum, eſt $\frac{n+m+1}{m+2} = 1 + \frac{n-1}{m+2}$; et exponens rationis coëfficientis tertii ad ſecundum eſt $\frac{n+m+2}{m+3} = 1 + \frac{n-1}{m+3}$: et quoniam $\frac{n-1}{m+3} < \frac{n-1}{m+2}$, prædicti coëfficientes lentius quam in progreſſione geometrica creſcunt. Sic igitur ſumta ratione $\frac{\Delta x}{x}$, ut primus terminus ſeriei præcedentis major ſit ſecundo ejus termino bis ſumto, tanto magis quilibet alius terminus major erit duplo termini ſequentis; proindeque applicari poteſt demonſtratio §. 16ti.

4°. Exponens n ſit numerus negativus minor unitate.

Singuli factores coëfficientium $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \ldots \frac{n+m}{m+1}$, ſunt fractiones veræ, & proinde coëfficientes continue decreſcunt. Quare tanto magis applicari poteſt demonſtratio §. 16ti.

In omni igitur caſu mutatio $\frac{(x+\Delta x)^n - x^n}{x^n - (x-\Delta x)^n}$ poteſt reddi minor quacunque quantitate propoſita.

§. 18.

§. 18. *Theorema.* Omnibus ut in theoremate §. 16. positis, sit $Q = P + Ax^a + Bx^b + Cx^c + Dx^d + \ldots\ldots + Lx^l + Mx^m + Nx^n$. Dico, rationem æqualitatis esse limitem rationis $Q : P$ decrescentis aut crescentis, prouti A est quantitas positiva aut negativa.

Etenim (per theorema §. 16.) fiat priore casu $Q < P + 2Ax^a$
altero $Q > P - 2Ax^a$, Ratio æqualitatis est limes rationis decrescentis $P + 2Ax^a : P$
& limes rationis crescentis $P - 2Ax^a : P$. Ergo, a fortiori, ratio æqualitatis limes est rationis decrescentis aut crescentis $Q : P$, prouti A est quantitas positiva aut negativa.

Observatio. Quæ de casu, quo numerus terminorum functionis Q est illimitatus, præcepta fuerunt (§. 16. Observ. 2.) huc pariter valent.

Exemplum. Sit $Q \begin{matrix} = \dfrac{(x+\Delta x)^n - x^n}{\Delta x} \\ = \dfrac{x^n - (x-\Delta x)^n}{\Delta x} \end{matrix}$, & mutatio Δx possit fieri minor quacunque quantitate proposita; ratio æqualitatis est limes rationis $Q : nx^{n-1}$.

§. 19. *Applicatio.* Sint x & y duæ quantitates, quæ simul fieri possint quacunque quantitate proposita minores. Sint Q & Q' duæ functiones prædictarum quantitatum conditionibus §§. 16—18. consentaneæ, ita ut sit

$$Q = P + Ax^a + Bx^b + Cx^c + Dx^d + \ldots\ldots$$
$$Q' = P' + A'x^{a'} + B'x^{b'} + C'x^{c'} + D'x^{d'} + \ldots\ldots$$

Dico, limitem $Q : Q' = P : P'$.

Etenim: $\lim. Q : P = 1 : 1$ (§. 18.)
$P : P' = P : P'$
$\lim. P' : Q' = 1 : 1$ (§. 18.)
Ergo $\lim. Q : Q' = P : P'$ (§. 14.)

§. 20. *Theorema.* Sit x quantitas mutabilis, quæ major reddi potest quacunque quantitate proposita. Sint A, B, C, D .... L, M, N coefficientes magnitudine dati. Sint a, b, c, d .... l, m, n, exponentes etiam dati continue decrescentes.

Et fit $Q = Ax^a + Bx^b + Cx^c + Dx^d + \ldots Lx^l + Mx^m + Nx^n$. Dico, rationem æqualitatis esse limitem rationis $Q : Ax^a$.

Demonstratio. $Q = x^a(A + B\frac{1}{x^{a-b}} + C\frac{1}{x^{a-c}} + D\frac{1}{x^{a-d}} + \ldots L\frac{1}{x^{a-l}} + M\frac{1}{x^{a-m}} + N\frac{1}{x^{a-n}}$.

Quoniam autem x major potest fieri quacunque quantitate proposita, $\frac{1}{x}$ potest fieri minor quacunque quantitate proposita; & quoniam $a, b, c, d \ldots l, m, n$, continue decrescunt, exponentes $a-b, a-c, a-d \ldots a-l, a-m, a-n$, continue crescunt; & proinde, quantitate mutabili x posita majore unitate, termini $\frac{1}{x^{a-b}}, \frac{1}{x^{a-c}}, \frac{1}{x^{a-d}}, \ldots \frac{1}{x^{a-l}}, \frac{1}{x^{a-m}}, \frac{1}{x^{a-n}}$, continue decrescunt; & proinde (§. 16.) quantitas $B\frac{1}{x^{a-b}} + C\frac{1}{x^{a-c}} + D\frac{1}{x^{a-d}} + \ldots + L\frac{1}{x^{a-l}} + M\frac{1}{x^{a-m}} + N\frac{1}{x^{a-n}}$ potest reddi minor quacunque quantitate proposita. Hinc quantitas A limes est quantitatis $A + B\frac{1}{x^{a-b}} + C\frac{1}{x^{a-c}} + D\frac{1}{x^{a-d}} + \ldots L\frac{1}{x^{a-l}} + M\frac{1}{x^{a-m}} + N\frac{1}{x^{a-n}}$, seu

$$\lim. A : A + B\frac{1}{x^{a-b}} + C\frac{1}{x^{a-c}} + D\frac{1}{x^{a-d}} + \ldots L\frac{1}{x^{a-l}} + M\frac{1}{x^{a-m}} + N\frac{1}{x^{a-n}} = 1 : 1$$

& $\lim. Ax^a : Ax^a + Bx^b + Cx^c + Dx^d + \ldots Lx^l + Mx^m + Nx^n = 1 : 1$.
seu tandem $\lim. Ax^a : Q = 1 : 1$.

Exemplum. Existente n numero integro positivo: vidimus (§. y'. *Introd.*) summam potestatum ordinis m numerorum naturalium ab unitate usque ad n continue crescentium, esse sequentem:

$$\int n^m = \frac{1}{m+1} n^{m+1} + Bn^m + Cn^{m-1} + Dn^{m-2} + En^{m-3} + \ldots\ldots$$

Proinde $\lim. \int n^m : \frac{1}{m+1} n^{m+1} = 1 : 1$,

seu $\lim. \int n^m : n \times n^m = 1 : m+1$.

Hoc est: limes rationis summæ potestatum cujuslibet ordinis numerorum naturalium continue crescentium inde ab unitate, ad maximum horum numerorum toties sumptum quot sunt prædicti numeri, est ratio unitatis ad exponentem prædictarum potestatum unitate auctum.

$$\begin{aligned} \text{Sic} \quad \lim. \int n &: n.n = 1 : 2 \\ \lim. \int n^2 &: n.n^2 = 1 : 3 \\ \lim. \int n^3 &: n.n^3 = 1 : 4 \\ \lim. \int n^4 &: n.n^4 = 1 : 5 \\ \&c. & \quad \&c. \end{aligned}$$

§. 21.

§. 21. *Theorema.* Sint duæ quantitates datæ, limites duarum quantitatum mutabilium. Multiplicentur in ſe invicem tam quantitates datæ quam quantitates mutabiles. Dico: productum ex priore multiplicatione ortum, eſſe limitem poſterioris producti.

Sint A & B duæ quantitates datæ, & ſint X & Y duæ quantitates mutabiles, quarum limites ſint A & B. Dico: productum AB eſſe limitem producti XY.

Primus Caſus. Sint quantitates A & B limites quantitatum decreſcentium X & Y. Sit ideo

$$\begin{array}{rcl} X & = & A+x \\ \underline{Y} & = & \underline{B+y} \\ XY & = & AB+Ay+Bx+xy. \end{array}$$

Dico: productum AB eſſe limitem producti (decreſcentis) XY.

Etenim	lim. B	:	$B+y$	$= 1 : 1$	(§. 3.)
Ergo	lim. Bx	:	$Bx+xy$	$= 1 : 1$	(§ 10.)
Pariterque	lim. $AB+Ay+Bx$	:	$AB+Ay+Bx+xy$	$= 1 : 1$	(§. 2.)

Sed per hyp. x & y poſſunt fieri ſimul minores quacunque quantitate propoſita; ergo & Ay, Bx poſſunt fieri ſimul minores quacunque quantitate propoſita (§. 6.), & etiam $Ay+Bx$ poteſt fieri minor quacunque quantitate propoſita (§. 6.). Proinde quantitas AB eſt limes quantitatis $AB+Ay+Bx$ (§. 6.), & eadem quantitas AB eſt limes quantitatis decreſcentis $AB+Ay+Bx+xy$, ſeu XY. Scilicet

$$\begin{array}{llcll} & \text{lim. } AB & : & AB+Ay+Bx & = 1 : 1 \\ & \text{lim. } AB+Ay+Bx & : & XY & = 1 : 1 \\ \text{Ergo} & \text{lim. } AB & : & XY & = 1 : 1. \end{array}$$

Er proinde AB eſt limes producti (decreſcentis) XY.

Secundus Caſus. Sint quantitates A & B limites quantitatum creſcentium X & Y. Sit

$$\begin{array}{rcl} X & = & A-x \\ \underline{Y} & = & \underline{B-y} \\ XY & = & AB-Ay-Bx+xy. \end{array}$$

lim. $B : B-y = 1 : 1$ (§. 3.)

Ergo & lim. $Bx : Bx-xy = 1 : 1$ (§. 10.)

Pariterque, lim. $AB-Ay-Bx : AB-Ay-Bx+xy\,(=XY) = 1 : 1.$ (§. 2.)

Sed

Sed ut prius, AB eſt limes quantitatis (creſcentis) $AB - Ay - Bx$.

$$\begin{array}{llcccl} \text{Hinc} & \text{lim.}\, AB & : & AB-Ay-Bx & = 1 : 1 & \\ & \text{lim.}\, AB-Ay-Bx & : & XY & = 1 : 1 & \\ \text{Ergo} & \text{lim.}\, AB & : & XY & = 1 : 1 & (\S.\ 14.) \end{array}$$

Proinde AB eſt limes producti creſcentis XY.

Tertius Caſus. Sit A limes quantitatis decreſcentis X.
B limes quantitatis creſcentis Y.

$$\begin{array}{lcl} \text{Sit ideo} \quad X & = & A+x \\ Y & = & B-y \\ \hline XY & = & AB-Ay+Bx-xy. \end{array}$$

$$\begin{array}{lll} \text{Erit} & \text{lim.}\, B : B-y & = 1 : 1 \\ \text{Ergo} & \text{lim.}\, Bx : Bx-xy & = 1 : 1. \\ \text{Et} & \text{lim.}\, AB-Ay+Bx : AB-Ay+Bx-xy\ (=XY) & = 1 : 1. \end{array}$$

Sed ut prius AB eſt limes quantitatis (ſive creſcentis ſive decreſcentis) $AB-Ay+Bx$.

$$\begin{array}{llccl} \text{Hinc} & \text{lim.}\, AB & : & AB-Ay+Bx & = 1 : 1 \\ & \text{lim.}\, AB-Ay+Bx & : & XY & = 1 : 1 \\ \text{Ergo} & \text{lim.}\, AB & : & XY & = 1 : 1 \end{array}$$

Et proinde AB eſt limes producti (ſive creſcentis ſive decreſcentis) XY.

Applicatio. Sint A, B, C, D, ... quantitates datæ; & ſint totidem quantitates mutabiles X, Y, Z, V, ... quarum priores ſint limites. Dico: productum $ABCD$ . . . . . eſſe limitem producti $XYZV$ . . . . .

Etenim A & B ſunt limites quantitatum mutabilium X & Y;
Ergo AB eſt limes producti mutabilis XY
Sed C eſt limes quantitatis mutabilis Z
Ergo ABC eſt limes producti mutabilis XYZ
Sed D eſt limes quantitatis mutabilis V;
Ergo $ABCD$ eſt limes producti mutabilis $XYZV$; & ſic deinceps.

§. 22. *Lemma notum.* In omni proportione geometrica continua ſumma terminorum extremorum major eſt termino medio bis ſumto.

Corollarium 1. In omni proportione geometrica continua; exceſſus quo maximus terminus medium ſuperat, major eſt exceſſu quo terminus medius ſuperat minimum.

Corol-

Corollarium 2. Sit progreſſio geometrica creſcens: differentia duorum terminorum contiguorum creſcit inde a minimo usque ad maximum.

Corollarium 3. Sint duæ progreſſiones creſcentes, una geometrica, altera arithmetica, quæ duos primos terminos habeant communes, & quarum terminorum numerus idem ſit: reliqui termini progreſſionis geometricæ majores ſunt terminis progreſſionis arithmeticæ æquè ab extremis diſtantibus. Idem a fortiorì valet, ſi terminus ſecundus progreſſionis geometricæ major ſit termino ſecundo progreſſionis arithmeticæ.

Corollarium 4. Sint duæ progreſſiones creſcentes, una geometrica, altera arithmetica, quæ duos terminos extremos (h. e. primum & ultimum) communes habent, & quorum numerus terminorum idem ſit. Reliqui termini progreſſionis geometricæ minores ſunt terminis progreſſionis arithmeticæ æque ab extremis diſtantibus.

Theorema. Datis duobus terminis extremis progreſſionis geometricæ, numerus terminorum ita poteſt augeri, ut minoris terminorum datorum & termini huic proximi differentia minor ſit qualibet quantitate data (hoc termino dato minore).

Sint P & Q duo termini dati, quorum P minor; & ſit a quantitas data. Sumatur differentia $Q-P$ terminorum Q & P. Repetatur quantitas data a toties, ut ſuperet differentiam $Q-P$; ſit n numerus vicium, quibus repetita fuit. Dividatur differentia $Q-P$ in n partes æquales. Tum inter terminos datos P & Q inſerantur n termini continue geometrice proportionales. Dico factum.

Etenim inter duos terminos P & Q inſerantur totidem termini n continue arithmetice proportionales: ſint p & p' termini ipſi P proximi in progreſſione geometrica & arithmetica.

Erit (per Coroll. 4. præc.) $p < p'$; & ideo $p - P < p' - P$. Atqui (per conſtructionem) $p' - P < a$; ergo a fortiori $p - P < a$.

Corollarium 1. Terminus itaque datus P limes eſt quantitatis mutabilis p; proinde ratio æqualitatis etiam limes eſt rationis creſcentis $P : p$ (§. 3.)

Corollarium 2. Sit q terminus, qui terminum Q immediate præcedit. Quoniam $P : p = q : Q$, ratio æqualitatis etiam limes eſt rationis creſcentis $q : Q$,

 ſeu

ſeu limes rationis decreſcentis $Q:q$ (§. 10.). Proinde quantitas data Q pariter eſt limes quantitatis mutabilis creſcentis q (§. 3.).

Corollarium 3. Quoniam ratio $\frac{P}{Q} = \left(\frac{P}{p}\right)^n = \left(\frac{q}{Q}\right)^n$; $\left(\frac{P}{Q}\right)^{\frac{1}{n}} = \frac{P}{p} = \frac{q}{Q}$: & pariter $\frac{Q}{P} = \left(\frac{p}{P}\right)^n = \left(\frac{Q}{q}\right)^n$; hinc $\left(\frac{Q}{P}\right)^{\frac{1}{n}} = \frac{p}{P} = \frac{Q}{q}$. Proinde aucto n ratio æqualitatis eſt limes rationis ſubmultiplicatæ rationis cujuslibet propoſitæ; ſeu tam lim. $\left(\frac{P}{Q}\right)^{\frac{1}{n}}$ quam lim. $\left(\frac{Q}{P}\right)^{\frac{1}{n}} = 1$. Quare, ſi $\frac{P}{Q}$ vel $\frac{Q}{P} = z$; lim. $z^{\frac{1}{n}} = 1$, quæcunque ſit z ſive major unitate, ſive minor eâdem.

Scholium. Breviter adhuc, quæ illuſtrandis tum præcedentibus quibusdam, tum ſequentibus neceſſaria ſunt, de quantitatum, quæ majores aut minores fieri poſſunt qualibet quantitate propoſita, poteſtatibus exponentis dati adjungam.

1°. Sit z quantitas, quæ major fieri poteſt qualibet quantitate propoſita; & ſit n exponens datus integer poſitivus. Creſcente z, creſcit a fortiore poteſtas z^n; & proinde, a fortiori, poteſtas z^n major fieri poteſt quacunque quantitate propoſita.

Sit autem exponens datus numerus fractus formæ $\frac{1}{n}$; ſeu proponatur functio $\sqrt[n]{z}$. Quæcunque ſit quantitas propoſita a, fiat $b \geqq a^n$; tum fiat (quod poſſibile per hyp.) $z > b > a^n$: erit etiam $\sqrt[n]{z} > \sqrt[n]{b} > a$.

Hoc pariter obtinet, ſi exponens datus fuerit numerus fractus $\frac{m}{n}$; ita ut functio propoſita ſit $\sqrt[n]{z^m} = z^{\frac{m}{n}}$. Fiat enim $b^m \geqq a^n$; tum fiat $z > b$, & proinde $z^m > b^m$: erit etiam $z^{\frac{m}{n}} > b^{\frac{m}{n}} > a$.

Si autem exponens datus fuerit numerus negativus; ſeu ſi functio propoſita fuerit $\frac{1}{z^n}$: creſcente z, functio hæc minor reddi poteſt quacunque quantitate propoſita. Etenim propoſita ſit quantitas $\frac{1}{a}$: fiat per caſum primum $z^n > a$; & erit $\frac{1}{z^n} < \frac{1}{a}$.

2°. Sit

2°. Sit quantitas z talis, quæ minor reddi poteſt quacunque quantitate propoſita; dico, etiam quamlibet ejus poteſtatem z^n poſſe quacunque quantitate propoſita fieri minorem, ſi n fuerit numerus poſitivus; & contra, eandem quantitatem fieri poſſe quacunque quantitate propoſita majorem, ſi n fuerit numerus negativus.

Caſus hic ad primum facile reducitur. Quoniam enim z poteſt quacunque quantitate propoſita fieri minor; $\frac{1}{z}$ & proinde etiam $\left(\frac{1}{z}\right)^n$ major fieri poteſt quacunque quantitate propoſita; unde z^n minor erit quacunque quantitate propoſita.

Caput Secundum.

De rationibus differentialibus et integralibus.

§. 23.

Sit x^n potentia quæcunque quantitatis variabilis x. Sit etiam $a^{n-1}x$, productum ex eadem quantitate mutabili per quantitatem datam a^{n-1}. Quantitas variabilis x recipiat mutationem quamcunque Δx; unde duæ quantitates $a^{n-1}x$ & x^n fiant $a^{n-1}(x+\Delta x)$ & $(x+\Delta x)^n$; & proinde prædictæ quantitates recipiant mutationes ſimultaneas

$$a^{n-1}\Delta x \quad \& \quad \frac{n}{1}x^{n-1}\Delta x+\frac{n}{1}.\frac{n-1}{2}x^{n-2}\Delta x^2+\frac{n}{1}\ldots\frac{n-2}{3}x^{n-3}\Delta x^3+\frac{n}{1}\ldots\frac{n-3}{4}x^{n-4}\Delta x^4+\ldots$$

Ratio harum mutationum ſimultanearum æqualis eſt ideo rationi

$$a^{n-1} \quad : \quad \frac{n}{1}x^{n-1} + \frac{n}{1}.\frac{n-1}{2}x^{n-2}\Delta x +\frac{n}{1}\ldots\frac{n-2}{3}x^{n-3}\Delta x^2+\frac{n}{1}\ldots\frac{n-3}{4}x^{n-4}\Delta x^3+\ldots$$

Proinde, quamdiu n eſt numerus quicunque ab unitate diverſus; ratio prædictarum mutationum differt a ratione $a^{n-1} : nx^{n-1}$. At vero x manente eadem, decreſcente Δx, quantitas $\frac{n}{1}.\frac{n-1}{2}x^{n-2}\Delta x+\frac{n}{1}\ldots\frac{n-2}{3}x^{n-3}\Delta x^2+\frac{n}{1}\ldots\frac{n-3}{4}x^{n-4}\Delta x^3+\ldots$ poteſt fieri minor quacunque quantitate propoſita (§. 17.); proinde quantitas $\frac{n}{1}x^{n-1}$ limes eſt quantitatis

$$\frac{n}{1}.x^{n-1} + \frac{n}{1}.\frac{n-1}{2}x^{n-2}\Delta x+\frac{n}{1}\ldots\frac{n-2}{3}x^{n-3}\Delta x^2+\frac{n}{1}\ldots\frac{n-3}{4}x^{n-4}\Delta x^3+\ldots\ldots$$

Et ratio $a^{n-1} : \frac{n}{1}x^{n-1}$ limes eſt rationis mutabilis

$$a^{n-1}: \frac{n}{1} x^{n-1} + \frac{n}{1} \cdot \frac{n-1}{2} x^{n-2} \Delta x + \frac{n}{1} \cdots \frac{n-2}{3} x^{n-3} \Delta x^2 + \frac{n}{1} \cdots \frac{n-3}{4} x^{n-4} \Delta x^3 + \cdots\cdots$$

ſeu lim. $\frac{\Delta . x^n}{a^{n-1} \Delta x} = \frac{n x^{n-1}}{a^{n-1}}$; & proinde etiam lim. $\frac{\Delta . x^n}{\Delta . x} = n x^{n-1}$.

Quoniam autem hic notandi modus lim. $\frac{\Delta . x^n}{a^{n-1} \Delta x}$ ſeu lim. $\frac{\Delta . x^n}{\Delta . x}$, calculo minus eſt commodus; & tamen hoc ſignum aliaque ſimilia ſæpiſſime occurrunt: facilitatis calculi gratia (& quidem, quod probe notandum, unice hac de cauſa), ſigno minus commodo lim. $\frac{\Delta . x^n}{\Delta x}$ ſubſtituatur hoc $\frac{d . x^n}{d x}$; & ſic conſtituatur æquatio $\frac{d . x^n}{d x} = n x^{n-1}$.

§. 24. Maximopere vero cavendum eſt, ne credamus ſymbolum $\frac{d . x^n}{d x}$, quod formam magnitudinis ex duabus compoſitæ præ ſe fert, revera eſſe ſymbolum compoſitum; ac deſignare fractionem, cujus termini ſint $d . x^n$ & dx; quaſi $d . x^n$ & $d x$ denotent certas quantitates, & ut ita dicam, minuſcula (miniatures) quantitatum verarum $\Delta . x^n$ & Δx: aut ne credamus, ex æquatione $\frac{d . x^n}{d x} = n x^{n-1}$ poſſe deduci hanc $d . x^n = n x^{n-1} d x$. Expreſſio $\frac{d . x^n}{d x}$ incomplexa eſt atque peculiaris, ad deſignandos exponentes limitum rationum ſimultaneorum quantitatum mutabilium x^n & x incrementorum facilitatis cauſa introducta: & quamvis veſtigia conſervet originis ſuæ $\frac{\Delta . x^n}{\Delta x}$; ſigna $d . x^n$ & $d x$ ſigillatim ſpectata nullam amplius ad quantitates veras $\Delta . x^n$ & Δx relationem habere firmiter tenendum eſt. Quæ monita eo magis urgenda eſſe cenſeo, quod doctrina hæc quæſtionibus nonnullis inanibus fuit exagitata; quæ ne motæ quidem fuiſſent, ſi ad genuinam ſymboli illius indolem Mathematici ſemper attendiſſent. (*a*)

Hoc exemplo præeunte, quæ ſequuntur definitiones, facile intelligentur.

§. 24.

(*a*) Nonnulli etiam, qui veram ſigni hujus naturam agnoviſſe videntur, auſi non ſunt, quam hic profiteor ſententiam amplecti. Videatur v. gr. Diſſertatio Abbatis Caluso in *novis Commentariis Academiæ Taurinenſis* T. II. ubi ſic ſtatuit Auctor: *Il ne ſuffit pas de voir clairement que* $\frac{dv}{dx}$ *etant la limite du raport* $\frac{\Delta v}{\Delta x}$*; et* $\frac{dz}{dx}$ *la limite du raport* $\frac{\Delta z}{\Delta x}$*; on aura,* $\frac{dv}{dx} = z + x\frac{dz}{dx}$ *lorsque* $v = xz$*: mais, il faut d'abord*

§. 25. *Definitiones.* Limes rationis mutationum, quas duæ pluresve quantitates mutabiles ſimul ſuſcipiunt, dicatur earum *ratio differentialis;* & exponens hujus rationis dicatur *exponens differentialis.* Operatio, qua exponens differentialis quæritur, dicatur *differentiatio.* Item: calculus, qui occupatur rationibus differentialibus inveſtigandis, dicatur *calculus differentialis.*

Sit P functio quantitatis mutabilis x; exponens differentialis harum quantitatum deſignetur $\frac{dP}{dx}$. Item, ſint P & Q duæ quantitates mutabiles quæcunque, exponens differentialis harum quantitatum deſignetur $\frac{dP}{dQ}$. Et ſymbola $\frac{dP}{dx}$, $\frac{dP}{dQ}$ .... ſpectentur tanquam ſigna ſimplicia, nulla apparentis ipſorum compoſitionis ratione habita.

Hinc, ſi P eſt potentia ordinis cujuscunque n quantitatis mutabilis x, $\frac{dP}{dx} = nx^{n-1}$ (§. 23.) Pariter ſit Q potentia alterius ordinis m quantitatis mutabilis x; erit $\frac{dQ}{dx} = mx^{m-1}$. Hinc $\frac{dP}{dQ} = \frac{n}{m}x^{n-m}$.

$$\begin{array}{lllll} \text{Etenim} & \lim.\Delta P : \Delta x & = nx^{n-1} & : & 1 \\ \text{item} & \lim.\Delta x : \Delta Q & = 1 & : & mx^{m-1} \\ \text{Ergo} & \lim.\Delta P : \Delta Q & = nx^{n-1} & : & mx^{m-1} \quad (\S.\ 14.) \end{array}$$

$$\text{ſeu } \frac{dP}{dQ} = \frac{n}{m}x^{n-m}.$$

Item: ſit P functio quantitatis mutabilis x, qualis ſequitur

$$P = Ax^a + Bx^b + Cx^c + Dx^d + \ldots + Lx^l + Mx^m + Nx^n$$

$$\text{fit } \frac{dP}{dx} = Aax^{a-1} + Bbx^{b-1} + Ccx^{c-1} + Ddx^{d-1} + \ldots + Llx^{l-1} + Mmx^{m-1} + Nnx^{n-1}.$$

§. 26. Exercitii cauſa adjungam nonnulla exempla variarum functionum.

Sint P & Q duæ quantitates mutabiles, quæ referantur v. gr. ad eandem quantitatem mutabilem x: quæritur exponens differentialis $\frac{d.PQ}{dx}$.

d'abord attacher une idée nette et precise à dv, dx, dz; *afin que ces expreſſions ſignifient quelque choſe par elles-mêmes et independamment les unes des autres.* Cum nuperrime vir adeo ſagax ejusmodi difficultatem moverit, e re profecto eſſe cenſendum eſt, in limine, & cum prima illius ſigni introductione, lectorem de vera ejus ſignificatione monere.

$$(P+\Delta P)(Q+\Delta Q) = PQ + P\Delta Q + Q\Delta P + \Delta P.\Delta Q$$

Hinc $\Delta.PQ = P\Delta Q + Q\Delta P + \Delta P.\Delta Q$

et . $\dfrac{\Delta.PQ}{\Delta x} = P\dfrac{\Delta Q}{\Delta x} + Q\dfrac{\Delta P}{\Delta x} + \dfrac{\Delta P}{\Delta x}.\Delta Q.$

Hinc lim. $\dfrac{\Delta.PQ}{\Delta x} = P.\text{lim.}\dfrac{\Delta Q}{\Delta x} + Q\,\text{lim.}\dfrac{\Delta P}{\Delta x},$

feu $\dfrac{\mathrm{d}PQ}{\mathrm{d}x} = P\dfrac{\mathrm{d}Q}{\mathrm{d}x} + Q\dfrac{\mathrm{d}P}{\mathrm{d}x}.$

Eodem modo $\dfrac{\mathrm{d}.PQR}{\mathrm{d}x} = PQ.\dfrac{\mathrm{d}R}{\mathrm{d}x} + R\dfrac{\mathrm{d}.PQ}{\mathrm{d}x}$

$$= PQ\,\frac{\mathrm{d}R}{\mathrm{d}x} + PR\frac{\mathrm{d}Q}{\mathrm{d}x} + QR\frac{\mathrm{d}P}{\mathrm{d}x}$$

$$\frac{\mathrm{d}.PQRS}{\mathrm{d}x} = PQR\frac{\mathrm{d}S}{\mathrm{d}x} + S\frac{\mathrm{d}.PQR}{\mathrm{d}x}$$

$$= PQR\frac{\mathrm{d}S}{\mathrm{d}x} + PQS.\frac{\mathrm{d}R}{\mathrm{d}x} + PRS\frac{\mathrm{d}Q}{\mathrm{d}x} + QRS\frac{\mathrm{d}P}{\mathrm{d}x}.$$

Et fic deinceps. Scilicet exponens differentialis producti quotcunque quantitatum mutabilium, & alius cujuscunque quantitatis mutabilis x, æqualis eft fummæ productorum omnium priorum quantitatum una excepta, per exponentem differentialem hujus quantitatis & prædictæ quantitatis x.

Eademque regula applicatur ad inveftigandos exponentes differentiales quantitatum fractarum.

Etenim $\dfrac{P}{Q} = PQ^{-1}$; ideo $\dfrac{\mathrm{d}.\frac{P}{Q}}{\mathrm{d}x} = P\dfrac{\mathrm{d}Q^{-1}}{\mathrm{d}x} + Q^{-1}\dfrac{\mathrm{d}P}{\mathrm{d}x}.$ Sed (§. 25.) $\dfrac{\mathrm{d}Q^{-1}}{\mathrm{d}x} = -Q^{-2}\dfrac{\mathrm{d}Q}{\mathrm{d}x}$; & ideo $\dfrac{\mathrm{d}.\frac{P}{Q}}{\mathrm{d}x} = Q^{-1}\dfrac{\mathrm{d}P}{\mathrm{d}x} - PQ^{-2}\dfrac{\mathrm{d}Q}{\mathrm{d}x} = \dfrac{Q\frac{\mathrm{d}P}{\mathrm{d}x} - P\frac{\mathrm{d}Q}{\mathrm{d}x}}{QQ}.$

Exponens hic differentialis obtinetur etiam modo fequenti.

Sit $\dfrac{P}{Q} = Z$, erit $P = QZ$; hinc $\dfrac{\mathrm{d}P}{\mathrm{d}x} = Q\dfrac{\mathrm{d}Z}{\mathrm{d}x} + Z\dfrac{\mathrm{d}Q}{\mathrm{d}x}$

$$= Q\frac{\mathrm{d}Z}{\mathrm{d}x} + \frac{P}{Q}\frac{\mathrm{d}Q}{\mathrm{d}x}$$

$$\frac{\mathrm{d}Z}{\mathrm{d}x} = \frac{1}{Q}\frac{\mathrm{d}P}{\mathrm{d}x} - \frac{P}{QQ}\frac{\mathrm{d}Q}{\mathrm{d}x}$$

$$= \frac{Q\frac{\mathrm{d}P}{\mathrm{d}x} - P\frac{\mathrm{d}Q}{\mathrm{d}x}}{QQ}.$$

Hoc

Hoc eſt: multiplicetur denominator fractionis per exponentem differentialem numeratoris, ab hoc facto ſubtrahatur productum numeratoris per exponentem differentialem denominatoris; oritur exponens differentialis fractionis.

Hisce exemplis continetur univerſus calculus differentialis, quatenus ad functiones tantum algebraicas pertinet. (De quantitatibus tranſcendentibus & exponentialibus poſtea dicemus.)

Sit v. gr. quantitas $\sqrt[3]{(axx-x^3)}$, cujus quæritur exponens differentialis.

Sit $P = \sqrt[3]{(axx-x^3)} = (axx-x^3)^{\frac{1}{3}}$.

$$\frac{dP}{dx} = \tfrac{1}{3}(axx-x^3)^{-\frac{2}{3}}(2ax-3x^2)$$

$$= \frac{2ax-3xx}{3\sqrt[3]{(axx-x^3)^2}} = \frac{2ax-3xx}{3x\sqrt[3]{(x(a-x)^2)}} = \frac{2a-3x}{3\sqrt[3]{(x(a-x)^2)}}.$$

$$\text{Sit } P = x^m y^n + Ax^p y^q + Bx^r y^s + Cx^t y^v + \ldots\ldots$$

$$\frac{dP}{dx} = mx^{m-1}y^n + pAx^{p-1}y^q + rBx^{r-1}y^s + tCx^{t-1}y^v + \ldots\ldots$$

$$+ \frac{dy}{dx}(nx^m y^{n-1} + qAx^p y^{q-1} + sBx^r y^{s-1} + vCx^t y^{v-1} + \ldots\ldots)$$

§. 27. Quemadmodum ex data relatione mutua duarum pluriumve quantitatum mutabilium quæritur earum ratio differentialis: ſic viciſſim ex data ratione differentiali duarum pluriumve quantitatum mutabilium, quæri poteſt relatio mutua ipſarum quantitatum. Hæc relatio dicatur *ratio integralis;* operatio qua ratio integralis quæritur, dicatur *integratio;* & calculus, qui occupatur inveſtigatione rationum integralium, dicatur *calculus integralis.*

Exempla. Data ſit ratio differentialis $\frac{dP}{dx} = x^n = \frac{1}{n+1}(n+1)x^n$: ſit ratio integralis $P = \frac{1}{n+1}x^{n+1}$.

Sit ratio differentialis $\frac{dP}{dx} = y + x\frac{dy}{dx}$; ſit ratio integralis $P = xy$.

Sit ratio differentialis $\frac{dP}{dx} = \frac{y - x\frac{dy}{dx}}{yy}$; ſit ratio integralis $P = \frac{x}{y}$.

Obſervatio 1. Data relatione mutua duarum pluriumve quantitatum mutabilium complexarum, quarum termini partim ſunt conſtantes: hi termini conſtantes non afficiunt exponentem differentialem. Quare viciſſim, data ratione diffe-

differentiali duarum pluriumve quantitatum mutabilium, non unica ei respondet ratio integralis; sed hæc ratio, præter quantitates mutabiles per rationem differentialem determinatas, assumere etiam potest terminum constantem, qui vulgo per C designatur.

Ita si $\frac{dP}{dx} = x^n$; fit $P = C + \frac{1}{n+1}x^{n+1}$

$$\frac{dP}{dx} = y + x\frac{dy}{dx}; \quad P = C + xy$$

$$\frac{dP}{dx} = \frac{y - x\frac{dy}{dx}}{yy}; \quad P = C + \frac{x}{y}.$$

Quantitas hæc constans C, quæ, rationis differentialis tantum ratione habita, est indeterminata, plerumque per naturam quæstionis propositæ determinatur.

Exemplum. Sit $\frac{dP}{dx} = (x-a)^n$; & quæratur ratio integralis huic rationi differentiali respondens, talis, ut ratio illa evanescat, quando $x = a$.

Ideo $P = C + \frac{1}{n+1}(x-a)^{n+1}$. Atqui quantitas $\frac{1}{n+1}(x-a)^{n+1}$ evanescit, posito $x = a$; ergo $C = 0$: & $P = \frac{1}{n+1}(x-a)^{n+1}$, quæ nunc dicitur ratio integralis completa.

Sit vero $\frac{dP}{dx} = (a+x)^n$; & quæratur ratio integralis talis, ut evanescat posito $x = 0$.

$P = C + \frac{1}{n+1}(a+x)^{n+1}$. Atqui, posito $x = 0$, $(a+x)^{n+1}$ fit a^{n+1}. Ergo $C = -\frac{1}{n+1}a^{n+1}$. Proinde $P = \frac{1}{n+1}((a+x)^{n+1} - a^{n+1})$

$$= \frac{1}{n+1}\left(\frac{n+1}{1}a^n x + \frac{n+1}{1}\cdot\frac{n}{2}a^{n-1}x^2 + \frac{n+1}{1}\cdots\frac{n-1}{3}a^{n-2}x^3 + \frac{n+1}{1}\cdots\frac{n-2}{4}a^{n-3}x^4 + \ldots\ldots\right.$$

$$= a^n x + \frac{n}{1.2}a^{n-1}x^2 + \frac{n.n-1}{1.2.3}a^{n-2}x^3 + \frac{n\ldots n-2}{1\ldots 4}a^{n-3}x^4 + \ldots\ldots$$

Exemplum. Sit $\frac{dP}{dx} = \frac{1}{a+x} = (a+x)^{-1}$; & sit $P = 0$, quando $x = 0$.

$$P = \frac{x}{a} - \tfrac{1}{2}\frac{x^2}{a^2} + \tfrac{1}{3}\frac{x^3}{a^3} - \tfrac{1}{4}\frac{x^4}{a^4} + \tfrac{1}{5}\frac{x^5}{a^5} - \tfrac{1}{6}\frac{x^6}{a^6} + \ldots\ldots\ldots$$

Scholium.

Scholium. Sic exceptionem tolli cenſeo communiter huc usque præceptam integrationis formulæ $\frac{\mathrm{d}P}{\mathrm{d}x} = \frac{1}{a+x} = (a+x)^{-1}$, quæ deducere videtur ad relationem integralem $P = C + \frac{1}{0}(a+x)^0$, ſenſu omnino caſſam, & quam ideo alia methodo explicare mathematici allaborarunt: cum oſtenderim, formulam illam eodem prorſus modo evolvi, quo formula generalis $\frac{\mathrm{d}P}{\mathrm{d}x} = (a+x)^n$, ſeu $P = C + \frac{1}{n+1}(a+x)^{n+1}$; addita conditione, quod evaneſcat, ſi $x = 0$. (Vid. præter alios, EULERI *Calculus integralis*, T. I. pag. 26. & 27. ubi illuſtris hic Mathematicus iterata vice exceptionem illam affirmat.) Ipſe eandem ſententiam profeſſus fui in Diſſertatione: *Expoſition elementaire des Principes des Calculs ſuperieurs* pag. 93. Fatendum tamen, caſum hunc a reliquis eatenus eſſe diſtinguendum, quod ſubſtractio indicata $(a+x)^{n+1} - a^{n+1}$ peragi, & diviſione actû inſtituta factor $n+1$ tolli neceſſario debeat, ut formula intelligibilis emergere poſſit.

Obſervatio 2. Data relatione mutua duarum pluriumve quantitatum mutabilium, ratio differentialis earundem ſemper poteſt obtineri, ideoque calculus differentialis poteſt dici omnimode completus. Idem non valet de calculo integrali. Scilicet data ratione differentiali duarum pluriumve quantitatum mutabilium non ſemper obtineri poteſt relatio earum integralis. Nempe non obſtantibus aſſiduis ſagaciſſimorum mathematicorum ſtudiis calculus integralis adhuc in cunabulis jacet; nec ulla regula omnino generalis præſto eſſe dici poteſt, quà omnes omnino formulæ poſſint ad integrationem perduci. (Quidquid hoc reſpectu ſperaverit ſagaciſs. Dn. PACCASSI. Vide *Phyſikaliſche Arbeiten der einträchtigen Freunde zu Wien.* 1786.)

Nimium a ſcopo meo abluderet, formulas, quarum integrationes in promtu ſunt, exponere, pariter atque artificia, quorum ſubſidio plurimæ formulæ integrantur, quæ primas calculi integralis regulas effugere videntur, tum & inventa varia, quibus integratio formularum complexarum ad integrationem formularum (ſpecie ſaltem) ſimplicium reducuntur. Conſulendi eam in rem ſunt auctores, qui de calculo integrali ſcripſerunt, eo fine, ut ipſum perficerent aut tironibus explanarent: quos inter nominare ſufficiet COTESII Tractatum *de*

Harmonia mensurarum, aut WALMESLEY *Analyse de la mesure des raports et des angles*, & præ omnibus EULERI *Institutiones calculi integralis*. (*a*)

§. 28. Cum exponentes differentiales quantitatum mutabilium plerumque & ipsi quantitates mutabiles sint; quæ de qualibet quantitate mutabili dicta sunt, ejusmodi quoque exponentibus differentialibus possunt applicari. Nominatim quæri possunt exponentes differentiales istiusmodi exponentium differentialium.

Sit v. gr. P functio quantitatis mutabilis x, cujus exponens differentialis $\frac{\mathrm{d}P}{\mathrm{d}x}$. Quod si exponens hic & ipse mutabilis est: indagari pariter potest tam ratio mutationum simultanearum quantitatum variabilium $\frac{\mathrm{d}P}{\mathrm{d}x}$ & x; quam ratio, quæ est limes istius rationis, atque ejus exponens. Ratio, quæ limes est rationis mutationum simultanearum exponentis differentialis duarum pluriumve quantitatum mutabilium & unius ex illis, dicatur *ratio differentialis secundi ordinis*; & exponens prædictæ rationis dicatur *exponens differentialis secundi ordinis*.

Exponens itaque differentialis secundi ordinis functionis P & quantitatis mutabilis x est lim. $\frac{\Delta\frac{\mathrm{d}P}{\mathrm{d}x}}{\Delta x}$; & eadem de causa, quâ prius, nempe majoris calculi facilitatis causa, exponens iste denotetur signo $\frac{\mathrm{dd}P}{\mathrm{d}x^2}$. Et hîc (imo etiam, si fieri posset, a fortiori) repetenda sunt, quæ de simplicitate symboli hujus monui, non obstante ejus apparenti compositione. Ne quis quærat: quid sit $\mathrm{dd}P$, neque quid sit $\mathrm{d}x^2$? Nullum, quod satisfaciat, obtinebit responsum. Consideret potius symbolum $\frac{\mathrm{dd}P}{\mathrm{d}x^2}$ tanquam expressionem unicam & sui generis, cujus termini apparentes nullam servant relationem aut nexum cum quantitatibus $\Delta^2 P$, & Δx^2, ex quibus incautiores mathematici eos derivare aggredi possent, & reipsa nimium frequenter aggressi sunt.

Si

(*a*) Utut imperfectus etiamnum sit calculus integralis: ea tamen est multitudo formularum, tam quas integrare, quam quas ad simpliciores reducere licet, & formulæ hæ in tot voluminibus, vel a privatis mathematicis, vel ab academiis editis, dispersæ sunt; ut gratam æque ac utilem mathematicis navaret operam, qui formulas jam integratas (COTESII & WALMESLEY exemplo), in tabulas apte dispositas redigeret, quarum ope sæpenumero a labore tædioso & superfluo vacare liceret.

Si exponens differentialis secundi ordinis $\frac{\mathrm{dd}P}{\mathrm{d}x^2}$ sit & ipse quantitas mutabilis: operationes, quæ cuilibet quantitati mutabili applicatæ fuerunt, huic etiam exponenti poterunt applicari; & nominatim quæri poterit tam ratio mutationum simultanearum quantitatum mutabilium $\frac{\mathrm{dd}P}{\mathrm{d}x^2}$ & x, quam ratio limes istius rationis, & ejus exponens. Hæc ratio dicatur *ratio differentialis tertii ordinis*, & exponens ejusdem dicatur *exponens differentialis tertii ordinis*.

Exponens igitur differentialis tertii ordinis prædictæ functionis P est lim. $\frac{\Delta \frac{\mathrm{dd}P}{\mathrm{d}x^2}}{\Delta x}$; & majoris calculi facilitatis causa designatur symbolo $\frac{\mathrm{d}^3 P}{\mathrm{d}x^3}$, quod pariter spectandum est tanquam signum unicum, incomplexum, non quasi compositum ex duabus partibus $\mathrm{d}^3 P$ & $\mathrm{d}x^3$. Neque movenda est quæstio absona, quid sint termini illi apparentes; neque comparatio aliqua signorum $\mathrm{d}^3 P$ & $\mathrm{d}x^3$ cum quantitatibus $\Delta^3 P$ & Δx^3 est instituenda.

Hinc facile liquet, quid sint rationes & exponentes differentiales altiorum ordinum, 4^{ti}, 5^{ti}, 6^{ti} . . . n^{ti}; & quid denotare censenda sint signa illis respondentia, $\frac{\mathrm{d}^4 P}{\mathrm{d}x^4}$, $\frac{\mathrm{d}^5 P}{\mathrm{d}x^5}$, $\frac{\mathrm{d}^6 P}{\mathrm{d}x^6}$ . . . $\frac{\mathrm{d}^n P}{\mathrm{d}x^n}$.

En aliqua exempla præcedentibus illustrandis apta.

Sit $P = x^n$.

Erunt
$$\frac{\mathrm{d}P}{\mathrm{d}x} = nx^{n-1}$$
$$\frac{\mathrm{dd}P}{\mathrm{d}x^2} = n.\overline{n-1}.x^{n-2}$$
$$\frac{\mathrm{d}^3 P}{\mathrm{d}x^3} = n.\overline{n-1}.\overline{n-2}.x^{n-3}$$
$$\frac{\mathrm{d}^4 P}{\mathrm{d}x^4} = n.\overline{n-1}\ldots\overline{n-3}.x^{n-4}$$
$$\frac{\mathrm{d}^5 P}{\mathrm{d}x^5} = n.\overline{n-1}\ldots\overline{n-4}.x^{n-5}$$
$$\vdots \qquad \vdots$$
$$\frac{\mathrm{d}^m P}{\mathrm{d}x^m} = n.\overline{n-1}\ldots\overline{n-(m-1)}\,x^{n-m}.$$

Si exponens n fuerit numerus integer positivus, series hæc exponentium differentialium abrumpetur.

Sci-

Scilicet: fit $m=n$, erit $\frac{d^n P}{dx^n} = n.n-1 \ldots 2.1.x^0 = 1.2.3 \ldots n$

et $\frac{d^{n+1} P}{dx^{n+1}} = 0.$

Nempe exiftente n numero integro pofitivo: exponens differentialis, cujus ordo per ipfum potentiæ exponentem defignatur, fit conftans; & exponentes differentiales ordinum fuperiorum evanefcunt.

In ceteris cafibus, ubi n non eft numerus integer pofitivus, nunquam pervenitur ad exponentem differentialem conftantem, nec ad evanefcentem.

Secundum Exemplum. Sit $P = zy$, & quærantur omnes exponentes differentiales refpectu ejusdem variabilis x, cujus z & y cenfeantur effe functiones.

Sit ideo $P = zy$

$$\frac{dP}{dx} = z\frac{dy}{dx} + y\frac{dz}{dx}$$

$$\frac{ddP}{dx^2} = z\frac{ddy}{dx^2} + 2\frac{dz}{dx}\cdot\frac{dy}{dx} + y\frac{ddz}{dx^2}$$

$$\frac{d^3P}{dx^3} = z\frac{d^3y}{dx^3} + 3\frac{dz}{dx}\cdot\frac{ddy}{dx^2} + 3\frac{ddz}{dx^2}\cdot\frac{dy}{dx} + y\frac{d^3z}{dx^3}$$

$$\frac{d^4P}{dx^4} = z\frac{d^4y}{dx^4} + 4\frac{dz}{dx}\cdot\frac{d^3y}{dx^3} + 6\frac{ddz}{dx^2}\cdot\frac{ddy}{dx^2} + 4\frac{d^3z}{dx^3}\cdot\frac{dy}{dx} + y\frac{d^4z}{dx^4}$$

$$\frac{d^5P}{dx^5} = z\frac{d^5y}{dx^5} + 5\frac{dz}{dx}\cdot\frac{d^4y}{dx^4} + 10\frac{ddz}{dx^2}\cdot\frac{d^3y}{dx^3} + 10\frac{d^3z}{dx^3}\cdot\frac{ddy}{dx^2} + 5\frac{d^4z}{dx^4}\cdot\frac{dy}{dx} + y\frac{d^5z}{dx^5}.$$

Generatim.

$$\frac{d^mP}{dx^m} = z\frac{d^my}{dx^m} + \frac{m}{1}\frac{dz}{dx}\cdot\frac{d^{m-1}y}{dx^{m-1}} + \frac{m}{1}\cdot\frac{m-1}{2}\frac{ddz}{dx^2}\cdot\frac{d^{m-2}y}{dx^{m-2}} + \ldots + \frac{m}{1}\frac{dy}{dx}\cdot\frac{d^{m-1}z}{dx^{m-1}} + y\frac{d^mz}{dx^m}.$$

Identitas (jam a LEIBNITZIO obfervata, *Mifcell. Berol.* 1710.) coëfficientium terminorum fucceffivorum harum formularum cum coëfficientibus formulæ binominalis Newtonianæ neminem attentum effugiet. Ipfeque differentiationum fucceffivarum proceffus identitatem hanc liquido manifeftat.

Sit

Sit $P = \frac{y}{x} = yx^{-1}$

$$\frac{dP}{dx} = x^{-1}\frac{dy}{dx} - yx^{-2}$$

$$\frac{ddP}{dx^2} = x^{-1}\frac{ddy}{dx^2} - 2\frac{dy}{dx}x^{-2} + 2yx^{-3}$$

$$\frac{d^3P}{dx^3} = x^{-1}\frac{d^3y}{dx^3} - 3\frac{ddy}{dx^2}x^{-2} + 6\frac{dy}{dx}x^{-3} - 6yx^{-4}$$

$$\frac{d^4P}{dx^4} = x^{-1}\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3}x^{-2} + 12\frac{ddy}{dx^2}x^{-3} - 24\frac{dy}{dx}x^{-4} + 24yx^{-5}.$$

&c. &c. &c.

Sit $P = (xx - aa)^m = (x+a)^m . (x-a)^m.$

Sit $y = (x+a)^m$	$z = (x-a)^m$
$\frac{dy}{dx} = m.(x+a)^{m-1}$	$\frac{dz}{dx} = m(x-a)^{m-1}$
$\frac{ddy}{dx^2} = m.m-1.(x+a)^{m-2}$	$\frac{ddz}{dx^2} = m.m-1.(x-a)^{m-2}$
$\frac{d^3y}{dx^3} = m....m-2.(x+a)^{m-3}$	$\frac{d^3z}{dx^3} = m....m-2.(x-a)^{m-3}$
$\frac{d^4y}{dx^4} = m....m-3.(x+a)^{m-4}$	$\frac{d^4z}{dx^4} = m....m-3.(x-a)^{m-4}$
$\frac{d^5y}{dx^5} = m....m-4.(x+a)^{m-5}$	$\frac{d^5z}{dx^5} = m....m-4.(x-a)^{m-5}$
$\vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots$
$\frac{d^ny}{dx^n} = m....m-(n-1)(x+a)^{m-n}.$	$\frac{d^nz}{dx^n} = m....m-(n-1)(x-a)^{m-n}.$

Hinc $$\frac{dP}{dx} = m . (x-a)^m . (x+a)^{m-1}$$
$$+ m . (x-a)^{m-1} . (x+a)^m$$

$$\frac{ddP}{dx^2} = m . m-1 . (x-a)^m . (x+a)^{m-2}$$
$$+ 2 . m . m . (x-a)^{m-1} . (x+a)^{m-1}$$
$$+ m . m-1 . (x-a)^{m-2} . (x+a)^m$$

$$\frac{d^3P}{dx^3} = m.m\text{-}1\;.\;m-2\;.\;(x-a)^m\quad.\;(x+a)^{m-3}$$
$$+\;3\;.\;m.\;m\quad.\;m-1\;.\;(x-a)^{m-1}.\;(x+a)^{m-2}$$
$$+\;3\;.\;m.m\text{-}1\;.\quad m\quad.\;(x-a)^{m-2}.\;(x+a)^{m-1}$$
$$+\;1\;.\;m\;.\;m\text{-}1\;.\;m-2\;.\;(x-a)^{m-3}.\;(x+a)^m$$

$$\frac{d^4P}{dx^4} = m\;.\;.\;.\;.\;.\;.\;m-3\;.\;(x-a)^m\quad.\;(x+a)^{m-4}$$
$$+\;4\;.\;m\;.\;m\quad.\;m-1\;.\;m-2\;.\;(x-a)^{m-1}.\;(x+a)^{m-3}$$
$$+\;6\;.\;m\;.\;m\text{-}1\;.\;m\quad.\;m-1\;.\;(x-a)^{m-2}.\;(x+a)^{m-2}$$
$$+\;4\;.\;m\;.\;m\text{-}1\;.\;m-2\;.\;m\quad.\;(x-a)^{m-3}.\;(x+a)^{m-1}$$
$$+\;1\;.\;m\;.\;.\;.\;.\;.\;m-3\;.\;(x-a)^{m-4}.\;(x+a)^m$$

Ordo regularis, juxta quem coëfficientes ſeſe ſubſequuntur, longe apertius patet, quam ſi quantitas $(xx-aa)^m$ non fuiſſet in factores ſuos ſoluta.

Utut regularis ſit poſtrema ſeries; eo tamen caſu, quo m eſt numerus integer negativus, differentiationes ſucceſſivas paulo aliter inſtituere juvabit.

Exempli cauſa: ſit $m = -1$; ideoque $P = \frac{1}{x-a.\,x+a} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$
$$= \frac{1}{2a}\left((x-a)^{-1} - (x+a)^{-1}\right).$$

Hinc $\frac{dP}{dx} = \frac{1}{2a}\begin{Bmatrix} -(x-a)^{-2} \\ +(x+a)^{-2} \end{Bmatrix}$

$$\frac{ddP}{dx^2} = \frac{1}{2a}\begin{Bmatrix} +\;1.2.(x-a)^{-3} \\ -\;1.2.(x+a)^{-3} \end{Bmatrix}$$
$$\frac{d^3P}{dx^3} = \frac{1}{2a}\begin{Bmatrix} -\;1.2.3.(x-a)^{-4} \\ +\;1.2.3.(x+a)^{-4} \end{Bmatrix}$$
$$\frac{d^4P}{dx^4} = \frac{1}{2a}\begin{Bmatrix} +\;1\ldots.4.(x-a)^{-5} \\ -\;1\ldots.4.(x+a)^{-5} \end{Bmatrix}$$
$$\frac{d^5P}{dx^5} = \frac{1}{2a}\begin{Bmatrix} -\;1\ldots.5.(x-a)^{-6} \\ +\;1\ldots.5.(x+a)^{-6} \end{Bmatrix}$$
&c. &c. &c.

Tironibus identitatem utriusque differentiandi proceſſus oſtendendam, uti varias alias abbreviationes evolvendas, relinquo.

§. 29.

§. 29. Quæcunque de imperfectione calculi integralis, quod ad exponentes differentiales primi gradus attinet, dicta sunt; tanto magis valent de integratione formularum differentialium secundi, tertii, altiorumque ordinum, quæ nonnisi perpaucis casibus institui potest; utcunque eam promovere allaboraverint mathematici, quorum hanc in rem inventa collecta exhibet celeb. EULERUS in opere jam laudato *Institutionum calculi integralis.* Illustrationis & exercitii causa pauca quædam exempla facillima hic sufficiant.

Sit $\frac{ddz}{dx^2} = (x+a)^m$

Hinc $\frac{dz}{dx} = C + \frac{1}{m+1}(x+a)^{m+1}$

Unde rursus $z = C' + Cx + \frac{1}{m+1 \cdot m+2}(x+a)^{m+2}$.

Quoniam quælibet integratio introducit aliquam quantitatem constantem, post secundam integrationem duæ occurrunt quantitates constantes C' & C, quæ plerumque per particularem aliquam quæstionis conditionem determinantur.

Sit

$$\frac{d^n z}{dx^n} = (x+a)^m$$

$$\frac{d^{n-1}z}{dx^{n-1}} = C + \frac{1}{m+1}(x+a)^{m+1}$$

$$\frac{d^{n-2}z}{dx^{n-2}} = C' + Cx + \frac{1}{m+1 \cdot m+2}(x+a)^{m+2}$$

$$\frac{d^{n-3}z}{dx^{n-3}} = C'' + C'x + \tfrac{1}{2}Cx^2 + \frac{1}{m+1 \ldots m+3}(x+a)^{m+3}$$

$$\frac{d^{n-4}z}{dx^{n-4}} = C''' + C''x + \tfrac{1}{2}C'x^2 + \frac{1}{1.2.3}Cx^3 + \frac{1}{m+1 \ldots m+4}(x+a)^{m+4}$$

$$\frac{d^{n-5}z}{dx^{n-5}} = C^{IV} + C'''x + \frac{1}{1 \cdot 2}C''x^2 + \frac{1}{1 \ldots 3}C'x^3 + \frac{1}{1 \ldots 4}Cx^4 + \frac{1}{m+1 \ldots m+5}(x+a)^{m+5}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\frac{ddz}{dx^2} = C^{N-III} + C^{N-IV}x + \frac{1}{1.2}C^{N-V}x^2 + \;.\;.\;.\;.\;.\; + \frac{1}{1.2 \ldots n-3}Cx^{n-3} + \frac{1}{m+1 \ldots m+n-2}(x+a)^{m+n-2}$$

$$\frac{dz}{dx} = C^{N-II} + C^{N-III}x + \frac{1}{1.2}C^{N-IV}x^2 + \frac{1}{1.2.3}C^{N-V}x^3 + \ldots + \frac{1}{1.2 \ldots n-2}Cx^{n-2} + \frac{1}{m+1 \ldots m+n-1}(x+a)^{m+n-1}$$

$$z = C^{N-I} + C^{N-II}x + \frac{1}{1.2}C^{N-III}x^2 + \frac{1}{1.2.3}C^{N-IV}x^3 + \ldots + \frac{1}{1.2 \ldots n-1}Cx^{n-1} + \frac{1}{m+1 \ldots m+n}(x+a)^{m+n}.$$

CAPUT

Caput Tertium.

De theoremate Tayloriano.

§. 30.

Theoremate, quod Taylorianum dicitur, data mutatione quantitatis mutabilis, determinatur mutatio, quam ejusdem quantitatis mutabilis functio quæcunque recipit.

Theorema hoc tam multiplicis est in calculo differentiali & integrali usus, ut accurate explicari, & quo par est rigore, demonstrari mereatur. Taylor (Mathematicus Anglus) illud exposuit in opere suo inscripto: *Methodus incrementorum directa & inversa* (Lond. 1715.); sed ex formula (§. *p. Introd.*)

$$P^m \doteq P + \frac{m}{1}\Delta P + \frac{m}{1}\cdot\frac{m-1}{2}\Delta^2 P + \frac{m}{1}\ldots\ldots\frac{m-2}{3}\Delta^3 P + \ldots\ldots\ldots$$

$$= P + \frac{m\Delta x}{1}\cdot\frac{\Delta P}{\Delta x} + \frac{m\Delta x}{1}\cdot\frac{(m-1)\Delta x}{2}\cdot\frac{\Delta^2 P}{\Delta x^2} + \frac{m\Delta x}{1}\cdot\frac{(m-1)\Delta x}{2}\cdot\frac{(m-2)\Delta x}{3}\cdot\frac{\Delta^3 P}{\Delta x^3} + \ldots$$

consequi tantum indicavit. Plerique post illum autores supplendæ huic deductioni ideam infiniti adhibuerunt, quam hoc opere penitus eliminare studui.

In Dissertatione inscripta: *Exposition elementaire des principes des calculs superieurs*, theorematis hujus demonstrationem ad notiones claras & mere elementares revocare allaboravi. Cum vero, quam pag. 47—51. proposui, demonstrastratio (inprimis autem quod attinet ad expressiones in pag. 51. contentas), minus firma mihi videretur, neque mihi omnino satisfaceret (quod innui pag. 207.); aliam pag. 208—210. subjunxi, Academiæ Berolinensis de prædicta Dissertatione judicio posteriorem; quam & ill. Kæstnerum (*Anfangsgründe der Analysis des Unendlichen*, *2te Aufl.* §. 144 seq.) jam tradidisse postea comperi. Nititur ea hoc fundamento, quod functio quælibet quantitatis mutabilis serie potentiarum hujus quantitatis in constantes quasdam ductarum exprimi possit, qua reductione admissa, demonstratio hæc mihi plane satisfacit.

§. 31. *Primus Casus.* Sit x^n potestas quæcunque quantitatis mutabilis x, quæ fiat $x+b$; ita ut x^n fiat $(x+b)^n$. Jam vidimus (§. *e. Introd.*) esse

$$(x+b)^n = x^n + \frac{n}{1}x^{n-1}b + \frac{n}{1}\cdot\frac{n-1}{2}x^{n-2}b^2 + \frac{n}{1}\ldots\frac{n-2}{3}x^{n-3}b^3 + \frac{n}{1}\ldots\frac{n-3}{4}x^{n-4}b^4 + \ldots$$

Sit

$$
\begin{aligned}
\text{Sit}\qquad & P = x^n \\
\text{Erit (§. 28.)}\quad & \frac{dP}{dx} = nx^{n-1} \\
& \frac{ddP}{dx^2} = n.\overline{n-1}.x^{n-2} \\
& \frac{d^3P}{dx^3} = n.\overline{n-1}.\overline{n-2}.x^{n-3} \\
& \frac{d^4P}{dx^4} = n \;.\;.\;.\;.\; \overline{n-3}.x^{n-4} \\
& \frac{d^5P}{dx^5} = n \;.\;.\;.\;.\; \overline{n-4}.x^{n-5} \\
& \&c. \qquad\qquad \&c.
\end{aligned}
$$

$$
\text{Hinc } (x+b)^n = x^n + \frac{b}{1}\frac{dP}{dx} + \frac{b^2}{1.2}\frac{ddP}{dx^2} + \frac{b^3}{1\ldots3}\frac{d^3P}{dx^3} + \frac{b^4}{1\ldots4}\frac{d^4P}{dx^4} + \frac{b^5}{1\ldots5}\frac{d^5P}{dx^5} + \ldots\ldots
$$

Proinde exiſtente P potentia qualibet quantitatis mutabilis x, valor hujus poteſtatis, reſpondens mutationi propoſitæ b quantitatis mutabilis x, exprimitur per hanc mutationem b, & per exponentes differentiales ordinum ſucceſſivorum ipſius illius poteſtatis & quantitatis mutabilis x.

§. 32. *Secundus Caſus.* Functio propoſita P quantitatis mutabilis x aut ſit immediate ex potentiis quantitatis mutabilis x compoſita, aut ad eas reducta.

Scilicet ſit $P = Ax^a + Bx^b + Cx^c + Dx^d + Ex^e \ldots\ldots$

ſeu ſit $P = Q + R + S + T + V \ldots\ldots$

ubi $Q,\; R,\; S,\; T,\; V \ldots\ldots$ ſunt potentiæ quælibet quantitatis mutabilis x, coëfficientibus datis affectæ.

Sint etiam $P',\; Q',\; R',\; S',\; T',\; V' \ldots\ldots$ valores omnium prædictarum functionum, quos recipiunt, quando x, accepta mutatione b, fit $x+b$.

$$
\begin{aligned}
\text{Erit ideo}\quad P &= Q + R + S + T + V + \ldots\ldots \\
\frac{dP}{dx} &= \frac{dQ}{dx} + \frac{dR}{dx} + \frac{dS}{dx} + \frac{dT}{dx} + \frac{dV}{dx} + \ldots\ldots \\
\frac{ddP}{dx^2} &= \frac{ddQ}{dx^2} + \frac{ddR}{dx^2} + \frac{ddS}{dx^2} + \frac{ddT}{dx^2} + \frac{ddV}{dx^2} + \ldots\ldots
\end{aligned}
$$

$$\frac{d^3P}{dx^3} = \frac{d^3Q}{dx^3} + \frac{d^3R}{dx^3} + \frac{d^3S}{dx^3} + \frac{d^3T}{dx^3} + \frac{d^3V}{dx^3} + \ldots\ldots$$

$$\frac{d^4P}{dx^4} = \frac{d^4Q}{dx^4} + \frac{d^4R}{dx^4} + \frac{d^4S}{dx^4} + \frac{d^4T}{dx^4} + \frac{d^4V}{dx^4} + \ldots\ldots$$

$$\frac{d^5P}{dx^5} = \frac{d^5Q}{dx^5} + \frac{d^5R}{dx^5} + \frac{d^5S}{dx^5} + \frac{d^5T}{dx^5} + \frac{d^5V}{dx^5} + \ldots\ldots$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\frac{d^mP}{dx^m} = \frac{d^mQ}{dx^m} + \frac{d^mR}{dx^m} + \frac{d^mS}{dx^m} + \frac{d^mT}{dx^m} + \frac{d^mV}{dx^m} + \ldots\ldots$$

Item $P' = Q' + R' + S' + T' + V' + \ldots\ldots$

Atqui (*Cas. I.*)

$$Q' = Q + \frac{b}{1}\frac{dQ}{dx} + \frac{b^2}{1.2}\frac{ddQ}{dx^2} + \frac{b^3}{1.2.3}\frac{d^3Q}{dx^3} + \frac{b^4}{1\ldots4}\frac{d^4Q}{dx^4} + \frac{b^5}{1\ldots5}\frac{d^4Q}{dx^5} + \ldots\ldots$$

$$R' = R + \frac{b}{1}\frac{dR}{dx} + \frac{b^2}{1.2}\frac{ddR}{dx^2} + \frac{b^3}{1.2.3}\frac{d^3R}{dx^3} + \frac{b^4}{1\ldots4}\frac{d^4R}{dx^4} + \frac{b^5}{1\ldots5}\frac{d^5R}{dx^5} + \ldots\ldots$$

$$S' = S + \frac{b}{1}\frac{dS}{dx} + \frac{b^2}{1.2}\frac{ddS}{dx^2} + \frac{b^3}{1.2.3}\frac{d^3S}{dx^3} + \frac{b^4}{1\ldots4}\frac{d^4S}{dx^4} + \frac{b^5}{1\ldots5}\frac{d^5S}{dx^5} + \ldots\ldots$$

$$T' = T + \frac{b}{1}\frac{dT}{dx} + \frac{b^2}{1.2}\frac{ddT}{dx^2} + \frac{b^3}{1.2.3}\frac{d^3T}{dx^3} + \frac{b^4}{1\ldots4}\frac{d^4T}{dx^4} + \frac{b^5}{1\ldots5}\frac{d^5T}{dx^5} + \ldots\ldots$$

$$V' = V + \frac{b}{1}\frac{dV}{dx} + \frac{b^2}{1.2}\frac{ddV}{dx^2} + \frac{b^3}{1.2.3}\frac{d^3V}{dx^3} + \frac{b^4}{1\ldots4}\frac{d^4V}{dx^4} + \frac{b^5}{1\ldots5}\frac{d^5V}{dx^5} + \ldots\ldots$$

$$- \quad - \quad - \quad - \quad - \quad - \quad - \quad -$$

$$- \quad - \quad - \quad - \quad - \quad - \quad - \quad -$$

Hinc $P' = P + \frac{b}{1}\frac{dP}{dx} + \frac{b^2}{1.2}\frac{ddP}{dx^2} + \frac{b^3}{1.2.3}\frac{d^3P}{dx^3} + \frac{b^4}{1\ldots4}\frac{d^4P}{dx^4} + \frac{b^5}{1\ldots5}\frac{d^5P}{dx^5} + \ldots\ldots$

Proinde, admiſſa functionis cujuslibet quantitatis mutabilis in poteſtates quantitatis hujus decompoſitione, rigoroſe omnino theorema Taylorianum demonſtratur. Deſiderari tamen poterat, ipſum, reductione illa non ſuppoſita, ſufficienter ſtabiliri: quod in Diſſertatione inſcripta *Theorematis Tayloriani demonſtratio* (Tubingæ 1789.) præſtitit clar. PFLEIDERER, poſtulando tantum, mutationem ΔP functionis P, quæ eſt functio mutationis Δx variabilis x, talem eſſe, ut exprimi poſſit per potentias hujus mutationis, quarum exponentes ſequuntur

tur progressionem numerorum naturalium. Hujus methodum succincte expositam hic subjungam.

§. 33. Sit P functio quæcunque quantitatis mutabilis x.

Sint P^{I}, P^{II}, P^{III}, P^{IV}, P^{V} . . . . P^{N}, P^{N+1}, valores, quos functio illa induit, si loco x substituantur quantitates

$$x+\Delta x,\ x+2\Delta x,\ x+3\Delta x,\ x+4\Delta x,\ x+5\Delta x\ \ldots\ldots x+n\Delta x,\ x+(n+1)\Delta x.$$

$$\begin{array}{lllllll}
\text{Sit etiam } P^{I} & = P + & A\Delta x + & B\Delta x^2 & + & C\Delta x^3 + & D\Delta x^4 + \ldots\ldots \\
\text{Hinc } P^{II} & = P + & 2A\Delta x + & 2^2 B\Delta x^2 & + & 2^3 C\Delta x^3 + & 2^4 D\Delta x^4 + \ldots\ldots \\
P^{III} & = P + & 3A\Delta x + & 3^2 B\Delta x^2 & + & 3^3 C\Delta x^3 + & 3^4 D\Delta x^4 + \ldots\ldots \\
P^{IV} & = P + & 4A\Delta x + & 4^2 B\Delta x^2 & + & 4^3 C\Delta x^3 + & 4^4 D\Delta x^4 + \ldots\ldots \\
P^{V} & = P + & 5A\Delta x + & 5^2 B\Delta x^2 & + & 5^3 C\Delta x^3 + & 5^4 D\Delta x^4 + \ldots\ldots \\
\vdots & & & \vdots & & & \vdots \\
P^{N} & = P + & nA\Delta x + & n^2 B\Delta x^2 & + & n^3 C\Delta x^3 + & n^4 D\Delta x^4 + \ldots\ldots \\
P^{N+1} & = P + & (n+1)A\Delta x + & (n+1)^2 B\Delta x^2 & + & (n+1)^3 C\Delta x^3 + & (n+1)^4 D\Delta x^4 + \ldots\ldots
\end{array}$$

Unde

$$\begin{array}{llllll}
\Delta P\,(=P^{I}-P) & = A\Delta x + & B\Delta x^2 + & C\Delta x^3 + & D\Delta x^4 & + \ldots\ldots \\
\Delta P^{I}\,(=P^{II}-P^{I}) & = A\Delta x + & (2^2-1)B\Delta x^2 + & (2^3-1)C\Delta x^3 + & (2^4-1)D\Delta x^4 & + \ldots\ldots \\
\Delta P^{II}\,(=P^{III}-P^{II}) & = A\Delta x + & (3^2-2^2)B\Delta x^2 + & (3^3-2^3)C\Delta x^3 + & (3^4-2^4)D\Delta x^4 & + \ldots\ldots \\
\Delta P^{III}\,(=P^{IV}-P^{III}) & = A\Delta x + & (4^2-3^2)B\Delta x^2 + & (4^3-3^3)C\Delta x^3 + & (4^4-3^4)D\Delta x^4 & + \ldots\ldots \\
\Delta P^{V}\,(=P^{V}-P^{IV}) & = A\Delta x + & (5^2-4^2)B\Delta x^2 + & (5^3-4^3)C\Delta x^3 + & (5^4-4^4)D\Delta x^4 & + \ldots\ldots \\
\vdots & & \vdots & & \vdots & \\
\Delta P^{N-1}\,(=P^{N}-P^{N-1}) & = A\Delta x + & (n^2-(n-1)^2)B\Delta x^2 + & (n^3-(n-1)^3)C\Delta x^3 + & (n^4-(n-1)^4)D\Delta x^4 & + \ldots\ldots \\
\Delta P^{N}\,(=P^{N+1}-P^{N}) & = A\Delta x + & ((n+1)^2-n^2)B\Delta x^2 + & ((n+1)^3-n^3)C\Delta x^3 + & ((n+1)^4-n^4)D\Delta x^4 & + \ldots\ldots
\end{array}$$

In his æquationibus coëfficientes τϰ $A\Delta x$ differentiæ sunt primi ordinis numerorum naturalium; proinde (§. 9. *Introd.*) inter se æquales, nempe $= 1$; & differentiæ ulteriorum ordinum evanescunt. Hinc fit

$\Delta 2 P$

$$\begin{array}{lllll}
\Delta^2 P(=\Delta P^{I}-\Delta P) & =\Delta^{II}(2^2\ldots 1)B\Delta x^2 & +\ \Delta^{II}(2^3\ldots 1)C\Delta x^3 & +\ \Delta^{II}(2^4\ldots 1)D\Delta x^4 & +\ldots \\
\Delta^2 P^{I}(=\Delta P^{II}-\Delta P^{I}) & =\Delta^{II}(3^2\ldots 1^2)B\Delta x^2 & +\ \Delta^{II}(3^3\ldots 1^3)C\Delta x^3 & +\ \Delta^{II}(3^4\ldots 1^4)D\Delta x^4 & +\ldots \\
\Delta^2 P^{II}(=\Delta P^{III}-\Delta P^{II}) & =\Delta^{II}(4^2\ldots 2^2)B\Delta x^2 & +\ \Delta^{II}(4^3\ldots 2^3)C\Delta x^3 & +\ \Delta^{II}(4^4\ldots 2^4)D\Delta x^4 & +\ldots \\
\Delta^2 P^{III}(=\Delta P^{IV}-\Delta P^{III}) & =\Delta^{II}(5^2\ldots 3^2)B\Delta x^2 & +\ \Delta^{II}(5^3\ldots 3^3)C\Delta x^3 & +\ \Delta^{II}(5^4\ldots 3^4)D\Delta x^4 & +\ldots \\
\vdots & & \vdots & & \vdots
\end{array}$$

$$\Delta^2 P^{N-1}(=\Delta P^{N}-\Delta P^{N-1})=\Delta^{II}(n+1)^2..(n-1^2)B\Delta x^2+\Delta^{II}(n+1)^3..(n-1)^3C\Delta x^3+\Delta^{II}(n+1)^4..(n-1)^4D\Delta x^4+\ldots$$

$$\Delta^2 P^{N}(=\Delta P^{N+1}-\Delta P^{N})=\Delta^{II}(n+2)^2\ldots n^2)Bx^2+\Delta^{II}(n+2)^3\ldots n^3)C\Delta x^3+\Delta^{II}(n+2)^4\ldots n^4)D\Delta x^4+\ldots$$

In his æquationibus coëfficientes τȢ $B\Delta x^2$ ſunt differentiæ ſecundi ordinis ſecundarum poteſtatum numerorum naturalium; proinde (§. *q. Introd.*) conſtantes, nempe $= 1 . 2$; & differentiæ alteriorum ordinum evaneſcunt. Hinc fit

$$\begin{array}{lllll}
\Delta^3 P(=\Delta^2 P^{I}-\Delta^2 P) & =\ \Delta^{III}(3^3\ldots 1^3)C\Delta x^3 & +\Delta^{III}(3^4\ldots 1^4)D\Delta x^4 & +\Delta^{III}(3^5\ldots 1^5)E\Delta x^5 & +\ldots \\
\Delta^3 P^{I}(=\Delta^2 P^{II}-\Delta^2 P^{I}) & =\ \Delta^{III}(4^3\ldots 1^3)C\Delta x^3 & +\Delta^{III}(4^4\ldots 1^4)D\Delta x^4 & +\Delta^{III}(4^5\ldots 1^5)E\Delta x^5 & +\ldots \\
\Delta^3 P^{II}(=\Delta^2 P^{III}-\Delta^2 P^{II}) & =\ \Delta^{III}(5^3\ldots 2^3)C\Delta x^3 & +\Delta^{III}(5^4\ldots 2^4)D\Delta x^4 & +\Delta^{III}(5^5\ldots 2^5)E\Delta x^5 & +\ldots \\
\Delta^3 P^{III}(=\Delta^2 P^{IV}-\Delta^2 P^{III}) & =\ \Delta^{III}(6^3\ldots 3^3)C\Delta x^3 & +\Delta^{III}(6^4\ldots 3^4)D\Delta x^4 & +\Delta^{III}(6^5\ldots 3^5)E\Delta x^5 & +\ldots \\
\vdots & & \vdots & & \vdots
\end{array}$$

$$\Delta^3 P^{N-1}(=\Delta^2 P^{N}-\Delta^2 P^{N-1})=\Delta^{III}(n+2)^3..n-1)^3C\Delta x^3+\Delta^{III}(n+2)^4..n-1)^4D\Delta x^4+\Delta^{III}(n+2)^5..(n-1)^5E\Delta x^5+\ldots$$

$$\Delta^3 P^{N}(=\Delta^2 P^{N+1}-\Delta^2 P^{IV})=\Delta^{III}(n+3)^3\ldots n^3)C\Delta x^3+\Delta^{III}(n+3)^4\ldots n^4)D\Delta x^4+\Delta^{III}(n+3)^5\ldots n^5)E\Delta x^5+\ldots$$

In his æquationibus coëfficientes τȢ $C\Delta x^3$ ſunt differentiæ tertii ordinis tertiarum poteſtatum numerorum naturalium; proinde (§. *q. Introd.*) conſtantes, nempe $= 1 . 2 . 3$, & differentiæ ulteriorum ordinum illorum coëfficientium evaneſcunt. Hinc rurſus

$$\begin{array}{lllll}
\Delta^4 P(=\Delta^3 P^{I}-\Delta^3 P) & =\Delta^{IV}(4^4\ldots 1)D\Delta x^4 & +\Delta^{IV}(4^5\ldots 1)E\Delta x^5 & +\ \Delta^{IV}(4^6\ldots 1)F\Delta x^6 & +\ldots \\
\Delta^4 P^{I}(=\Delta^3 P^{II}-\Delta^3 P^{I}) & =\Delta^{IV}(5^4\ldots 1)D\Delta x^4 & +\Delta^{IV}(5^5\ldots 1)E\Delta x^5 & +\ \Delta^{IV}(5^6\ldots 1)F\Delta x^6 & +\ldots \\
\Delta^4 P^{II}(=\Delta^3 P^{III}-\Delta^3 P^{II}) & =\Delta^{IV}(6^4\ldots 2^4)D\Delta x^4 & +\Delta^{IV}(6^5\ldots 2^5)E\Delta x^5 & +\ \Delta^{IV}(6^6\ldots 2^6)F\Delta x^6 & +\ldots \\
\Delta^4 P^{III}(=\Delta^3 P^{IV}-\Delta^3 P^{III}) & =\Delta^{IV}(7^4\ldots 3^4)D\Delta x^4 & +\Delta^{IV}(7^5\ldots 3^5)E\Delta x^5 & +\ \Delta^{IV}(7^6\ldots 3^6)F\Delta x^6 & +\ldots \\
\vdots & & \vdots & & \vdots
\end{array}$$

$$\Delta^4 P^{N-1}(=\Delta^3 P^{N}-\Delta^3 P^{N-1})=\Delta^{IV}(n+3)^4..(n-1)^4D\Delta x^4+\Delta^{IV}(n+3)^5..(n-1)^5E\Delta x^5+\Delta^{IV}(n+3)^6..(n-1)^6F\Delta x^6+\ldots$$

$$\Delta^4 P^{N}(=\Delta^3 P^{N+1}-\Delta^3 P^{N})=\Delta^{IV}(n+4^4\ldots n)^4D\Delta x^4+\Delta^{IV}(n+4)^5\ldots n^5)E\Delta x^5+\Delta^{IV}(n+4)^6\ldots n^6)F\Delta x^6+\ldots$$

In his æquationibus coëfficientes τȢ $D\Delta x^4$ ſunt differentiæ quarti ordinis poteſtatum

tum quartarum numerorum naturalium; proinde (§. *q. Introd.*) constantes, nempe = 1. 2. 3. 4. & differentiae ulteriorum ordinum evanescunt.

Hinc sumtis differentiis quinti ordinis functionis P coëfficientes τῦ $E\Delta x^5$ fiunt differentiae quinti ordinis potestatum quintarum numerorum naturalium; proinde (§. *q. Introd.*) constantes, nempe = 1. 2. . . . 5. & differentiae ulteriorum ordinum evanescunt.

Unde procedendo ad differentias sexti, septimi, octavi . . . . ordinis functionis P; coëfficientes terminorum $F\Delta x^6$, $G\Delta x^7$, $H\Delta x^8$ . . . . qui respective sunt differentiae - - sexti, septimi, octavi . . . . ordinis potestatum - - - - sextarum, septimarum, octavarum . . . . numerorum naturalium fiunt respective quantitates constantes,

1. 2. . . 6, 1. 2. . . 7, 1. 2. . . 8 . . . . & differentiae ulteriorum ordinum evanescunt.

Sumtis itaque rationibus differentialibus successivis, quae ex praecedentibus aequationibus consequuntur, fit

$$\begin{array}{lll}
\frac{dP}{dx} = A & \text{Unde} & A = \frac{1}{1}\,\frac{dP}{dx} \\
\frac{ddP}{dx^2} = 1.2B & & B = \frac{1}{1.2}\,\frac{ddP}{dx^2} \\
\frac{d^3P}{dx^3} = 1.2.3C & & C = \frac{1}{1.2.3}\,\frac{d^3P}{dx^3} \\
\frac{d^4P}{dx^4} = 1.2..4D & & D = \frac{1}{1.2...4}\,\frac{d^4P}{dx^4} \\
\frac{d^5P}{dx^5} = 1.2...5E & & E = \frac{1}{1.2...5}\,\frac{d^5P}{dx^5} \\
\frac{d^6P}{dx^6} = 1.2....6F & & F = \frac{1}{1.2...6}\,\frac{d^6P}{dx^6} \\
\vdots \quad \vdots & & \vdots \quad \vdots \\
\frac{d^mP}{dx^m} = 1.2....mM & & M = \frac{1}{1.2...m}\,\frac{d^mP}{dx^m}.
\end{array}$$

Pro-

Proinde

$$P'(=P + A\Delta x + B\Delta x^2 + C\Delta x^3 + D\Delta x^4 + E\Delta x^5 + F\Delta x^6 + \ldots)$$
$$= P + \frac{\Delta x}{1}\frac{dP}{dx} + \frac{\Delta x^2}{1.\,2}\frac{ddP}{dx^2} + \frac{\Delta x^3}{1.2.3}\frac{d^3P}{dx^3} + \frac{\Delta x^4}{1.2..4}\frac{d^4P}{dx^4} + \frac{\Delta x^5}{1.2..5}\frac{d^5P}{dx^5} + \frac{\Delta x^6}{1.2..6}\frac{d^6P}{dx^6} + \ldots$$

§. 34. Uberrimi hujus propofitionis ufus in fequentibus patebunt. Heic unum exemplum mere algebraicum utilitatis ipfius afferre fufficiet: differentias omnium ordinum functionis cujuslibet quantitatis mutabilis per differentias poteftatum numerorum naturalium determinando.

Quoniam $P^N = P + \frac{n}{1}\Delta x\frac{dP}{dx} + \frac{n^2}{1.\,2}\Delta x^2\frac{ddP}{dx^2} + \frac{n^3}{1.\,2.\,3}\Delta x^3\frac{d^3P}{dx^3} + \frac{n^4}{1.2\ldots4}\Delta x^4\frac{d^4P}{dx^4} + \frac{n^5}{1.2\ldots5}\Delta x^5\frac{d^5P}{dx^5} + \text{- - - - - -} + \text{- - - - - -}$

et $P^{N+1} = P + \frac{n+1}{1}\Delta x\frac{dP}{dx} + \frac{(n+1)^2}{1.\,2}\Delta x^2\frac{ddP}{dx^2} + \frac{(n+1)^3}{1.\,2.\,3}\Delta x^3\frac{d^3P}{dx^3} + \frac{(n+1)^4}{1.2\ldots4}\Delta x^4\frac{d^4P}{dx^4} + \frac{(n+1)^5}{1.2\ldots5}\Delta x^5\frac{d^5P}{dx^5} + \text{- - - - - -} + \text{- - - - - -}$

Erit $\Delta P^N\,(=P^{N+1}-P^N) = \frac{n+1-n}{1}\Delta x\frac{dP}{dx} + \frac{(n+1)^2-n^2}{1.\,2}\Delta x^2\frac{ddP}{dx^2} + \frac{(n+1)^3-n^3}{1.\,2.\,3}\Delta x^3\frac{d^3P}{dx^3} + \frac{(n+1)^4-n^4}{1.\,2\ldots4}\Delta x^4\frac{d^4P}{dx^4} + \frac{(n+1)^5-n^5}{1.\,2\ldots5}\Delta x^5\frac{d^5P}{dx^5} + \text{- - - - - - -} + \text{- - - - - - -}$

$\Delta P^{N+1} = \frac{n+2-(n+1)}{1}\Delta x\frac{dP}{dx} + \frac{(n+2)^2-(n+1)^2}{1.\,2}\Delta x^2\frac{ddP}{dx^2} + \frac{(n+2)^3-(n+1)^3}{1.\,2.\,3}\Delta x^3\frac{d^3P}{dx^3} + \frac{(n+2)^4-(n+1)^4}{1.\,2\ldots4}\Delta x^4\frac{d^4P}{dx^4} + \frac{(n+2)^5-(n+1)^5}{1.\,2\ldots5}\Delta x^5\frac{d^5P}{dx^5} + \text{- - - - - - -} + \text{- - - - - - -}$

Hinc

Hinc $\Delta^2 P^{N} (= \Delta P^{N+1} - \Delta P^{N} =$

$$= \frac{n+2-2(n+1)+n}{1}\Delta x \frac{dP}{dx}$$
$$+ \frac{(n+2)^2-2(n+1)^2+n^2}{1.2}\Delta x^2 \frac{ddP}{dx^2}$$
$$+ \frac{(n+2)^3-2(n+1)^3+n^3}{1.2.3}\Delta x^3 \frac{d^3P}{dx^3}$$
$$+ \frac{(n+2)^4-2(n+1)^4+n^4}{1.2\ldots4}\Delta x^4 \frac{d^4P}{dx^4}$$
$$+ \frac{(n+2)^5-2(n+1)^5+n^5}{1.2\ldots5}\Delta x^5 \frac{d^5P}{dx^5}$$
$$+ \text{-} \quad \text{-} \quad \text{-} \quad \text{-} \quad \text{-}$$
$$+ \text{-} \quad \text{-} \quad \text{-} \quad \text{-} \quad \text{-}$$

$\Delta^2 P^{N+1} =$

$$= \frac{n+3-2(n+2)+n+1}{1}\Delta x \frac{dP}{dx}$$
$$+ \frac{(n+3)^2-2(n+2)^2+(n+1)^2}{1.2}\Delta x^2 \frac{ddP}{dx^2}$$
$$+ \frac{(n+3)^3-2(n+2)^3+(n+1)^3}{1.2.3}\Delta x^3 \frac{d^3P}{dx^3}$$
$$+ \frac{(n+3)^4-2(n+2)^4+(n+1)^4}{1.2\ldots4}\Delta x^4 \frac{d^4P}{dx^4}$$
$$+ \frac{(n+3)^5-2(n+2)^5+(n+1)^5}{1.2\ldots5}\Delta x^5 \frac{d^5P}{dx^5}$$
$$+ \text{-} \quad \text{-} \quad \text{-} \quad \text{-} \quad \text{-}$$
$$+ \text{-} \quad \text{-} \quad \text{-} \quad \text{-} \quad \text{-}$$

Hinc $\Delta^3 P^{N} (= \Delta^2 P^{N+1} - \Delta^2 P^{N}) =$

$$= \frac{(n+3)-3(n+2)+3(n+1)-n}{1}\Delta x \frac{dP}{dx}$$
$$+ \frac{(n+3)^2-3(n+2)^2+3(n+1)^2-n^2}{1.2}\Delta x^2 \frac{ddP}{dx^2}$$
$$+ \frac{(n+3)^3-3(n+2)^3+3(n+1)^3-n^3}{1.2.3}\Delta x^3 \frac{d^3P}{dx^3}$$
$$+ \frac{(n+3)^4-3(n+2)^4+3(n+1)^4-n^4}{1.2\ldots4}\Delta x^4 \frac{d^4P}{dx^4}$$
$$+ \frac{(n+3)^5-3(n+2)^5+3(n+1)^5-n^5}{1.2..5}\Delta x^5 \frac{d^5P}{dx^5}$$
$$+ \text{-} \quad \text{-} \quad \text{-} \quad \text{-} \quad \text{-} \quad \text{-}$$
$$+ \text{-} \quad \text{-} \quad \text{-} \quad \text{-} \quad \text{-} \quad \text{-}$$

$\Delta^3 P^{N+1} =$

$$= \frac{n+4-3(n+3)+3(n+2)-(n+1)}{1}\Delta x \frac{dP}{dx}$$
$$+ \frac{(n+4)^2-3(n+3)^2+3(n+2)^2-(n+1)^2}{1.2}\Delta x^2 \frac{ddP}{dx^2}$$
$$+ \frac{(n+4)^3-3(n+3)^3+3(n+2)^3-(n+1)^3}{1.1.2}\Delta x^3 \frac{d^3P}{dx^3}$$
$$+ \frac{(n+4)^4-3(n+3)^4+3(n+2)^4-(n+1)^4}{1.2\ldots4}\Delta x^4 \frac{d^4P}{dx^4}$$
$$+ \frac{(n+4)^5-3(n+3)^5+3(n+2)^5-(n+1)^5}{1.2\ldots4}\Delta x^5 \frac{d^5P}{dx^5}$$
$$+ \text{-} \quad \text{-} \quad \text{-} \quad \text{-} \quad \text{-} \quad \text{-} \quad \text{-}$$
$$+ \text{-} \quad \text{-} \quad \text{-} \quad \text{-} \quad \text{-} \quad \text{-} \quad \text{-}$$

Hinc

Hinc $\Delta^4 p\text{N} \left(=\Delta^3 p\text{N+1} - \Delta^3 p\text{N}\right)$

$$= \frac{n+4-4(n+3)+6(n+2)-4(n+1)+n}{1}\Delta x \frac{dP}{dx}$$
$$+ \frac{(n+4)^2-4(n+3)^2+6(n+2)^2-4(n+1)^2+n^2}{1.2}dx^2\frac{ddP}{dx^2}$$
$$+ \frac{(n+4)^3-4(n+3)^3+6(n+2)^3-4(n+1)^3+n^3}{1.2.3}dx^3\frac{d^3P}{dx^3}$$
$$+ \frac{(n+4)^4-4(n+3)^4+6(n+2)^4-4(n+1)^4+n^4}{1.2\ldots4}\Delta x^4\frac{d^4P}{dx^4}$$
$$+ \frac{(n+4)^5-4(n+3)^5+6(n+2)^5-4(n+1)^5+n^5}{1.2\ldots5}\Delta x^5\frac{d^5P}{dx^5}$$
$$+ - - - - - - - - -$$
$$+ - - - - - - - - -$$

Generatim

$$\Delta^m p\text{N} = \frac{n+m-\frac{m}{1}(n+m-1)+\frac{m}{1}.\frac{m-1}{2}(n+m-2)-\frac{m}{1}..\frac{m-2}{3}(n+m-3)\ldots\pm\frac{m}{1}(n+1)\mp n}{1}\Delta x\frac{dP}{dx}$$
$$+ \frac{(n+m)^2-\frac{m}{1}(n+m-1)^2+\frac{m}{1}.\frac{m-1}{2}(n+m-2)^2-\frac{m}{1}..\frac{m-2}{3}(n+m-3)^2\ldots\pm\frac{m}{1}(n+1)^2\mp n^2}{1.2}\Delta x^2\frac{ddP}{dx^2}$$
$$+ \frac{(n+m)^3-\frac{m}{1}(n+m-1)^3+\frac{m}{1}.\frac{m-1}{2}(n+m-2)^3-\frac{m}{1}..\frac{m-2}{3}(n+m-3)^3\ldots\pm\frac{m}{1}(n+1)^3\mp n^3}{1.2.3}\Delta x^3\frac{d^3P}{dx^3}$$
$$+ \frac{(n+m)^4-\frac{m}{1}(n+m-1)^4+\frac{m}{1}.\frac{m-1}{2}(n+m-2)^4-\frac{m}{1}..\frac{m-2}{3}(n+m-3)^4\ldots\pm\frac{m}{1}(n+1)^4\mp n^4}{1.2\ldots4}\Delta x^4\frac{d^4P}{dx^4}$$
$$+ \frac{(n+m)^5-\frac{m}{1}(n+m-1)^5+\frac{m}{1}.\frac{m-1}{2}(n+m-2)^5-\frac{m}{1}..\frac{m-2}{3}(n+m-3)^5\ldots\pm\frac{m}{1}(n+1)^5\mp n^5}{1.2\ldots5}\Delta x^5\frac{d^5P}{dx^5}$$
$$+ - - - - - - - - - -$$
$$+ - - - - - - - - - -$$

Nomi-

Nominatim igitur

$$\Delta^{m}P = \frac{m - \frac{m}{1}(m-1) + \frac{m}{1}\cdot\frac{m-1}{1}(m-2) - \frac{m}{1}\ldots\frac{m-2}{1}(m-3) \;\ldots\ldots \pm \frac{m}{1}1}{1}\Delta x\frac{dP}{dx}$$

$$+ \frac{m^2 - \frac{m}{1}(m-1)^2 + \frac{m}{1}\cdot\frac{m-1}{2}(m-2)^2 - \frac{m}{1}\ldots\frac{m-2}{3}(m-3)^2 \ldots\ldots \pm \frac{m}{1}1^2}{1.2}\Delta x^2\frac{ddP}{dx^2}$$

$$+ \frac{m^3 - \frac{m}{1}(m-1)^3 + \frac{m}{1}\cdot\frac{m-1}{2}(m-2)^3 - \frac{m}{1}\ldots\frac{m-2}{3}(m-3)^3 \ldots\ldots \pm \frac{m}{1}1^3}{1.2.3}\Delta x^3\frac{d^3P}{dx^3}$$

$$+ \frac{m^4 - \frac{m}{1}(m-1)^4 + \frac{m}{1}\cdot\frac{m-1}{2}(m-2)^4 - \frac{m}{1}\ldots\frac{m-2}{3}(m-3)^4 \ldots\ldots \pm \frac{m}{1}1^4}{1.2.3.4}\Delta x^4\frac{d^4P}{dx^4}$$

$$+ \frac{m^5 - \frac{m}{1}(m-1)^5 + \frac{m}{1}\cdot\frac{m-1}{2}(m-2)^5 - \frac{m}{1}\ldots\frac{m-2}{3}(m-3)^5 \ldots\ldots \pm \frac{m}{1}1^5}{1.2.\ldots 5}\Delta x^5\frac{d^5P}{dx^5}$$

$$+ \;-\; -\; -\; -\; -\; -$$

$$+ \;-\; -\; -\; -\; -\; -$$

$$= \frac{\Delta^{m}.m}{1}\Delta x\frac{dP}{dx}$$

$$+ \frac{\Delta^{m}.m^2}{1.2}\Delta x^2\frac{ddP}{dx^2}$$

$$+ \frac{\Delta^{m}.m^3}{1.2.3}\Delta x^3\frac{d^3P}{dx^3}$$

$$+ \frac{\Delta^{m}.m^4}{1.2.3.4}\Delta x^4\frac{d^4P}{dx^4}$$

$$+ \frac{\Delta^{m}.m^5}{1.2\ldots 5}\Delta x^5\frac{d^5P}{dx^5}$$

$$+ \;-\; -\; -\; -$$

$$+ \;-\; -\; -\; -$$

Atqui (§. q. *Introd.*) $\Delta^m . m^m = 1.2.3 \ldots m$, & differentiæ ulteriorum ordinum evaneſcunt.

$$\text{Ergo}\quad \Delta^m P = \frac{\Delta^m . m^m}{1.2\ldots m}\Delta x^m \frac{d^m P}{dx^m}$$
$$+ \frac{\Delta^m . m^{m+1}}{1.2\ldots(m+1)}\Delta x^{m+1}\frac{d^{m+1}P}{dx^{m+1}}$$
$$+ \frac{\Delta^m . m^{m+2}}{1.2\ldots(m+2)}\Delta x^{m+2}\frac{d^{m+2}P}{dx^{m+2}}$$
$$+ \frac{\Delta^m . m^{m+3}}{1.2\ldots(m+3)}\Delta x^{m+3}\frac{d^{m+3}P}{dx^{m+3}}$$
$$+ \frac{\Delta^m . m^{m+4}}{1.2\ldots(m+4)}\Delta x^{m+4}\frac{d^{m+4}P}{dx^{m+4}}$$
$$+ \ldots$$
$$+ \ldots$$

Itaque differentiæ omnium ordinum functionis cujuslibet alicujus quantitatis mutabilis reducuntur ad differentias poteſtatum numerorum naturalium.

Caput Quartum

De ſerie Bernoulliana.

§. 35.

Sit $\frac{dZ}{dx} = y$ ratio differentialis data inter quantitates mutabiles x, y, Z. Relatio earum integralis exprimi poteſt per exponentes differentiales omnium ordinum variabilium x & y, quod ſic inveſtigatur.

$\frac{dZ}{dx}$

$$\frac{dZ}{dx} = y$$
$$= y + x\frac{dy}{dx}$$
$$-\left(x\frac{dy}{dx} + \tfrac{1}{2}xx\frac{ddy}{dx^2}\right)$$
$$+\left(\tfrac{1}{2}xx\frac{ddy}{dx^2} + \frac{1}{1.2.3}x^3\frac{d^3y}{dx^3}\right)$$
$$-\left(\frac{1}{1.2.3}x^3\frac{d^3y}{dx^3} + \frac{1}{1.2\ldots4}x^4\frac{d^4y}{dx^4}\right)$$
$$+\left(\frac{1}{1.2\ldots4}x^4\frac{d^4y}{dx^4} + \frac{1}{1.2\ldots5}x^5\frac{d^5y}{dx^5}\right)$$
$$- \quad - \quad - \quad -$$
$$- \quad - \quad - \quad -$$

Itaque

$$Z = C + xy - \frac{x^2}{1.2}\frac{dy}{dx} + \frac{x^3}{1.2.3}\frac{ddy}{dx^2} - \frac{x^4}{1.2\ldots4}\frac{d^3y}{dx^3} + \frac{x^5}{1.2\ldots5}\frac{d^4y}{dx^4} - \frac{x^6}{1.2\ldots6}\frac{d^5y}{dx^5} + \ldots.$$

Series hæc notatu admodum digna, & cujus fœcundæ applicationes in ſequentibus occurrent, *Bernoulliana* dicitur; quod JOH. BERNOULLI primus eam ſub forma hac ſimpliciſſima tradidit. (Vide *ipſius Opera*, T. I. p. 125 ſeqq.)

Exempla. Sit $y = x^m$.

$$\frac{dy}{dx} = mx^{m-1}$$
$$\frac{ddy}{dx^2} = m.m-1x^{m-2}$$
$$\frac{d^3y}{dx^3} = m..m-2x^{m-3}$$
$$\frac{d^4y}{dx^4} = m\ldots m-3x^{m-4}$$
$$\frac{d^5y}{dx^5} = m\ldots.m-4x^{m-5}$$
$$\frac{d^6y}{dx^6} = m\ldots..m-5x^{m-6}$$
$$- \quad - \quad - \quad -$$
$$- \quad - \quad - \quad -$$

 Hinc

Hinc posito $\frac{dZ}{dx} = x^m$

$$Z = C + x^{m+1}\left(1 - \frac{m}{1.2} + \frac{m.m-1}{1.2.3} - \frac{m..m-2}{1.2..4} + \frac{m...m-3}{1.2...5} - \frac{m...m-4}{1.2...6} + \frac{m...m-5}{1.2...7} - ...\right)$$

Atqui posito $\frac{dZ}{dx} = x^m$ est $Z = C + \frac{1}{m+1} x^{m+1}$ (§. 27.)

Ergo $\frac{1}{m+1} = 1 - \frac{m}{1.2} + \frac{m.m-1}{1.2.3} - \frac{m..m-2}{1.2..4} + \frac{m...m-3}{1.2...5} - \frac{m...m-4}{1.2...6} + \frac{m...m-5}{1.2...7} - .....$

Observatio prima. Casu hoc facile est expressionum præcedentium identitatem demonstrare.

Etenim $1 - \frac{m}{1.2} + \frac{m.m-1}{1.2.3} - \frac{m..m-2}{1...4} + \frac{m...m-3}{1...5} - \frac{m...m-4}{1...6} + ...$

$$= \frac{1}{m+1}\left(\frac{m+1}{1} - \frac{m+1.m}{1.2} + \frac{m+1}{1}..\frac{m-1}{3} - \frac{m+1}{1}..\frac{m-2}{4} + \frac{m+1}{1}..\frac{m-3}{5} - \frac{m+1}{1}..\frac{m-4}{6} + ..\right)$$

$$= \frac{1}{m+1}\left(1 - (1-1)^{m+1}\right) = \frac{1}{m+1}.$$

Observatio secunda. Series Bernoulliana varias induere potest formas, prouti quantitas x augetur aut minuitur aliqua quantitate constanti a: fit enim eodem omnino modo

$$Z = C + (x+a)y - \frac{(x+a)^2}{1.2}\frac{dy}{dx} + \frac{(x+a)^3}{1.2.3}\frac{ddy}{dx^2} - \frac{(x+a)^4}{1...4}\frac{d^3y}{dx^3} + \frac{(x+a)^5}{1...5}\frac{d^4y}{dx^4} - ....$$

Observatio tertia. Cum series Bernoulliana relationem integralem sub forma omnium simplicissima raro exhibeat (quod ipso exemplo præcedente $\frac{dZ}{dx} = x^m$ fit satis manifestum); ad eam non est recurrendum, nisi prius tentatis aliis integrationem propositam perficiendi methodis. Seriei hujus collatio cum expressionibus, quæ aliis methodis obtinentur, ad varia ducit theoremata, quibus evolvendis non immorabor.

§. 36. In æquatione differentiali $\frac{dZ}{dx} = y$ exponens propositus y potest esse quantitas quælibet simplex aut composita, dummodo exponentes differentiales omnium ordinum duarum quantitatum variabilium y & x exhiberi possint. Hoc monere eo magis e re esse censui, quod mathematici etiam in calculis superioribus versatissimi non videntur illud satis agnovisse; uti ex dubiis patet,

patet, quæ ſagaciſſimus HOLLAND amico ſuo clariſs. LAMBERT (*Briefwechſel* I. *Band* pag. 105.) propoſuit his verbis: „*Ich habe beſonders durch die Bernoulliſche Reihe finden wollen, ob ſich nicht könnte* $\int . y^m dx$ *aus* $\int y\, dx$, *oder welches ich beſonders wünſchte*, $\int . xy dx$ *aus* $\int . y dx$ *finden laſſen. Mein Bemühen war fruchtlos, wie es noch alle meine Unterſuchungen in dieſem Punct geweſen ſind. Wenn dieſe Vergleichung allgemein möglich iſt (woran ich noch zweifle); ſo ſehe ich die Entdeckung derſelben für eine der wichtigſten an, die man zur Erweiterung der Integral-Rechnung machen kann.*„ (*a*)

Sit ideo conformiter voto celeb. HOLLAND $\frac{dZ}{dx} = xy$; & ſit $P = xy$.

$$\frac{dP}{dx} = y + x\frac{dy}{dx}$$

$$\frac{ddP}{dx^2} = 2\frac{dy}{dx} + x\frac{ddy}{dx^2}$$

$$\frac{d^3P}{dx^3} = 3\frac{ddy}{dx^2} + x\frac{d^3y}{dx^3}$$

$$\frac{d^4P}{dx^4} = 4\frac{d^3y}{dx^3} + x\frac{d^4y}{dx^4}$$

$$\frac{d^5P}{dx^5} = 5\frac{d^4y}{dx^4} + x\frac{d^5y}{dx^5}$$

$$\frac{d^6P}{dx^6} = 6\frac{d^5y}{dx^5} + x\frac{d^6y}{dx^6}$$

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 Hinc

(*a*) Hoc eſt: „*Quærebam inprimis, anne ope ſeriei Bernoullianæ ex* $\int . y dx$ *liceret* $\int y^m dx$ *aut (quod præcipue optabam)* $\int . xy dx$ *eruere. Conatus mei fuerunt irriti, uti hactenus omnes meæ hanc in rem inveſtigationes. Quod ſi hœc comparatio generatim poſſibilis ſit (de quo adhuc dubito); inventio ejus mihi una ex potiſſimis eſſe videtur, quibus calculus integralis perfici poſſit.*

Hinc
$$Z = C + xxy$$
$$- \frac{x^2}{1.2}\left(y + x\frac{dy}{dx}\right]$$
$$+ \frac{x^3}{1.2.3}\left(2\frac{dy}{dx} + x\frac{ddy}{dx^2}\right]$$
$$- \frac{x^4}{1.2..4}\left(3\frac{ddy}{dx^2} + x\frac{d^3y}{dx^3}\right]$$
$$+ \frac{x^5}{1.2..5}\left(4\frac{d^3y}{dx^3} + x\frac{d^4y}{dx^4}\right]$$
$$- \frac{x^6}{1.2..6}\left[\frac{d^4y}{dx^4} + x\frac{d^5y}{dx^5}\right]$$
$$+ \frac{x^7}{1.2..7}\left[\frac{d^5y}{dx^5} + x\frac{d^6y}{dx^6}\right]$$
$$- \quad - \quad - \quad - \quad - \quad -$$
$$+ \quad - \quad - \quad - \quad - \quad -$$
$$= C + \frac{1}{1.2}xxy - \frac{x^3}{1.2.3}\frac{dy}{dx} + \frac{x^4}{1.2..4}\frac{ddy}{dx^2} - \frac{x^5}{1.2..5}\frac{d^3y}{dx^3} + \frac{x^6}{1.2..6}\frac{d^4y}{dx^4} - \frac{x^7}{1.2..7}\frac{d^5y}{dx^5} + \ldots\ldots$$

Sit $P = y^{m}$

$$\frac{dP}{dx} = my^{m-1}\frac{dy}{dx}$$

$$\frac{ddP}{dx^2} = m.m-1y^{m-2}\left[\frac{dy}{dx}\right]^2 + my^{m-1}\frac{ddy}{dx^2}$$

$$\frac{d^3P}{dx^3} = m\ldots m-2y^{m-3}\left[\frac{dy}{dx}\right]^3 + 3m.m-1y^{m-2}\frac{dy}{dx}\cdot\frac{ddy}{dx^2} + my^{m-1}\frac{d^3y}{ax^3}$$

$$\frac{d^4P}{dx^4} = m\ldots m-3y^{m-4}\left[\frac{dy}{dx}\right]^4 + 6m\ldots m-2y^{m-3}\left[\frac{dy}{dx}\right]^2\frac{ddy}{dx^2} + \begin{matrix} 3m.m-1y^{m-2}\left[\dfrac{ddy}{dx^2}\right]^2 \\ 4m.m-1y^{m-2}\dfrac{dy}{dx}\cdot\dfrac{d^3y}{dx^3}\end{matrix} + my^{m-1}\frac{d^4y}{dx^4}$$

$$\frac{d^5P}{dx^5} = m\ldots m-4y^{m-5}\left[\frac{dy}{dx}\right]^5 + 10m\ldots m-3y^{m-4}\left[\frac{dy}{dx}\right]^3\frac{ddy}{dx^2} + \begin{matrix} 15m\ldots m-2y^{m-3}\dfrac{dy}{dx}\left[\dfrac{ddy}{dx^2}\right]^2 \\ 10m\ldots m-2y^{m-3}\left[\dfrac{dy}{dx}\right]^2\dfrac{d^3y}{dx^3}\end{matrix} +$$

$$+ \begin{matrix} 10m.m-1y^{m-2}\dfrac{ddy}{dx^2}\cdot\dfrac{d^3y}{dx^3} \\ 5m.m-1y^{m-2}\dfrac{dy}{dx}\,\dfrac{d^4y}{dx^4}\end{matrix} + m.y^{m-1}\frac{d^5y}{dx^5}$$

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Quibus

Quibus ſubſtitutionibus factis obtinetur valor exponentis integralis per exponentes differentiales omnium ordinum quantitatum y & x.

Sit iterum $\frac{dZ}{dx} = vy$.

Sit $P = vy$

$$\frac{dP}{dx} = v\frac{dy}{dx} + y\frac{dv}{dx}$$

$$\frac{ddP}{dx^2} = v\frac{ddy}{dx^2} + 2\frac{dy}{dx}\frac{dv}{dx} + y\frac{ddv}{dx^2}$$

$$\frac{d^3P}{dx^3} = v\frac{d^3y}{dx^3} + 3\frac{dv}{dx}\frac{ddy}{dx^2} + 3\frac{ddv}{dx^2}\frac{dy}{dx} + y\frac{d^3v}{dx^3}$$

$$\frac{d^4P}{dx^4} = v\frac{d^4y}{dx^4} + 4\frac{dv}{dx}\frac{d^3y}{dx^3} + 6\frac{ddv}{dx^2}\frac{ddy}{dx^2} + 4\frac{d^3v}{dx^3}\frac{dy}{dx} + y\frac{d^4v}{dx^4}$$

$$\frac{d^5P}{dx^5} = v\frac{d^5y}{dx^5} + 5\frac{dv}{dx}\frac{d^4y}{dx^4} + 10\frac{ddv}{dx^2}\frac{d^3y}{dx^3} + 10\frac{d^3v}{dx^3}\frac{ddy}{dx^2} + 5\frac{d^4v}{dx^4}\frac{dy}{dx} + y\frac{d^5v}{dx^5}$$

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Hinc $Z = C + vxy$

$$- \frac{x^2}{1.2}\left[v\frac{dy}{dx} + y\frac{dv}{dx}\right]$$

$$+ \frac{x^3}{1.2.3}\left[v\frac{ddy}{dx^2} + 2\frac{dv}{dx}\frac{dy}{dx} + y\frac{ddv}{dx^2}\right]$$

$$- \frac{x^4}{1.2\ldots4}\left[v\frac{d^3y}{dx^3} + 3\frac{dv}{dx}\frac{ddy}{dx^2} + 3\frac{dy}{dx}\frac{ddv}{dx^2} + y\frac{d^3v}{dx^3}\right]$$

$$+ \frac{x^5}{1.2\ldots5}\left[v\frac{d^4y}{dx^4} + 4\frac{dv}{dx}\frac{d^3y}{dx^3} + 6\frac{ddv}{dx^2}\frac{ddy}{dx^2} + 4\frac{d^3v}{dx^3}\frac{dy}{dx} + y\frac{d^4v}{dx^4}\right]$$

$$- \frac{x^6}{1.2\ldots6}\left[v\frac{d^5y}{dx^5} + 5\frac{dv}{dx}\frac{d^4y}{dx^4} + 10\frac{ddv}{dx^2}\frac{d^3y}{dx^3} + 10\frac{d^3v}{dx^3}\frac{ddy}{dx^2} + 5\frac{d^4v}{dx^4}\frac{dy}{dx} + y\frac{d^5v}{dx^5}\right]$$

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§. 37.

§. 37. Quemadmodum relatio differentialis primi ordinis $\frac{dZ}{dx} = y$ ducit ad relationem integralem, qua Z per x & y determinatur: ita etiam relatio differentialis incomplexa $\frac{d^m Z}{dx^m} = y$ ducit ad relationem integralem inter $\frac{d^{m-1}Z}{dx^{m-1}}$ & easdem variabiles; ac deinde ad determinationem aliorum exponentium differentialium inferiorum ipsarum Z & x.

1°. Sit $\frac{ddZ}{dx^2} = y$.

Erit (§. 35.)

$$\frac{dZ}{dx} = C + xy - \frac{x^2}{1.2}\frac{dy}{dx} + \frac{x^3}{1.2.3}\frac{ddy}{dx^2} - \frac{x^4}{1.2...4}\frac{d^3y}{dx^3} + \frac{x^5}{1.2...5}\frac{d^4y}{dx^4} - \frac{x^6}{1.2...6}\frac{d^5y}{dx^5} + ...$$

Sit $\frac{dZ}{dx} = C + P$.

Itaque
$$\frac{dP}{dx} = y$$
$$\frac{ddP}{dx^2} = \frac{dy}{dx}$$
$$\frac{d^3P}{dx^3} = \frac{ddy}{dx^2}$$
$$\frac{d^4P}{dx^4} = \frac{d^3y}{dx^3}$$
$$\frac{d^5P}{dx^5} = \frac{d^4y}{dx^4}$$

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Hinc
$$Z = C' + Cx + xxy - \frac{x^3}{1.2}\frac{dy}{dx} + \frac{x^4}{1.2.3}\frac{ddy}{dx^2} - \frac{x^5}{1.2...4}\frac{d^3y}{dx^3} + \frac{x^6}{1.2...5}\frac{d^4y}{dx^4} - ...$$
$$- \frac{x^2}{1.2}y + \frac{x^3}{1.2.3}\frac{dy}{dx} - \frac{x^4}{1.2...4}\frac{ddy}{dx^2} + \frac{x^5}{1.2...5}\frac{d^3y}{dx^3} - \frac{x^6}{1.2...6}\frac{d^4y}{dx^4} + ...$$
$$= C' + Cx + \frac{x^2}{1.2}y - \frac{2x^3}{1.2.3}\frac{dy}{dx} + \frac{3x^4}{1.2...4}\frac{ddy}{dx^2} - \frac{4x^5}{1.2...5}\frac{d^3y}{dx^3} + \frac{5x^6}{1.2...6}\frac{d^4y}{dx^4} - ...$$

Sit

Sit $\dfrac{d^3Z}{dx^3} = y$.

Hinc $\dfrac{dZ}{dx} = C' + Cx + \dfrac{x^2}{1.2}y - \dfrac{2x^3}{1.2.3}\dfrac{dy}{dx} + \dfrac{3x^4}{1.2...4}\dfrac{ddy}{dx^2} - \dfrac{4x^5}{1.2...5}\dfrac{d^3y}{dx^3} + \dfrac{5x^6}{1.2...6}\dfrac{d^4y}{dx^4} - \dots$

$= C' + Cx + P.$

$$\frac{dP}{dx} = xy - \frac{x^2}{1.2}\frac{dy}{dx} + \frac{x^3}{1.2.3}\frac{ddy}{dx^2} - \frac{x^4}{1.2...4}\frac{d^3y}{dx^3} + \frac{x^5}{1.2...5}\frac{d^4y}{dx^4} - \frac{x^6}{1.2...6}\frac{d^5y}{dx^5} + \dots$$

$$\frac{ddP}{dx^2} = y$$

$$\frac{d^3P}{dx^3} = \frac{dy}{dx}$$

$$\frac{d^4P}{dx^4} = \frac{ddy}{dx^2}$$

$$\frac{d^5P}{dx^5} = \frac{d^3y}{dx^3}$$

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Proinde $Z = C'' + C'x + \tfrac{1}{2}Cx^2$

$$+ \frac{x^3}{1.2.3}y - \frac{3x^4}{1.2..4}\frac{dy}{dx} + \frac{6x^5}{1.2...5}\frac{ddy}{dx^2} - \frac{10x^6}{1.2...6}\frac{d^3y}{dx^3} + \frac{15x^7}{1.2...7}\frac{d^4y}{dx^4} - \frac{21x^8}{1.2...8}\frac{d^5y}{dx^5} + \dots$$

Hinc porro, si $\dfrac{d^4Z}{dx^4} = y$

erit $Z = C''' + C''x + \tfrac{1}{2}C'x^2 + \dfrac{1}{1.2.3}Cx^3$

$$+ \frac{x^4}{1.2...4}y - \frac{4x^5}{1.2...5}\frac{dy}{dx} + \frac{10x^6}{1.2...6}\frac{ddy}{dx^2} - \frac{20x^7}{1.2...7}\frac{d^3y}{dx^3} + \frac{35x^8}{1.2...8}\frac{d^4y}{dx^4} - \frac{56x^9}{1.2...9}\frac{d^5y}{dx^5} + \dots$$

Generatim sit $\dfrac{d^mZ}{dx^m} = y$.

Fiet $Z = C^{M-I} + C^{M-II}x + \dfrac{1}{1.2}C^{M-III}x^2 + \dfrac{1}{1.2.3}C^{M-IV}x^3 + \dots \dfrac{1}{1.2...m-2}C'x^{m-2} + \dfrac{1}{1.2...m-1}Cx^{m-1}$

$$+ \frac{x^m}{1.2...m}y - \frac{m.x^{m+1}}{1.2...m+1}\frac{dy}{dx} + \frac{\frac{m}{1}.\frac{m+1}{2}x^{m+2}}{1.2...m+2}\frac{ddy}{dx^2} - \frac{\frac{m}{1}\dots\frac{m+2}{3}x^{m+3}}{1.2...m+3}\frac{d^3y}{dx^3} +$$

$$+ \frac{\frac{m}{1}\dots\frac{m+3}{4}x^{m+4}}{1.2...m+4}\frac{d^4y}{dx^4} - \frac{\frac{m}{1}\dots\frac{m+4}{5}x^{m+5}}{1.2...m+5}\frac{d^5y}{dx^5} + \dots$$

Obser-

Observatio. Ex his formulis plurima curiosa saltem theoremata deduci possent; conferendo eas cum formulis, quae cum ipsis identicae esse debent, & quae aut integratione immediata, aut methodis a priore diversis obtinentur. Et hic repetendum est (a fortiori) monitum, ut ad series Bernoullianas non prius recurratur, quam alia subsidia fuerint exhausta.

Exemplum.

Sit $\frac{d^m Z}{dx^m} = x^n.$

$$\frac{d^{m-1}Z}{dx^{m-1}} = C + \frac{1}{n+1} x^{n+1}$$

$$\frac{d^{m-2}Z}{dx^{m-2}} = C' + Cx + \frac{1}{n+1\,.\,n+2} x^{n+2}$$

$$\frac{d^{m-3}Z}{dx^{m-3}} = C'' + C'x + \frac{1}{1.2} Cx^2 + \frac{1}{n+1\ldots n+3} x^{n+3}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$Z = C^{M-I} + C^{M-II}x + \frac{1}{1.2} C^{M-III}x^2 + \ldots + \frac{1}{1.2..m-1} Cx^{m-1} + \frac{1}{n+1\ldots n+m} x^{n+m}.$$

Caput Quintum.

De Tangentibus.

§. 38.

Definitio. Sit arcus curvae, isque totus aut concavus aut convexus versus diametrum, ad quam refertur: per punctum aliquod hujus arcus ducatur linea recta, quae ipsi ita occurrat, ut ad utramque puncti hujus partem extra curvam jaceat. Haec recta dicitur *tangens* curvae, & punctum, ubi curvae occurrit, dicitur *punctum contactus.*

Per punctum contactus ducatur recta diametro ordinatim applicata. Si tangens occurrat diametro, ad quam curva refertur: pars diametri ordinatam inter a puncto contactus ductam & punctum, ubi tangens occurrit diametro, comprehensa, dicitur *subtangens.*

Sci-

Scilicet curva AMM' referatur ad diametrum AP per rectas ipsi ordinatim applicatas MP, $M'P'$. Per punctum M arcus MM', qui ex utraque puncti M parte ad diametrum sit concavus aut convexus, ducatur recta MT, quæ curvæ in puncto M ita occurrat, ut ad utramque puncti M partem extra curvam cadat: hæc recta MT dicitur tangens curvæ, & punctum M dicitur punctum contactus. Sit MP diametro ordinatim applicata, & tangens occurrat diametro in T: linea PT dicitur subtangens. Fig. 11.

Quæcunque dicentur de arcubus versus diametrum concavis, facile (mutatis mutandis) ad arcus versus diametrum convexos applicantur: quare (brevitatis causa) de prioribus tantum dicere sufficiat.

Observatio. Ex puncto M ad aliud quodpiam punctum M' arcus AMM' ducatur chorda MM', quæ producta diametro occurrat in S. Per punctum M' ducantur recta $M'P'$ diametro ordinatim applicata; & recta $M'Q$ diametro parallela, quæ ipsi MP occurrat in Q. Linea PS dicitur *subsecans.*

Fit semper $MP : PS = MQ : M'Q$; hoc est: ratio mutationum simultanearum ordinatæ & abscissæ diametri æquatur rationi ordinatæ ad subsecantem. Prior itaque ratio major est aut minor ratione ordinatæ ad subtangentem, prouti subsecans minor est aut major subtangente: nempe in casu figuræ, ubi arcus MM' concavus est versus diametrum, prouti punctum M' situm est inter puncta M & S, aut secus. Quoniam vero puncto M' propius propiusque ad punctum M accedente (ita ut mutationes simultaneæ ordinatæ & abscissæ diametri continue fiant minores), punctum S continue accedit ad punctum T; ita ut subsecans contiuue minus differat a subtangente: proclive est suspicari, rationem ordinatæ ad subtangentem posse ex ratione differentiali (si qua fuerit) ordinatæ ad subsecantem deduci; quod theoremate sequente stabilitur.

§. 39. ***Theorema.*** Ex puncto aliquo cujuslibet arcus, qui totus sit versus diametrum, ad quam refertur, concavus (aut convexus), ducatur recta, quæ arcum tangat & diametro occurrat. Dico: rationem mutationum simultanearum rectæ diametro ab eodem puncto ordinatim applicatæ & abscissæ diametri fieri posse minorem quacunque ratione data, quæ major sit ratione data ordi-

natæ ad ſubtangentem; majorem vero quacunque ratione data, quæ minor ſit prædicta ratione.

Sit MP diametro AP ordinatim applicata; & tangens MT, quæ diametro occurrat in T. Detur ratio $MP : PS$ $\begin{matrix}\text{major}\\ \text{minor}\end{matrix}$ ratione $MP : PT$; ideoque ſit PS $\begin{matrix}\text{minor}\\ \text{major}\end{matrix}$ ipſa PT. Dico: rationem mutationum ſimultanearum ordinatæ MP & abſciſſæ diametri AP poſſe fieri $\begin{matrix}\text{minorem}\\ \text{majorem}\end{matrix}$ ratione propoſita $MP : PS$.

Arcus propoſiti ſumatur punctum quodcunque m, priore quidem caſu ſitum ad easdem ordinatæ MP partes, ad quas eſt punctum T; poſteriore ad partes oppoſitas. Jungatur recta Mm; quæ, cum arcus (hyp.) ſit verſus diametrum concavus, tota intra eum cadet. Per punctum m ducantur rectæ mp ordinatim diametro applicata; atque mq diametro parallela, quæ ordinatæ MP ipſi vel productæ in puncto q occurrat.

Fig. 11. 1°. Erit ratio $Mq : mq$ vel $\left\{\begin{matrix}\text{minor priore caſu}\\ \text{major poſteriori caſu}\end{matrix}\right\}$ quam data ratio $MP : PS$; & tunc effectum erit, quod proponitur.

Vel ratio $Mq : mq$ erit rationi datæ $MP : PS$ æqualis, aut ipſa $\left\{\begin{matrix}\text{major priore caſu}\\ \text{minor caſu poſteriore}\end{matrix}\right\}$.

Si prius: ob $Mq : mq = MP : PS$ recta producta Mm incidet in punctum S.

Si poſterius: recta Mm producta diametrum in puncto s ita ſecabit, ut

2°. & 3°. punctum S cadat inter puncta s & T, ob $MP : Ps (= Mq : mq) \gtrless MP : PS$; proinde $Ps \lessgtr PS$. Tum igitur recta SM, utpote intra angulum TMs, eique ad verticem oppoſitum tMm ſita, arcum Mm verſus diametrum (hyp.) concavum in puncto aliquo M' citra vel ultra M ſito ita ſecabit, ut ſegmentum ipſius MM' intra arcum jaceat. Per M' agantur recta $M'P'$ diametro ordinatim applicata, atque $M'Q$ diametro parallela, quæ ordinatæ MP ipſi vel productæ in puncto Q occurrat. Porro ſumatur punctum quodcunque arcus Mm quidem, ſi producta Mm in punctum S incidit; arcus autem MM', ſi ſecus: & per hoc punctum N agatur diametro parallela NR, quæ ordinatæ MP in puncto R, chordæ vero Mm vel MM' in puncto N' occurrat.

Erit ſemper $MR : N'R = MP : PS$.

Atqui

Atqui $NR > N'R$, si puncta m, M', N ad easdem ordinatæ MP partes jacent, ad quas est punctum T; tum ergo $MR : NR < MR : N'R < MP : PS$.

Et $NR < N'R$, si puncta m, M', N, atque punctum T jacent ad partes ordinatæ MP oppositas; tum itaque $MR : NR > MR : N'R > MP : PS$.

Sed ratio $MR : NR$ est ratio mutationum simultanearum ordinatæ MP & abscissæ diametri AP. Ergo ratio harum mutationum simultanearum potest reddi $\begin{matrix}\text{minor}\\\text{major}\end{matrix}$ quacunque ratione proposita, quæ $\begin{matrix}\text{major}\\\text{minor}\end{matrix}$ fuerit ratione ordinatæ ad subtangentem.

§. 40. *Corollaria.* 1°. Ratio ordinatæ ad subtangentem est limes rationis mutationum simultanearum rectæ a puncto contactus ad diametrum ordinatim applicatæ, & lineæ ex diametro abscissæ; seu prior ratio est ratio differentialis harum linearum.

Scilicet sit $MP = y$; $AP = x$; $\frac{MP}{PT} = \frac{\mathrm{d}y}{\mathrm{d}x}$; seu $PT = y\frac{\mathrm{d}x}{\mathrm{d}y}$.

Ex constructione liquet: subtangentem esse limitem (si quis detur) subsecantis; quod etiam evinci potest, ut sequitur.

Sit $MP = y$; $M'P' = y'$; $PP' = \Delta x$.

Priore casu est $M'P' = y - \Delta x \cdot \frac{dy}{dx} + \frac{\Delta x^2}{1.2} \cdot \frac{ddy}{dx^2} - \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} + \frac{\Delta x^4}{1.2...4} \cdot \frac{d^4y}{dx^4} - \ldots\ldots$ (§. 33.)

Altero casu $M'P' = y + \Delta x \cdot \frac{dy}{dx} + \frac{\Delta x^2}{1.2} \cdot \frac{ddy}{dx^2} + \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} + \frac{\Delta x^4}{1.2...4} \cdot \frac{d^4y}{dx^4} + \ldots\ldots$

Hinc $MQ = \Delta x \cdot \frac{dy}{dx} \mp \frac{\Delta x^2}{1.2} \cdot \frac{ddy}{dx^2} + \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} \mp \frac{\Delta x^4}{1.2...4} \cdot \frac{d^4y}{dx^4} + \ldots\ldots$

$$PS = y \times \frac{1}{\frac{dy}{dx} \mp \frac{\Delta x}{1.2} \cdot \frac{ddy}{dx^2} + \frac{\Delta x^2}{1.2.3} \cdot \frac{d^3y}{dx^3} \mp \frac{\Delta x^3}{1.2...4} \cdot \frac{d^4y}{dx^4} + \ldots\ldots}$$

Si quis sit limes posterioris membri, erit etiam aliquis limes prioris. Sed (§. 16.) limes posterioris membri (si quis sit) est $y\frac{1}{\frac{dy}{dx}} = y\frac{dx}{dy}$. Ergo: limes prioris membri, nempe PT, est $y\frac{dx}{dy}$.

2°. Ratio æqualitatis est etiam limes rationis tangentis ad secantem; quatenus utraque recta hinc puncto contactus, inde punctis, ubi diametro occurrunt, terminatur. Et proinde tangens est limes secantis.

3°. Li-

3°. Limes rationis ſecantis ad ſubſecantem æqualis eſt rationi tangentis ad ſubtangentem.

§. 41. Hinc quoque determinantur tam angulus, ſub quo tangens occurrit (ſi fieri poſſit) diametro, quam angulus, quem tangens facit cum recta, quæ a puncto contactus ordinatim diametro applicatur.

Nempe ſi angulus P fuerit rectus: erit $\text{tang.} MTP = \frac{MP}{PT} = \frac{dy}{dx}$

$$\& \ \text{cot.} MTP = \text{tang.} PMT = \frac{PT}{MP} = \frac{dx}{dy}$$

Si vero angulus APM non fuerit rectus: erit

$$\text{cot.} MTP = \frac{PT}{MP} \text{coſec.} APM - \text{cot.} APM = \frac{dx}{dy} \text{coſec.} P - \text{cot.} P;$$

$$\& \ \text{cot.} \, 180° - MTP = \text{cot.} APM - \frac{dx}{dy} \text{coſec.} APM$$

$$\text{cot.} PMT = \frac{MP}{PT} \text{coſec.} APM - \text{cot.} APM = \frac{dy}{dx} \text{coſec.} P - \text{cot.} P;$$

$$\& \ \text{cot.} \, 180° - PMT = \text{cot.} APM - \frac{dy}{dx} \text{coſec.} APM. \ (a)$$

§. 42. Tranſeo ad exempla, quibus uſus præcedentium formularum fiat familiaris.

Sit æquatio propoſita $y^{m+n} = a^m x^n$ (quæ eſt æquatio parabolarum, m & n exiſtentibus numeris poſitivis).

Erit $(m+n) y^{m+n-1} \frac{dy}{dx} = n a^m x^{n-1}$

$$\frac{dy}{dx} = \frac{n}{m+n} \cdot \frac{a^m x^{n-1}}{y^{m+n-1}} = \frac{n}{m+n} \cdot \frac{y}{x} = \frac{n}{m+n} \left(\frac{a}{x}\right)^{\frac{m}{m+n}} = \frac{n}{m+n} \left(\frac{a}{y}\right)^{\frac{m+n}{n}}$$

$$\frac{dx}{dy} = \frac{m+n}{n} \cdot \frac{x}{y} = \frac{m+n}{n} \left(\frac{x}{a}\right)^{\frac{m}{m+n}} = \frac{m+n}{n} \left(\frac{y}{a}\right)^{\frac{m+n}{n}}$$

$$y \frac{dx}{dy} = PT = \frac{m+n}{n} x.$$

cot.

(a) Formulæ hæ fluunt ex hoc theoremate noto trigonometriæ planæ.
Sint A, B, C . . . latera trianguli rectilinei,
a, b, c . . . anguli his lateribus reſpective oppoſiti.
Datis lateribus A & B, & angulo c, quem continent; reliqui anguli a, b, immediate determinantur, ut ſequitur: $\text{cot.} a = \frac{B}{A} \text{coſec.} c - \text{cot.} c$

$$\text{cot.} b = \frac{A}{B} \text{coſec.} c - \text{cot.} c.$$

$$\cot. MTP = \frac{m+n}{n}\left(\frac{x}{a}\right)^{\frac{m}{m+n}} \operatorname{cosec}. APM - \cot. APM;$$

$$\cot. 180^\circ - MTP = \cot. APM - \frac{m+n}{n}\,\frac{x}{a}\,.\,\frac{m}{m+n} \operatorname{cosec}. APM$$

$$\cot. PMT = \frac{n}{m+n}\left(\frac{a}{x}\right)^{\frac{m}{m+n}} \operatorname{cosec}. APM - \cot. APM;$$

$$\cot. 180^\circ - PMT = \cot. APM - \frac{n}{m+n}\left(\frac{a}{x}\right)^{\frac{m}{m+n}} \operatorname{cosec}. APM.$$

Sit $x = 0$; erit $PT = \frac{m+n}{n}\,.\,0$: proinde altero puncto T, quod ductum tangentis determinet, deficiente; ejus positio foret indeterminata, nisi in promtu esset angulus, sub quo tangens diametro ad verticem occurrit. Quoniam autem $x = 0$; $\left(\frac{x}{a}\right)^{\frac{m}{m+n}} = 0$; proinde $\cot. 180^\circ - MTP = \cot. APM$; & $180^\circ - MTP = APM$. Ideo recta, quæ parabolas in vertice diametri tangit, parallela est rectis, quæ huic diametro ordinatim applicantur.

Etenim si fieri posset, ut recta, quæ ex vertice diametri alicujus parabolæ ducitur parallela rectis huic diametro ordinatim applicatis, occurreret iterum parabolæ; sit y pars istius rectæ intra parabolam comprehensa. Esset $y^{m+n} = a^m 0^n$; & proinde $y = 0$, contra hyp.

Crescente x, quantitas $\left(\frac{a}{x}\right)^{\frac{m}{m+n}}$ continue decrescit, nunquam vero fit $= 0$. Hinc est semper $\cot. 180^\circ - PMT < \cot. APM$. Proinde angulo APM existente recto aut acuto, est semper $180^\circ - PMT > APM$; & $PMT + APM < 180^\circ$. Sit autem APM obtusus, cujus supplementum sit MPA'.

Erit $\cot. PMT = \frac{n}{m+n}\left(\frac{a}{x}\right)^{\frac{m}{m+n}} \operatorname{cosec}. APM + \cot. MPA'$; Fig. 12.

hinc est semper $\cot. PMT > \cot. MPA'$; $PMT < MPA' < 180^\circ - MPT$;
unde iterum $PMT + MPT < 180^\circ$.

Tangentes ideo cujusvis parabolæ nunquam fiunt diametro parallelæ; sed ad parallelismum accedunt eo magis, quo remotiora a vertice sunt puncta curvæ, ad quæ ducuntur.

Exem-

Exemplum 2. Sit $y^m x^n = a^{m+n}$, quæ eſt æquatio hyperbolarum ad aſymptotos relatarum, exiſtentibus m & n numeris poſitivis.

Eſt ideo $m y^{m-1} x^n \frac{dy}{dx} + n y^m x^{n-1} = 0$;

$$m x \frac{dy}{dx} + n y = 0;$$

$$\frac{dy}{dx} = -\frac{n}{m}\frac{y}{x} = -\frac{n}{m}\cdot\left(\frac{a}{x}\right)^{\frac{m+n}{m}} = -\frac{n}{m}\left(\frac{y}{a}\right)^{\frac{m}{n}}$$

$$\frac{dx}{dy} = -\frac{m}{n}\frac{x}{y} = -\frac{m}{n}\cdot\left(\frac{x}{a}\right)^{\frac{m+n}{m}} = -\frac{m}{n}\left(\frac{a}{y}\right)^{\frac{m}{n}}$$

$$y\frac{dx}{dy} = -\frac{m}{n}x; \quad PT = -\frac{m}{n}AP.$$

Scilicet ratio ſubtangentis ad abſciſſam diametri eſt etiam ratio data; ſed ſubtangens & abſciſſa diametri ad partes ordinatæ MP oppoſitas jacent.

Fig. 13. $\cot.MTP = \frac{PT}{MP}\operatorname{coſec}.MPT - \cot.MPT = \frac{m}{n}\frac{AP}{MP}\operatorname{coſec}.MPA + \cot.MPA$

$$= \frac{m}{n}\left(\frac{x}{a}\right)^{\frac{m+n}{m}}\operatorname{coſec}.MPA + \cot.MPA$$

$$\cot.PMT = \frac{MP}{PT}\operatorname{coſec}.MPT - \cot.MPT = \frac{n}{m}\left(\frac{a}{x}\right)^{\frac{m+n}{m}}\operatorname{coſec}.MPA + \cot.MPA.$$

Sit MPA rectus aut acutus: $\cot.MTP$ ſemper major eſt quam $\cot.MPA$; ſed ad eam eo propius accedit, quo minor x. Sit autem MPA obtuſus;

$$\cot.MTP = \frac{m}{n}\left(\frac{x}{a}\right)^{\frac{m+n}{n}}\operatorname{coſec}.MPT - \cot.MPT$$

$$\cot.180^\circ - MTP = \cot.MPT - \frac{m}{n}\left(\frac{x}{a}\right)^{\frac{m+n}{n}}\operatorname{coſec}.MPT.$$

Ideo $\cot.180^\circ - MTP$ ſemper minor $\cot.MPT$; ſed ab ea differt eo minus, quo minor x. Scilicet tangens MT nunquam fit parallela ordinatis MP, ſed ad parallelismum accedit eo magis, quo minor abſciſſa AP.

Sit MPA rectus aut acutus: $\cot.PMT$ ſemper major quam $\cot.MPA$; ab ea differt eo magis, quo major x. Sit MPA obtuſus:

$$\cot.180^\circ - PMT = \cot.MPT - \frac{n}{m}\left(\frac{a}{x}\right)^{\frac{m+n}{n}}\operatorname{coſec}.P.$$

Proinde $\cot.180^\circ - PMT$ ſemper minor $\cot.MPT$ (unde $MPT + PMT < 180^\circ$); ſed

ſed ab ea differt eo minus, quo major eſt x ſeu AP. Scilicet tangens MT nunquam fit parallela aſymptoto AP; ſed ad parallelismum accedit eo magis, quo major AP.

Exemplum 3. Sit $y^m = \frac{b^m}{a^m}(x^m - a^m)$, quæ eſt etiam æquatio hyperbolarum ad diametros relatarum.

$$\text{Erit } my^{m-1}\frac{dy}{dx} = m\frac{b^m}{a^m}x^{m-1}$$

$$\frac{dy}{dx} = \frac{b^m}{a^m}\cdot\frac{x^{m-1}}{y^{m-1}}$$

$$\frac{dx}{dy} = \frac{a^m}{b^m}\cdot\frac{y^{m-1}}{x^{m-1}}$$

$$y\frac{dx}{dy} = PT = \frac{\frac{a^m}{b^m}\cdot\frac{b^m}{a^m}(x^m - a^m)}{x^{m-1}}$$

$$= \frac{x^m - a^m}{x^{m-1}} = x - \frac{a^m}{x^{m-1}}.$$

$$\text{Hinc } x - PT = \frac{a^m}{x^{m-1}} = a\left(\frac{a}{x}\right)^{m-1}$$

Subtangens igitur PT ſemper eſt minor quam abſciſſa diametri. Quoniam autem ex natura curvæ $x > a$; quo major x, eo minor $\frac{a}{x}$, & a fortiori eo minor $\left(\frac{a}{x}\right)^{m-1}$.

Proinde x poteſt fieri adeo magna, ut exceſſus, quo abſciſſa diametri ſubtangentem ſuperat, minor fiat quacunque quantitate propoſita.

$$\cot. MTP = \frac{a}{b}\left(\frac{x^m - a^m}{x^m}\right)^{\frac{m-1}{m}} \operatorname{coſec.} P - \cot. P$$

$$= \frac{a}{b}\left(1 - \frac{a^m}{x^m}\right)^{\frac{m-1}{m}} \operatorname{coſec.} P - \cot. P.$$

Hinc eſt ſemper $\cot. MTP + \cot. P > \frac{a}{b} \operatorname{coſec.} P$.

Sit $P = 90^\circ$; cot. $MTP > \frac{a}{b}$

tang. $MTP < \frac{b}{a}$: sed $\frac{b}{a}$ eſt limes tangentis creſcentis anguli MTP.

Et in genere: ſin. $MTP + P > \frac{a}{b}$ ſin. MTP

Sed ſin. PMT : ſin. $MPT = TP : MP$; ergo eſt ſemper $TP > MP \cdot \frac{a}{b}$.

Sit $x = a$; $a - PT = a$; $PT = 0$; unde altero puncto T, quod ductum tangentis determinat, deficiente, ejus poſitio foret indeterminata, niſi in promtu eſſet angulus, ſub quo tangens diametro ad verticem occurrit. Fit autem cot. $MTP = -$ cot. P; cot. $180^\circ - MTP =$ cot. P; $180^\circ - MTP = P$; & proinde tangens ad verticem eſt parallela rectis diametro ordinatim applicatis.

Exemplum 4. Sit $y^m = \frac{b^m}{a^m}(a^m - x^m)$, quæ eſt æquatio ellipſium.

$$my^{m-1}\frac{dy}{dx} = -m\frac{b^m}{a^m}x^{m-1}$$

$$\frac{dy}{dx} = -\frac{b^m}{a^m}\left(\frac{x}{y}\right)^{m-1}$$

$$\frac{dx}{dy} = -\frac{a^m}{b^m}\left(\frac{y}{x}\right)^{m-1}$$

$y\frac{dx}{dy} = PT = -\frac{a^m}{b^m}\,\frac{y^m}{x^{m-1}} = -\frac{a^m - x^m}{x^{m-1}}$. Signum $-$ indicat ſubtangentem & abſciſſas ſitas eſſe ad partes diverſas lineæ diametro ordinatim applicatæ.

$$\text{cot. } 180^\circ - T = \text{cot. } P + \frac{a}{b}\left(\frac{a^m}{x^m} - 1\right)^{\frac{m-1}{m}} \text{coſec. } P$$

$$\text{cot. } 180^\circ - M = \text{cot. } P + \frac{b}{a}\left(\frac{a^m}{a^m - x^m} - 1\right)^{\frac{m-1}{m}} \text{coſec. } P.$$

Sit $x = a$; $PT = 0$; unde ductus tangentis foret indeterminatus. Sed propter $\frac{a^m}{x^m} - 1 = 0$, fit cot. $180^\circ - T =$ cot. P, ſeu $T + P = 180^\circ$. Unde ad verticem diametri tangens eſt rectis diametro ordinatim applicatis parallela.

Quem-

Quemadmodum $y^m = \frac{b^m}{a^m}(a^m - x^m)$

eſt etiam $x^m = \frac{a^m}{b^m}(b^m - y^m)$.

Unde facto $x = a$, ſeu $y = b$, pariter eſt tangens priori diametro parallela.

Ex præcedentibus facile ſolvuntur (ſi fieri poſſit) problemata ſequentia.

§. 43. *Problema.* Ducere lineam rectæ poſitione datæ parallelam, quæ (ſi fieri poſſit) curvam datam contingat.

Solutio. Ducatur recta quævis, ad quam tanquam diametrum curva referatur per rectas huic diametro ordinatim applicatas. Jam ex natura data curvæ determinetur ejus æquatio reſpectu illius diametri, & inde determinetur ratio differentialis $\frac{dx}{dy}$ vel $\frac{dy}{dx}$. Tum in æquatione $\cot. 180^\circ - T = \cot. P - \frac{dx}{dy}$ coſec. P datur angulus T, & eliminari poteſt $\frac{dx}{dy}$; hinc determinatio quantitatum x & y ſemper reducitur ad ſolutionem æquationis alicujus datæ; proinde problema tanquam ſolutum haberi poteſt (quantum permittit imperfecta æquationum theoria).

Exemplum 1. Sit parabola, cujus æquatio $y^{m+n} = a^m x^n$, ad axem relata rectis huic perpendiculariter ordinatim applicatis. Et ducenda ſit recta, quæ parabolam hanc tangat, & axi occurrat ſub angulo dato T.

Eſt igitur $\cot. T = \frac{dx}{dy}$.

Sed ex æquatione parabolæ eſt $\frac{dx}{dy} = \frac{m+n}{n} \cdot \frac{y^{m+n-1}}{a^m x^{n-1}} = \frac{m+n}{n} \cdot \frac{x}{y}$.

Hinc $\frac{x}{y} = \frac{n}{m+n} \cot. T$

ſeu $x : y = n \cot. T : m+n$

$x^n : y^n = n^n \cot.^n T : (m+n)^n$ $\qquad$ $x^{m+n} : y^{m+n} = n^{m+n} \cot.^{m+n} T : (m+n)^{m+n}$

unde $y^m : a^m = n^n \cot.^n T : (m+n)^n$ $\quad$ unde $x^m : a^m = n^{m+n} \cot.^{m+n} T : (m+n)^{m+n}$

$$y = a\left(\frac{n}{m+n}\right)^{\frac{n}{m}} \cot.^{\frac{n}{m}} T \qquad\qquad x = a\left(\frac{n}{m+n}\right)^{\frac{m+n}{m}} \cot.^{\frac{m+n}{m}} T.$$

Exemplum 2. Curva propoſita ſit hyperbola, cujus æquatio $y^m = \frac{b^m}{a^m}(x^m - a^m)$; unde $\frac{dx}{dy} = \frac{a^m}{b^m} \cdot \frac{y^{m-1}}{x^{m-1}}$.

Fit ideo $\cot. T = \frac{a^m}{b^m} \cdot \frac{y^{m-1}}{x^{m-1}}$

$$= \frac{a^m}{b^m} \cdot \frac{y^m}{yx^{m-1}} = \frac{x^m - a^m}{yx^{m-1}}$$

$$y = \frac{x^m - a^m}{x^{m-1}} \text{tang}. T;$$

$$\frac{b}{a} x^{m-1} = (x^m - a^m)^{\frac{m-1}{m}} \text{ tang}. T$$

$$\left(\frac{b}{a}\right)^{\frac{m}{m-1}} x^m = (x^m - a^m) \text{tang}.^{\frac{m}{m-1}} T.$$

$$x^m : x^m - a^m = \text{tang}.^{\frac{m}{m-1}} T : \left(\frac{b}{a}\right)^{\frac{m}{m-1}}.$$

Quoniam autem $x^m > x^m - a^m$

debet eſſe $\text{tang}.^{\frac{m}{m-1}} T > \left(\frac{b}{a}\right)^{\frac{m}{m-1}}$

& proinde $\text{tang}. T > \frac{b}{a}$, uti in Ex. 3. §. 42.

§. 44. *Problema.* Ab puncto poſitione dato ducere rectam, quæ (ſi fieri poſſit) curvam poſitione datam contingat.

Solutio. Per punctum datum T ducatur recta quævis, ad quam tanquam diametrum curva data referatur per rectas huic ordinatim applicatas. Ex data curvæ æquatione, quatenus ad hanc diametrum refertur, determinetur ratio differentialis $\frac{dy}{dx}$ vel $\frac{dx}{dy}$, quæ ſubſtituatur in æquatione ſubtangentis $PT = y\frac{dx}{dy}$. Quoniam diſtantia puncti T ab origine abſciſſarum data ſupponitur, æquatio inde nata afficietur tantum quantitatibus incognitis x & y, quæ cum æquatione curvæ combinata ducet ad ſolutionem problematis propoſiti (quousque per theoriam æquationum licet).

Exemplum 1. Curva data ſit ellipſis, cujus æquatio $y^m = \frac{b^m}{a^m}(a^m - x^m)$; & punctum T ſitum ſit ſuper diametro, ad quam curva refertur.

Eſt

Eſt ideo $\frac{dy}{dx} = -\frac{b^m}{a^m}\frac{x^{m-1}}{y^{m-1}}$, quod indicat ſubtangentem & abſciſſam diametri ſitas eſſe ad diverſas partes ordinatæ.

$y\frac{dy}{dx} = PT = \frac{a^m}{b^m} \cdot \frac{y^m}{x^{m-1}} = \frac{a^m - x^m}{x^{m-1}} = a\left(\frac{a}{x}\right)^{m-1} - x$. Diſtantia puncti T ab origine abſciſſarum ſit l: fit $l - x = a\left(\frac{a}{x}\right)^{m-1} - x$; $\left(\frac{x}{a}\right)^{m-1} = \frac{a}{l}$; $x = a\sqrt[m-1]{\frac{a}{l}}$.

Exemplum 2. Curva data ſit hyperbola, cujus æquatio $y^m = \frac{b^m}{a^m}(x^m - a^m)$; & punctum T ſitum ſit ſuper diametro, ad quam curva refertur.

Eſt ut ſupra, $\frac{dy}{dx} = \frac{b^m}{a^m} \cdot \frac{y^m}{x^{m-1}}$.

$$y\frac{dx}{dy} = \frac{a^m}{b^m} \cdot \frac{y^m}{x^{m-1}} = \frac{x^m - a^m}{x^{m-1}} = x - a\left(\frac{a}{x}\right)^{m-1} = x - l; \ \& \ x = a\sqrt[m-1]{\frac{a}{l}}.$$

Scholium. Quemadmodum ſolutio ſemper poſſibilis eſt pro ellipſi, ſi fuerit $l > a$; ita etiam ſemper poſſibilis eſt pro hyperbola, ſi fuerit $l \begin{matrix} < a \\ > o. \end{matrix}$

§. 45. Cum expreſſio ſubtangentis t ſit $y\frac{dx}{dy}$: ſi ſubtangens data fuerit per quantitatem conſtantem, vel per functionem quantitatum x & y; ideo datus erit exponens differentialis $\frac{dx}{dy}$ per functiones earumdem variabilium, & proinde determinatio quantitatum x & y pertinebit ad calculum integralem.

Exempla. Sit $t = \frac{m}{n}y$; hinc $y\frac{dx}{dy} = \frac{m}{n}y$; $\frac{dx}{dy} = \frac{m}{n}$; $x = C + \frac{m}{n}$, quæ eſt æquatio ad lineam rectam.

Sit $t = \frac{y^m}{a^{m-1}}$; $y\frac{dx}{dy} = \frac{y^m}{a^{m-1}}$; $\frac{dx}{dy} = \frac{y^{m-1}}{a^{m-1}}$; $x = C + \frac{1}{m}\frac{y^m}{a^{m-1}}$. Sit $x = o$, quando $y = o$: tum $C = o$; $x = \frac{1}{m} \cdot \frac{y^m}{a^m}$. Sit vero $x = b$, quando $y = o$: erit $C = b$, & $x - b = \frac{1}{m} \cdot \frac{y^m}{a^{m-1}}$.

Sit $t = \frac{x^m y^n}{a^{m+n-1}}$; unde $\frac{x^m y^n}{a^{m+n-1}} = y\frac{dx}{dy}$; $\frac{y^{n-1}}{a^{m+n-1}} = \frac{1}{x^m}\frac{dx}{dy} = x^{-m}\frac{dx}{dy}$;

$\frac{\frac{1}{n}y^n}{a^{m+n-1}} = C - \frac{1}{m-1}x^{-m+1}$; $y^n = C - \frac{n}{m-1} \cdot \frac{a^{m+n-1}}{x^{m-1}}$.

Ob calculi autem integralis imperfectionem plerumque huic problemati (quod *methodus tangentium inversa* vocatur) æquatione terminis finitis expressa satisfieri nequit. Imo sunt casus nonnulli etiam simplicissimi, qui methodis huc usque expositis tractari non posse videntur, & de quibus inferius dicemus. Exempli causa sit $t = \frac{x^m}{a^{m-1}}$: erit $\frac{x^m}{a^{m-1}} = y\frac{dx}{dy}$; de qua formula, & aliis ipsi analogis, infra agemus.

§. 46. Quæcunque sit origo curvæ alicujus algebraicæ, ea semper potest ad æquationem reduci inter rectas coordinatas respectu alicujus diametri. Ex. gr. constantia summæ distantiarum cujusvis puncti ellipseos ab ejus focis pro fundamentali ipsius proprietate assumta, ex illa deducitur relatio coordinatarum alterutrius axium imo & respectu duarum quarumlibet diametrorum conjugatarum: quod scilicet quadrata rectarum diametro cuicunque ordinatim applicatarum proportionalia sint rectangulis sub abscissis ejusdem diametri. Idem valet de hyperbola; pariterque de omnibus sectionibus conicis, proprietatis fundamentalis loco sumta ratione data distantiarum cujusvis puncti ab alterutro foco & a directrice. Proinde, quæ jam præcepta fuerunt de ductu tangentium, quatenus curvæ ad aliquam diametrum referuntur per rectas huic ordinatim applicatas, sufficere possunt (pro curvis saltem algebraicis). Quoniam autem frequenter evenit, ut curvarum affectiones, & nominatim proprietates earum ad contactus pertinentes, modo lucidiori & simpliciori ex alia quadam ipsarum origine aut proprietate deduci possint, quam si æquatio inter coordinatas rectilineas instituatur; e re esse censeo, præcipuas curvarum origines perpendere, & methodos tangentes ducendi iis applicare, non adhibita æquatione, qua curvæ ad diametrum aliquam per rectas huic ordinatim applicatas referuntur. Sed ut hunc scopum modo sufficiente adimplere valeam: nonnulla præmittenda sunt lemmata, quæ & in sequentibus capitibus suas habebunt applicationes.

§. 47. *Theorema.* Ratio æqualitatis limes est tam rationis decrescentis arcus (qui totus versus chordam concavus supponitur) ad chordam; quam rationis crescentis arcus ad summam tangentium per extrema ejus ductarum, & concursu suo mutuo terminatarum.

Sit

Sit AMZ arcus curvæ (totus verſus chordam ſuam concavus). Dico: ex uno arcus illius extremo A duci poſſe chordam AM talem, ut ratio arcus ALM ad chordam AM (quæ ſemper major eſt ratione æqualitatis) minor ſit quacunque ratione propoſita majoris ad minorem; item, ut ductis ex A & M tangentibus, quæ ſibi mutuo occurrant in N, ratio arcus AM ad ſummam $AN + NM$ duarum tangentium (quæ ſemper minor eſt ratione æqualitatis) major ſit quacunque ratione propoſita minoris ad majorem. Fig. 14.

Sit ratio propoſita majoris ad minorem ea, quam ſumma crurum alicujus trianguli æquicruri $CE + ED$ habet ad baſim CD; ſeu ratio minoris ad majorem ea, quam habet CD ad $CE + ED$. Ad punctum A ducatur tangens BA, & chorda AZ, quæ faciat cum tangente BA angulum BAZ ipſo CED majorem. Tum (§. 43.) ducatur arcui AMZ tangens chordæ AZ parallela, quæ ipſi BA occurret. Sit ea MN. Dico factum.

Etenim ſuper CD, tanquam baſi, conſtituatur triangulum $CE'D$ ipſi ANM ſimile.

In triangulo $CE'D$ angulus $CE'D (= N = BAZ) > CED$. Ergo punctum E' ſitum eſt intra ſegmentum circuli, cujus baſis CD, & quod capax eſt anguli CED. Proinde $CE' + E'D < CE + ED$ (quod facile demonſtratur).

Eſt ideo $CE' + E'D : CD < CE + ED : CD$
ſed $CE' + E'D : CD = AN + NM : AM$
ergo $AN + NM : AM < CE + ED : CD.$

Atqui 1°. arcus ALM minor eſt ſumma $AN + NM$; ergo (a fortiori)

$$ALM : AM < CE + ED : CD$$

2°. arcus ALM major eſt chorda AM; ergo (a fortiori)

$$AN + NM : ALM < CE + ED : CD.$$

Proinde (§. 1.) ratio æqualitatis limes eſt rationis decreſcentis cujusvis arcus ad ſuam chordam; & limes rationis creſcentis ejusdem arcus ad ſummam tangentium, quæ ab extremis arcus ducuntur, & occurſu ſuo mutuo terminantur.

§. 48. *Corollarium* 1. Nominatim ratio æqualitatis limes eſt rationis decre-

cre-

crescentis arcus circuli ad suam chordam; & proinde etiam dimidii arcus ad dimidam chordam, seu arcus ad sinum rectum.

Corollarium 2. Rursus ratio æqualitatis limes est rationis crescentis arcus circuli ad tangentem ejus trigonometricam.

Ceterum corollaria hæc de circulo potuissent ex propositionibus ARCHIMEDIS de figuris ordinatis, quæ circulo inscribuntur & circumscribuntur, deduci.

Corollarium 3. Posito igitur sinu alicujus arcus s, ipse arcus functio est sinus hujus formæ (quæ deinceps accuratius determinabitur) $s + As^2 + Bs^3 + Cs^4 + Ds^5 + \ldots\ldots$ ideoque ratio arcus circuli ad sinum rectum est $1 + As + Bs^2 + Cs^3 + Ds^4 + \ldots : 1$ & ratio æqualitatis est limes hujus rationis decrescentis (§. 18.)

Item sit t tangens trigonometrica alicujus arcus circuli, erit arcus functio tangentis hujus formæ $t - At^2 + Bt^3 + Ct^4 + Dt^5 + \ldots\ldots$ Atque ratio arcus circuli ad tangentem trigonometricam $1 - At + Bt^2 + Ct^3 + Dt^4 + \ldots\ldots : 1.$ Hujus rationis crescentis limes est ratio æqualitatis (§. 18.)

Vicissim, posito arcu circuli z, erit
$$t = z + Az^2 + Bt^3 + Ct^3 + Dt^5 + \ldots\ldots$$
$$t : z = 1 + Az + Bt^2 + Ct^3 + Dt^4 + \ldots\ldots : 1.$$
$$s = z - Az^2 + Bz^3 + Cz^4 + Dz^5 + \ldots\ldots$$
$$s : z = 1 - Az + Bz^2 + Cz^3 + Dz^4 + \ldots\ldots : 1.$$

Corollarium 4. Quoniam sin. vers. $z = 2ss$, erit sin. vers. $z = A'zz + B'z^3 + C'z^4 + D'z^5 + \ldots\ldots$ Sed hæc satius est sequenti modo deducere.

Ratio chordæ ad sinum versum potest fieri major quacunque ratione data.

Etenim ratio radii dupli ad chordam æqualis est rationi chordæ ad sinum versum: sed prior ratio major reddi potest quacunque ratione data; ergo & posterior.

Hinc etiam rationes arcus & sinus recti ad sinum versum possunt reddi majores quacunque ratione data. Et ratio æqualitatis limes est rationum, quas sinus rectus, chorda, arcus - - - habent ad summam aut differentiam quantitatum harum & sinus versi (imo & quantitatum, quæ ad sinum versum datas habent rationes §. 18.)

§. 49.

§. 49. *Generatim* per punctum quodlibet M curvæ alicujus ducatur tangens MT, & recta quævis MP; per aliud quodcunque punctum M' ejusdem curvæ ducatur recta alicui rectæ positione datæ parallela, quæ ipsis MP, MT occurrat in punctis Q & Q'. Dico: rationem æqualitatis limitem esse arcus MNM' & segmenti tangentis MQ'. Fig. 11.

Etenim per punctum quodcunque rectæ MP ducatur recta ipsi $M'Q$ parallela, cui chorda MM' & tangens MQ' occurrant in S & T.

$$\begin{array}{lllll}
 & \lim. MNM' : MM' & = 1 : 1 & (\S.\ 47.) \\
\text{sed} & \lim. MM' : MQ' (= MS : TS) & = 1 : 1 & (\S.\ 40.) \\
\text{ergo} & \lim. MNM' : MQ' & = 1 : 1 & (\S.\ 14.) \\
\text{item} & \lim. QQ' : QM' (= PT : PS) & = 1 : 1 & (\S.\ 40.)
\end{array}$$

Corollaria. Sit Q angulus rectus.

$$\begin{array}{lll}
1^\circ. & MQ' : MQ = 1 : \text{cos.} TMP & \\
 & \lim. MNM' : MQ' = 1 : 1 & \\
\text{ergo} & \lim. MNM' : MQ = 1 : \text{cos.} TMP & (\S.\ 14.)
\end{array}$$

Scilicet: ratio differentialis arcus cujuslibet curvæ & rectæ axi sub angulo recto ordinatim applicatæ, æqualis est rationi radii ad cosinum anguli, sub quo tangens curvæ ad rectam axi ordinatim applicatam inclinatur.

$$\begin{array}{lll}
2^\circ. & \lim. MNM' : MQ' = 1 : 1 & \\
 & MQ' : QQ' = 1 : \text{sin.} M & \\
 & \lim. QQ' : M'Q = 1 : 1 & \\
\text{ergo} & \lim. MNM' : M'Q = 1 : \text{sin.} M & (\S.\ 14.)
\end{array}$$

Scilicet: ratio differentialis arcus cujuslibet curvæ & abscissæ axis æqualis est rationi radii ad sinum anguli, quem tangens facit cum recta axi ordinatim applicata.

3°. Curva referatur ad aliquod punctum F per radios vectores MF, $M'F$. Sit arcus MM'; & radio vectori FM perpendicularis $M'Q$, quæ tangenti in Q' occurrat. Centro F, radio FM', describatur arcus circuli $M'q$. Item sit PP' arcus, cujus radius FA constans. Fig. 15.

 lim.

$$\lim. MNM' : MQ' = 1 : 1$$
$$MQ' : QQ' = 1 : \text{fin.} M$$
$$\lim. QQ' : M'Q = 1 : 1$$
$$\lim. MQ : M'q = 1 : 1 \qquad (\S.\ 40.)$$
$$M'q : PP' = FM : FA \qquad (\S.\ 48.)$$
$$\text{ergo} \quad \lim. MNM' : PP' = FM : FA\,\text{fin.} M. \qquad (\S.\ 14.)$$

Hoc eſt: ratio differentialis arcus curvæ & arcus circuli, qui metitur angulum duobus radiis vectoribus comprehenſum, æqualis eſt rationi radii vectoris ad radium prædicti circuli, multiplicatum per ſinum anguli, ſub quo tangens curvæ ad radium vectorem inclinatur.

4°.
$$\lim. MNM : MQ' = 1 : 1$$
$$MQ' : MQ = 1 : \text{cof.} M$$
$$\lim. MQ : Mq = 1 : 1 \qquad (\S.\ 48.)$$
$$\text{ergo} \quad \lim. MNM' : Mq = 1 : \text{cof.} M.$$

Hoc eſt: ratio differentialis arcus alicujus curvæ & radii vectoris æqualis eſt rationi radii ad coſinum anguli, quem tangens curvæ cum radio vectore comprehendit.

5°.
$$\lim. Mq : MQ = 1 : 1 \qquad (\S.\ 48.)$$
$$\lim. MQ : M'Q = 1 : \text{tang.} M$$
$$\lim. M'Q : M'q = 1 : 1 \qquad (\S.\ 48.)$$
$$\lim. M'q : PP' = FM : FA$$
$$\text{ergo} \quad \lim. Mq : PP' = FM : FA\,\text{tang.} M \qquad (\S.\ 14.)$$

Hoc eſt: ratio differentialis radii vectoris & arcus circuli, qui metitur angulum duobus radiis vectoribus comprehenſum, æqualis eſt rationi radii vectoris ad radium hujus circuli multiplicatum per tangentem anguli, quem tangens curvæ cum radio vectore comprehendit.

Tranſeo ad applicationes.

§. 50. 1°. Curva referatur ad plures rectas poſitione datas per aliquam relationem datam inter perpendiculares, in rectas poſitione datas ex quolibet curvæ puncto demiſſas.

Rectæ

Rectæ perpendiculares in rectas positione datas demissæ, dicantur $MP,\quad MP',\quad MP'',\quad MP''',\quad MP^{\text{iv}} \ldots\ldots$

seu $y,\quad y',\quad y'',\quad y''',\quad y^{\text{iv}} \ldots\ldots$

Arcus curvæ dicatur z. Ex relatione data dabitur etiam relatio exponentium differentialium $\frac{dy}{dz},\quad \frac{dy'}{dz},\quad \frac{dy''}{dz},\quad \frac{dy'''}{dz},\quad \frac{dy^{\text{iv}}}{dz} \ldots\ldots$

proinde etiam dabitur relatio inter quantitates $\cos.TMP, \cos.TMP', \cos.TMP'', \cos.TMP''', \cos.TMP^{\text{iv}} \ldots$

Atqui punctum M & perpendicula $MP,\quad MP',\quad MP'',\quad MP''',\quad MP^{\text{iv}} \ldots\ldots$ dantur positione; ergo dabitur aliqua affectio tangentis, per quam determinabitur.

Exemplum 1. Summa rectangulorum prædictorum perpendiculorum per rectas magnitudine datas $a,\ a',\ a'',\ a''',\ a^{\text{iv}} \ldots\ldots$ datur magnitudine.

Erit ideo

$$a\cos.TMP + a'\cos.TMP' + a''\cos.TMP'' + a'''\cos.TMP''' + a^{\text{iv}}\cos.TMP^{\text{iv}} + \ldots\ldots = 0.$$

Locus propositus non est curva aliqua, sed linea recta (uti videre est ex opusculo meo, *Polygonometrie*, Geneve 1789.)

Exemplum 2. Summa spatiorum, quæ ad quadrata perpendiculorum datas habent rationes, datur magnitudine.

Erit ideo

$$ay\cos.TMP + a'y'\cos.TMP' + a''y''\cos.TMP'' + a'''y'''\cos.TMP''' + a^{\text{iv}}y^{\text{iv}}\cos.TMP^{\text{iv}} + \ldots\ldots = 0.$$

Quam proprietatem ad sectiones conicas pertinere demonstrare possum.

Exemplum 3. Summa rectangulorum prædictis perpendiculis binis sumptis factorum datur magnitudine.

Fit ideo

$$\left.\begin{array}{llllll}
 & y'\cos.TMP & + y''\cos.TMP & + y'''\cos.TMP & + y^{\text{iv}}\cos.TMP & + \ldots\ldots \\
y\cos.TMP' & - & + y''\cos.TMP' & + y'''\cos.TMP' & + y^{\text{iv}}\cos.TMP' & + \ldots\ldots \\
y\cos.TMP'' & + y'\cos.TMP'' + & - & + y'''\cos.TMP'' & + y^{\text{iv}}\cos.TMP'' & + \ldots\ldots \\
y\cos.TMP''' & + y'\cos.TMP''' & + y''\cos.TMP''' + & - & + y^{\text{iv}}\cos.TMP''' & + \ldots\ldots \\
y\cos.TMP^{\text{iv}} & + y'\cos.TMP^{\text{iv}} & + y''\cos.TMP^{\text{iv}} & + y'''\cos.TMP^{\text{iv}} + & - & + \ldots\ldots \\
- & - & - & - & - & - \\
- & - & - & - & - & -
\end{array}\right\} = 0$$

quæ proprietas ad sectiones conicas pertinet.

 §. 51.

§. 51. 2°. Sit curva ad plura puncta relata, per datam relationem inter radios vectores ad hæc puncta ductos.

Puncta data sint	$F,$	$F',$	$F'';$	$F''',$	F^{iv} ....
Radii vectores dicantur	$r,$	$r',$	$r'',$	$r''',$	r^{iv} ....
Ex relatione data ducatur relatio inter exponentes differentiales	$\frac{dr}{dz},$	$\frac{dr'}{dz},$	$\frac{dr''}{dz},$	$\frac{dr'''}{dz},$	$\frac{dr^{\text{iv}}}{dz}$ ....

seu inter quantitates $\text{cof.}FMT$, $\text{cof.}F'MT$, $\text{cof.}F''MT$, $\text{cof.}F'''MT$, $\text{cof.}F^{\text{iv}}MT$.... unde positio tangentis determinatur.

Exemplum 1. Sint duo puncta data; & summa radiorum rectorum detur magnitudine.

Quoniam $r + r'$ datur magnitudine;

est $\frac{dr}{dz} + \frac{dr'}{dz} = 0$.

seu $\text{cof.}FMT + \text{cof.}F'MT = 0$

hinc $\text{cof.}FMT = \text{cof.}180° - F'MT$, quæ est proprietas nota ellipseos.

Exemplum 2. Differentia radiorum vectorum detur magnitudine.

Quoniam $r - r'$ datur magnitudine;

fit $\frac{dr}{dz} - \frac{dr'}{dz} = 0$

$\text{cof.}FMT - \text{cof.}F'MT = 0$

$FMT = F'MT$, quæ est proprietas nota hyperbolæ.

Exemplum 3. Ratio radiorum vectorum sit semper eadem.

Quoniam ratio $r : r'$ est semper eadem,

est etiam ratio $\frac{dr}{dz} : \frac{dr'}{dz}$ semper eadem;

hinc ratio $\text{cof.}FMT : \text{cof.}F'MT$ semper eadem, quæ est proprietas circuli.

Exemplum 4. Rectangulum ex radiis vectoribus detur magnitudine.

Quoniam rr' datur magnitudine; $r'\frac{dr}{dz} + r\frac{dr'}{dz} = 0$

$r'\text{cof.}FMT + r\,\text{cof.}FMT = 0$

$r'\text{cof.}FMT = r\,\text{cof.}180° - F'MT$, quæ est proprietas curvæ Cassinianæ.

Exem-

Exemplum 5. Summa ſpatiorum, quæ datas habent rationes ad quadrata ex quotlibet radiis vectoribus, detur magnitudine.

Quoniam
$$ar^2 + a'r'^2 + a''r''^2 + a'''r'''^2 + a^{IV}r^{IV2} + \ldots$$ datur magnitudine; fit
$$2ar\frac{dr}{dz} + 2a'r'\frac{dr'}{dz} + 2a''r''\frac{dr''}{dz} + 2a'''r'''\frac{dr'''}{dz} + 2a^{IV}r^{IV}\frac{dr^{IV}}{dz} + \ldots = 0.$$
hinc
$$ar\,\text{coſ.}FMT + a'r'\,\text{coſ.}F'MT + a''r''\,\text{coſ.}F''MT + a'''r'''\,\text{coſ.}F'''MT + a^{IV}r^{IV}\,\text{coſ.}F^{IV}MT + \ldots = 0.$$
quæ eſt proprietas circuli. (Vid. *De relatione mutua capacitatis et terminorum figurarum.* Vars. 1782.)

§. 52. 3°. Curva referatur ad aliquod punctum datum, & ad rectam, per angulum, quem radius vector facit cum recta poſitione data.

Angulus mutabilis dicatur x.

Ex relatione data inter radium vectorem & angulum mutabilem dabitur exponens differentialis $\frac{dr}{dx}$, ſeu $r\,\text{cot.}\,FMT$.

Exemplum 1. Sit radius vector r datus magnitudine.

Fit ideo $\frac{dr}{dx} = 0$; $\text{coſ.}FMT = 0$, $FMT = 90°$, quæ eſt proprietas circuli.

Exemplum 2. Sit $r = ax$
$$\frac{dr}{dx} = a;\ r\,\text{cot.}FMT = a;\ \text{tang.}FMT = \frac{r}{a},$$ quæ eſt proprietas ſpiralis Archimedeæ.

Exemplum 3. Sit $r\,\text{coſ.}^2\tfrac{1}{2}x = a$.
$$\text{Erit}\ \frac{dr}{dx}\text{coſ.}^2\tfrac{1}{2}x - r\,\text{coſ.}\tfrac{1}{2}x\,\text{ſin.}\tfrac{1}{2}x = 0$$
$$r\,\text{cot.}FMT\,\text{coſ.}\tfrac{1}{2}x = r\,\text{ſin.}\tfrac{1}{2}x$$
$$\text{cot.}FMT = \text{tang.}\tfrac{1}{2}x$$
$$FMT = 90° - \tfrac{1}{2}x,$$ quæ eſt proprietas parabolæ.

Exemplum 4. Sit $r = \frac{bb}{a+e\,\text{coſ.}x}$
$$\text{erit}\ \frac{dr}{dx} = \frac{bbe\,\text{ſin.}x}{(a+e\,\text{coſ.}x)^2}$$
$$r\,\text{cot.}FMT = \frac{bbe\,\text{ſin.}x}{(a+e\,\text{coſ.}x)^2} = r\,.\,\frac{e\,\text{ſin.}x}{a+e\,\text{coſ.}x}$$
$$\text{cot.}FMT = \frac{e\,\text{ſin.}x}{a+e\,\text{coſ.}x},$$ quæ eſt proprietas ellipſeos.

§. 53. 4°. Curva referatur ad aliquod punctum & ad aliquam rectam, aut etiam ad plura puncta simul & ad plures rectas.

Ex relatione data inter quantitates

$$r, \quad r', \quad r'', \quad r''', \ldots \quad y, \quad y', \quad y'', \quad y''' \ldots$$

dabitur relatio inter exponentes differentiales

$$\frac{dr}{dz}, \quad \frac{dr'}{dz}, \quad \frac{dr''}{dz}, \quad \frac{dr'''}{dz}, \ldots \quad \frac{dy}{dz}, \quad \frac{dy'}{dz}, \quad \frac{dy''}{dz}, \quad \frac{dy'''}{dz} \ldots$$

seu inter quantitates

$$\text{cos.} FMT, \text{cos.} F'MT, \text{cos.} F''MT, \text{cos.} F'''MT \ldots \text{cos.} PMT, \text{cos.} P'MT, \text{cos.} P''MT, \text{cos.} P'''MT \ldots$$

unde positio tangentis determinabitur.

1°. Sit unicum punctum & unica recta.

Exemplum 1. Sit $r^2 = ay$

$$\text{fit} \quad 2r\frac{dr}{dz} = a\frac{dy}{dz}$$

$2r\,\text{cos.} FMT = a\,\text{cos.} PMT$, quæ proprietas pertinet ad circulum.

Exemplum 2. Sit $r = \frac{m}{n}y$

$$\text{fit} \quad \frac{dr}{dz} = \frac{m}{n}\frac{dy}{dz}$$

$\text{cos.} FMT = \frac{m}{n}\text{cos.} PMT$, quæ proprietas pertinet ad sectiones conicas.

2°. Sint plura puncta & plures rectæ.

Exemplum.

$$\text{Sit} \quad mr^2 + m'r'^2 + m''r''^2 + m'''r'''^2 + \ldots$$
$$= ay + a'y' + a''y'' + a'''y''' + \ldots$$
$$\text{erit} \quad 2mr\frac{dr}{dz} + 2m'r'\frac{dr'}{dz} + 2m''r''\frac{dr''}{dz} + 2m'''r'''\frac{dr'''}{dz} + \ldots$$
$$= a\frac{dy}{dz} + a'\frac{dy'}{dz} + a''\frac{dy''}{dz} + a'''\frac{dy'''}{dz} + \ldots$$

$$\text{seu} \quad 2mr\,\text{cos.} FMT + 2m'r'\text{cos.} F'MT + 2m''r''\text{cos.} F''MT + 2m'''r'''\text{cos.} F'''MT + \ldots$$
$$= a\,\text{cos.} PMT + a'\text{cos.} P'MT + a''\text{cos.} P''MT + a'''\text{cos.} P'''MT + \ldots$$

quæ proprietas pertinet ad circulum.

Scholium.

Scholium. Ex his exemplis patet, ſæpe potius eſſe, proprietates curvarum ex proprietate quapiam primaria deducere; operationesque mere algebraicas poſſe fieri & longiores & minus lucidas, quam quæ magis geometrice inſtituuntur.

Caput Sextum.

De Logarithmis.

§. 54.

In diſſertatione jam nominata *Expoſition elementaire des principes des calculs ſuperieurs* cap. VI. logarithmorum theoriam & calculum ex contemplatione curvæ logarithmicæ deduxi. Et quidem curva hæc omnino mihi apta videtur ad dilucidandum hoc caput, quod tanti eſt per univerſam matheſin momenti. Cum vero idem ſpectari poſſit tanquam ad calculum proprie pertinens, methodus mere algebraica, & progreſſionum geometricarum natura immediate nixa, potior videri poteſt, quam methodus mixta (nempe partim geometrica & partim algebraica), qua in prædicta diſſertatione uſus ſum.

Methodus autem, quam hic proſecuturus ſum, eſt mere elementaris, neque ulla infiniti notione affecta. Eandemque eo lubentius evolvam, cum (quod ſciam) prorſus ſit nova, principiis prius poſitis omnino conſentanea, fæcunditati eorum luculenter oſtendendæ apta, & demonſtrationi Pfleidererianæ theorematis Tayloriani valde analoga.

§. 55. Sit a^z quantitas exponentialis, quæ proinde functio eſt exponentis mutabilis z. Et quoniam imminuta z (poſito a numero poſitivo unitate majore) unitas eſt limes quantitatis decreſcentis a^z (§. 22.)

Sit $a^z = 1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + Fz^6 + Gz^7 + \ldots.$

Erit ideo $a^{2z} = 1+2Az+2^2Bz^2+2^3Cz^3+2^4Dz^4+2^5Ez^5+2^6Fz^6+2^7Gz^7+\ldots.$

$a^{3z} = 1+3Az+3^2Bz^2+3^3Cz^3+3^4Dz^4+3^5Ez^5+3^6Fz^6+3^7Gz^7+\ldots.$

$a^{4z} = 1+4Az+4^2Bz^2+4^3Cz^3+4^4Dz^4+4^5Ez^5+4^6Fz^6+4^7Gz^7+\ldots.$

$a^{5z} = 1+5Az+5^2Bz^2+5^3Cz^3+5^4Dz^4+5^5Ez^5+5^6Fz^6+5^7Gz^7+\ldots.$

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Hinc

Hinc ſumtis differentiis primis erit (§. *s. Introd.*)

$$(a^z-1)a^z = Az+(2^2-1)Bz^2+(2^3-1)Cz^3+(2^4-1)Dz^4+(2^5-1)Ez^5+(2^6-1)Fz^6+\dots$$
$$(a^z-1)a^{2z}=Az+(3^2-2^2)Bz^2+(3^3-2^3)Cz^3+(3^4-2^4)Dz^4+(3^5-2^5)Ez^5+(3^6-2^6)Fz^6+\dots$$
$$(a^z-1)a^{3z}=Az+(4^2-3^2)Bz^2+(4^3-3^3)Cz^3+(4^4-3^4)Dz^4+(4^5-3^5)Ez^5+(4^6-3^6)Fz^6+\dots$$
$$(a^z-1)a^{4z}=Az+(5^2-4^2)Bz^2+(5^3-4^3)Cz^3+(5^4-4^4)Dz^4+(5^5-4^5)Ez^5+(5^6-4^6)Fz^6+\dots$$

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Quoniam autem $a^z = 1 + Az+Bz^2+Cz^3+Dz^4+Ez^5+\dots$

et $a^z-1 = Az+Bz^2+Cz^3+Dz^4+Ez^5+\dots$

primus terminus ſingulorum productorum, $(a^z-1)a^z$, $(a^z-1)a^{2z}$, $(a^z-1)a^{3z}$, $(a^z-1)^{4z}$, ... ipſe fit Az: unde in æquationibus præcedentibus dividendo per z, & æquando terminos conſtantes (juxta methodum reverſionis ſerierum), fit $A=A$. Quare A manet quantitas indeterminata.

Sumantur differentiæ ſecundæ. Erit (§. *s. Introd.*)

$$(a^z-1)^2a^z = \Delta''(3^2\dots1^2)Bz^2+\Delta''(3^3\dots1^3)Cz^3+\Delta''(3^4\dots1^4)Dz^4+\Delta''(3^5\dots1^5)Ez^5+\dots$$
$$(a^z-1)^2a^{2z} = \Delta''(4^2\dots2^2)Bz^2+\Delta''(4^3\dots2^3)Cz^3+\Delta''(4^4\dots2^4)Dz^4+\Delta''(4^5\dots2^5)Ez^5+\dots$$
$$(a^z-1)^2a^{3z} = \Delta''(5^2\dots3^2)Bz^2+\Delta''(5^3\dots3^3)Cz^3+\Delta''(5^4\dots3^4)Dz^4+\Delta''(5^5\dots3^5)Ez^5+\dots$$
$$(a^z-1)^2a^{4z} = \Delta''(6^2\dots4^2)Bz^2+\Delta''(6^3\dots4^3)Cz^3+\Delta''(6^4\dots4^4)Dz^4+\Delta''(6^5\dots4^5)Ez^5+\dots$$

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Jam vero, quoniam $a^z-1 = Az+Bz^2+Cz^3+Dz^4+Ez^5+\dots$ primus terminus tam quadrati $(a^z-1)^2$, quam ſingulorum productorum hujus quadrati per terminos a^z, a^{2z}, a^{3z}, a^{4z}.... eſt AAz^2. Coëfficiens autem primi termini Bz^2 ſecundi membri cujusvis æquationum præcedentium eſt differentia ſecunda quadratorum numerorum naturalium; & proinde (§. *q. Introd.*) quantitas conſtans 1.2. Hinc dividendo utrinque per z^2, & æquando invicem terminos conſtantes (juxta methodum reverſionis ſerierum), fit $AA = 1.2B$; & proinde $B = \frac{1}{1.2}AA$.

Suman-

Sumantur differentiæ tertiæ; erit (§. 5. *Introd.*)

$$(a^z-1)^3a^z \;=\Delta'''(4^3\ldots1^3)Cz^3+\Delta'''(4^4\ldots1^4)Dz^4+\Delta'''(4^5\ldots1^5)Ez^5+\Delta'''(4^6\ldots1^6)Fz^6+\ldots$$
$$(a^z-1)^3a^{2z}=\Delta'''(5^3\ldots2^3)Cz^3+\Delta'''(5^4\ldots2^4)Dz^4+\Delta'''(5^5\ldots2^5)Ez^5+\Delta'''(5^6\ldots2^6)Fz^2+\ldots$$
$$(a^z-1)^3a^{3z}=\Delta'''(6^3\ldots3^3)Cz^3+\Delta'''(6^4\ldots3^4)Dz^4+\Delta'''(6^5\ldots3^5)Ez^5+\Delta'''(6^6\ldots3^6)Fz^6+\ldots$$
$$(a^z-1)^3a^{4z}=\Delta'''(7^3\ldots4^3)Cz^3+\Delta'''(7^4\ldots4^4)Dz^4+\Delta'''(7^5\ldots4^5)Ez^5+\Delta'''(7^6\ldots4^6)Fz^6+\ldots$$

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Quoniam $a^z-1 = Az+Bz^2+Cz^3+Dz^4+Ez^5+\ldots\ldots$ primus terminus tam cubi $(a^z-1)^3$, quam ſingulorum productorum prædicti cubi per terminos a^z, a^{2z}, a^{3z}, a^{4z}. . . . eſt A^3z^3.

Coëfficiens autem primi termini Cz^3 ſecundi membri cujusvis æquationum præcedentium eſt differentia tertia cuborum numerorum naturalium; & proinde (§. 9. *Introd.*) quantitas conſtans $=1.2.3$: hinc eadem methodo, quá coëfficiens B fuit determinatus, fit $A^3=1.2.3C$; & $C=\frac{1}{1.2.3}A^3$.

Sumantur differentiæ quartæ; erit (§. 5. *Introd.*)

$$(a^z-1)^4a^z \;=\Delta^{IV}(5^4\ldots1^4)Dz^4+\Delta^{IV}(5^5\ldots1^5)Ez^5+\Delta^{IV}(5^6\ldots1^6)Fz^6+\Delta^{IV}(5^7\ldots1^7)Gz^7+\ldots$$
$$(a^z-1)^4a^{2z}=\Delta^{IV}(6^4\ldots2^4)Dz^4+\Delta^{IV}(6^5\ldots2^5)Ez^5+\Delta^{IV}(6^6\ldots2^6)Fz^6+\Delta^{IV}(6^7\ldots2^7)Gz^7+\ldots$$
$$(a^z-1)^4a^{3z}=\Delta^{IV}(7^4\ldots3^4)Dz^4+\Delta^{IV}(7^5\ldots3^5(Ez^5+\Delta^{IV}(7^6\ldots3^6)Fz^6+\Delta^{IV}(7^7\ldots3^7)Gz^7+\ldots$$
$$(a^z-1)^4a^{4z}=\Delta^{IV}(8^4\ldots4^4)Dz^4+\Delta^{IV}(8^5\ldots4^5)Ez^5+\Delta^{IV}(8^6\ldots4^6)Fz^6+\Delta^{IV}(8^7\ldots4^7)Gz^7+\ldots$$

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Quoniam $a^z-1 = Az+Bz^2+Cz^3+Dz^4+Ez^5+\ldots\ldots$ primus terminus tam quartæ poteſtatis $(a^z-1)^4$, quam ſingulorum productorum hujus poteſtatis per terminos a^z, a^{2z}, a^{3z}, a^{4z}, . . . , eſt A^4z^4.

Coëfficiens autem primi termini Dz^4 ſecundi membri cujusvis æquationum præcedentium eſt differentia quarta quartarum poteſtatum numerorum naturalium; & proinde (§. 9. *Introd.*) quantitas conſtans $=1.2.3.4$. Hinc priore methodo fit $A^4=1.2.3.4D$; & $D=\frac{1}{1.2.3.4}A^4$.

Eodem prorſus modo, ſumtis differentiis quintis, primus terminus omnium productorum $(a^z-1)^5a^{nz}$ eſt A^5z^5; & coëfficiens primi termini Ez^5 poſterio-

ris

ris membri ſingularum expreſſionum horum productorum eſt differentia quinta quintarum poteſtatum numerorum naturalium, nempe $1.2.3.4.5$. Hinc $A^5 = 1.2.3.4.5E$; & $E = \frac{1}{1.2\ldots5}A^5$.

Tum ſumtis differentiis ſextis, primus terminus omnium productorum $(a^z - 1)^6 a^{nz}$, eſt $A^6 z^6$. Coëfficiens autem termini Fz^6 eſt differentia ſexta ſextarum poteſtatum numerorum naturalium; & proinde (§. *q. Introd.*) quantitas conſtans $1.2\ldots6$. Hinc $A^6 = 1.2\ldots5.6F$; & $F = \frac{1}{1.2\ldots5.6}A^6$.

Eadem methodo determinantur coëfficientes potentiarum ſucceſſivarum ipſius z; unde conſequitur formula generalis.

$$a^z (= 1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + Fz^6 + \ldots\ldots)$$
$$= 1 + Az + \frac{1}{1.2}A^2z^2 + \frac{1}{1.2.3}A^3z^3 + \frac{1}{1.2..4}A^4z^4 + \frac{1}{1.2..5}A^5z^5 + \ldots\ldots$$

§. 56. Quantitas ideo exponentialis a^z exprimi poteſt juxta legem admodum regularem per exponentem z, & aliam quamdam quantitatem A, quam ab ipſa a pendere deinceps videbimus.

Si vero z ſit numerus negativus, erit eodem omnino modo $\frac{1}{a^z}(= a^{-z}) =$

$$= 1 - Az + \frac{1}{1.2}A^2z^2 - \frac{1}{1.2.3}A^3z^3 + \frac{1}{1.2\ldots4}A^4z^4 - \frac{1}{1.2\ldots5}A^5z^5 + \ldots\ldots$$

Obſervatio. Si Az unitate major non eſt: ſeries hæc non modo convergit; ſed etiam termini ipſius, inde a termino quocunque propoſito ſumti, celerius decreſcunt, quam termini progreſſionis alicujus geometricæ decreſcentis.

Etenim ſit $Az = 1$; ſintque tres coëfficientes ſucceſſivi $\frac{1}{1.2\ldots p}$, $\frac{1}{1.2\ldots p+1}$, $\frac{1}{1.2\ldots p+2}$: termini hi ſunt inter ſe, uti 1, $\frac{1}{p+1}$, $\frac{1}{p+1}\cdot\frac{1}{p+2}$. Jam vero $\frac{1}{p+1}\cdot\frac{1}{p+2} < \frac{1}{(p+1)^2}$; ergo termini hi velocius decreſcunt quam in progreſſione geometrica, cujus exponens $\frac{1}{p+1}$. Idemque tanto magis obtinet, ſi $Az < 1$.

Quodſi autem $Az > 1$. Sit p numerus integer non major ipſo Az, ſed illi proximus. Termini Az, $\frac{1}{1.2}A^2z^2$, $\frac{1}{1.2.3}A^3z^3$, $\frac{1}{1.2\ldots4}A^4z^4\ldots\frac{1}{1.2\ldots p}A^pz^p$ continue

tinue crescunt: qui vero hos sequuntur termini, continue decrescunt, & rapidius quidem quam in progressione geometrica. Etenim tres termini successivi, $\frac{1}{1.2\ldots p+m}A^{p+m}z^{p+m}$, $\frac{1}{1.2\ldots p+m+1}A^{p+m+1}z^{p+m+1}$, $\frac{1}{1.2\ldots p+m+2}A^{p+m+2}z^{p+m+2}$ sunt inter se uti quantitates 1, $\frac{Az}{p+m+1}$, $\frac{A^2z^2}{p+m+1.p+m+2}$. Quoniam autem $\frac{Az}{p+1} < 1$, tanto magis est $\frac{Az}{p+m+1} < 1$; pariterque $\frac{A^2z^2}{p+m+1.p+m+2} < \frac{A^2z^2}{(p+m+1)^2}$. Proinde termini successivi rapidius decrescunt quam in progressione geometrica, cujus exponens est numerus fractus $\frac{Az}{p+m+1}$. Nominatim prædicta series inde a termino $\frac{A^pz^p}{1.2\ldots p}$ rapidius decrescit quam in progressione geometrica, cujus exponens $\frac{Az}{p+1}$: itaque summa terminorum ab hoc termino decrescentium minor est quam $\frac{A^pz^p}{1.2\ldots p}\left[\frac{1}{1-\frac{Az}{p+1}}\right]$, seu $\frac{A^pz^p}{1.2\ldots p}\cdot\frac{p+1}{p+1-Az}$. Proinde utut magnus sit numerus Az, series hæc ultra certum limitem non excurrit.

Ceterum quod ad applicationes seriei hujus attinet; fatendum, eam usus facilis esse iis tantum casibus, quibus $Az \overline{\overline{<}} 1$.

§. 57. Applicationum quarundam hujus seriei gratia, quæ deinceps exponentur, è re erit in ejus naturam penitius inquirere.

Sit binomium $\left(1+\frac{Az}{n}\right)^n$; erit semper

$$\left(1+\frac{Az}{n}\right)^n = 1+\frac{n}{1}\frac{Az}{n}+\frac{n}{1}\cdot\frac{n-1}{2}\frac{A^2z^2}{n^2}+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\frac{A^3z^3}{n^3}+\frac{n}{1}\ldots\frac{n-3}{4}\frac{A^4z^4}{n^4}+\ldots\ldots$$

$$= 1+Az+\frac{1-\frac{1}{n}}{1.2}A^2z^2+\frac{1-\frac{1}{n}}{1.2}\cdot\frac{1-\frac{2}{n}}{3}A^3z^3+\frac{1-\frac{1}{n}}{1.2}\ldots\frac{1-\frac{3}{n}}{4}A^4z^4+\ldots\ldots$$

Sit autem n numerus positivus, qui major fieri possit quocunque numero proposito.

1°. Sit $Az = 1$. Unitas est limes factorum $1-\frac{1}{n}$, $1-\frac{2}{n}$, $1-\frac{3}{n}$, ... $1-\frac{p}{n}$, quorum numerus p; & proinde etiam unitas est limes producti

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$1-\frac{1}{n}.\,1-\frac{2}{n}.\,1-\frac{3}{n}.\,1-\frac{4}{n}.\;\ldots\;1-\frac{p}{n}$ (§. 21.); unde feries

$1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2\ldots4}+\ldots+\frac{1}{1.2\ldots p+1}$ eft limes feriei

$1+1+\frac{1-\frac{1}{n}}{1.2}+\frac{1-\frac{1}{n}}{1.2}.\frac{1-\frac{2}{n}}{3}+\frac{1-\frac{1}{n}}{1.2}\cdots\frac{1-\frac{3}{n}}{4}+\ldots\ldots\frac{1-\frac{1}{n}}{1.2}\cdots\frac{1-\frac{p}{n}}{p+1}.$

Atqui, quo major p, eo minores funt factores fequentes $1-\frac{p+m}{n}$; & eo majores funt denominatores $1.2\ldots p+m+1$ terminorum fucceffivorum; proinde, a fortiori, feries $1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2..4}+\frac{1}{1.2..5}+\ldots$ eft limes feriei

$1+1+\frac{1-\frac{1}{n}}{1.2}+\frac{1-\frac{1}{n}}{1.2}.\frac{1-\frac{2}{n}}{3}+\frac{1-\frac{1}{n}}{1.2}\cdots\frac{1-\frac{3}{n}}{4}+\frac{1-\frac{1}{n}}{1.2}\cdots\frac{1-\frac{4}{n}}{5}+\ldots$; feu quantitatis $\left(1+\frac{1}{n}\right)^n$.

2°. Hæc valent a fortiori, fi $Az < 1$.

3°. Sit $Az > 1$; & fit p numerus integer non major quam z, fed ei proximus.

Sumto termino $\frac{1-\frac{1}{n}}{1.2}\cdots\frac{1-\frac{p-1}{n}}{p}A^pz^p$; termini, qui illum fequuntur, rapidius decrefcunt quam in progreffione geometrica, cujus exponens $\frac{Az}{p+1}$: tum quia factores fucceffivi denominatoris $p+2$, $p+3$, $p+4\ldots p+m$ majores funt factore $p+1$; tum etiam quia factores numeratoris $1-\frac{p}{n}$, $1-\frac{p+1}{n}\ldots$ $1-\frac{p+n-1}{n}$ fiunt continue minores: & proinde, a fortiori, fumma omnium terminorum, inde a termino $\frac{1-\frac{1}{n}}{1.2}.\frac{1-\frac{2}{n}}{3}\ldots\ldots\frac{1-\frac{p-1}{n}}{p}A^pz^p$, minor eft quantitate

$\frac{1-\frac{1}{n}}{1.2}.\frac{1-\frac{2}{n}}{3}\ldots\ldots\frac{1-\frac{p-1}{n}}{p}A^pz^p\left(\frac{1}{1-\frac{Az}{p+1}}\right)$, cujus limes eft

$\frac{1}{1.2\ldots p}A^pz^p.\frac{1}{1-\frac{Az}{p+1}}$; tantoque magis fumma omnium terminorum inde a termino

mino $\frac{1-\frac{1}{n}}{1.2}\cdots\frac{1-\frac{p+m-1}{n}}{p+m}A^{p+m}z^{p+m}$ minor eſt quantitate

$\frac{1-\frac{1}{n}}{1.2}\cdots\frac{1-\frac{p+m-1}{n}}{p+m}A^{p+m}z^{p+m}\left(\frac{1}{1-\frac{Az}{p+m}}\right)$, ſeu $\frac{1-\frac{1}{n}}{1.2}\cdots\frac{1-\frac{p+m-1}{n}}{p+m-1}A^{p+m}z^{p+m}\cdot\frac{1}{p+m-Az}$,

cujus limes eſt $\frac{1}{1.2\ldots.p+m-1}A^{p+m}z^{p+m}\cdot\frac{1}{p+m-Az}$; quæ cum minor ſit quantitate $\frac{A^p z^p}{1.2\ldots p}\cdot\frac{Az}{p+m-Az}\cdot\frac{A^{m-1}z^{m-1}}{(p+1)^{m-1}}$, poteſt reddi minor quacunque quantitate propoſita.

Proinde tandem in omni caſu ſeries

$1+Az+\frac{1}{1.2}A^2z^2+\frac{1}{1.2.3}A^3z^3+\frac{1}{1.2\ldots4}A^4z^4+\ldots.$ limes eſt ſeriei, quæ oritur ex quantitate $\left(1+\frac{Az}{n}\right)^n$ evoluta, ſeu eſt limes hujus quantitatis.

§. 58. Quoniam a exponens eſt rationis $a:1$; a^z exponens eſt rationis $(a:1)^z$. Scilicet ratio $(a:1)^z$ componitur ex z rationibus ipſi rationi $a:1$ æqualibus; ſeu z metitur rationem $(a:1)^z$. Hinc z dicitur *logarithmus* rationis $(a:1)^z$; ſeu omiſſo conſequente 1, communi & conſtanti, z dicitur logarithmus ipſius a^z, quatenus logarithmus ipſius a eſt unitas. Quantitas a dicitur *baſis* logarithmica; quantitas A autem dicitur *modulus*. Diverſitas ſyſtematum logarithmicorum pendet a baſi a; proindeque etiam a modulo A.

Baſis itaque a modulo ita pendet, ut ſit

$$a=1+A+\frac{1}{1.2}A^2+\frac{1}{1.2.3}A^3+\frac{1}{1.2\ldots4}A^4+\frac{1}{1.2\ldots5}A^5+\frac{1}{1.2\ldots6}A^6+\ldots.$$

Ex. gr. Sit $A=1$; erit $a=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2\ldots4}+\frac{1}{1.2\ldots5}+\frac{1}{1.2\ldots6}+\ldots$

$$=2,\ 718\ 281\ 828\ 4\tfrac{5}{6}\tfrac{+}{-}.$$

Syſtema logarithmicum huic ſuppoſitioni reſpondens dicitur *ſyſtema naturale*, ſeu etiam hyperbolicum; atque inter mathematicos convenit, baſim a hujus ſyſtematis per litteram e deſignare.

§. 59. Quoniam

$$a^z = 1 + Az + \frac{A^2}{1.2}z^2 + \frac{A^3}{1.2.3}z^3 + \frac{A^4}{1.2...4}z^4 + \frac{A^5}{1.2..5}z^5 + \frac{A^6}{1.2...6}z^6 + \ldots$$

et $$a^{-z} = 1 - Az + \frac{A^2}{1.2}z^2 - \frac{A^3}{1.2.3}z^3 + \frac{A^4}{1.2...4}z^4 - \frac{A^5}{1.2...5}z^5 + \frac{A^6}{1.2...6}z^6 + \ldots$$

erit $$\frac{a^z + a^{-z}}{2} = 1 + \frac{A^2}{1.2}z^2 + \frac{A^4}{1.2..4}z^4 + \frac{A^6}{1.2...6}z^6 + \frac{A^8}{1.2...8}z^8 + \frac{A^{10}}{1.2..10}z^{10} + \ldots$$

$$\frac{a^z - a^{-z}}{2} = Az + \frac{A^3}{1.2.3}z^3 + \frac{A^5}{1.2..5}z^5 + \frac{A^7}{1.2...7}z^7 + \frac{A^9}{1.2...9}z^9 + \frac{A^{11}}{1.2..11}z^{11} + \ldots$$

Nominatim

$$\frac{e^z + e^{-z}}{2} = 1 + \frac{1}{1.2}z^2 + \frac{1}{1.2..4}z^4 + \frac{1}{1.2...6}z^6 + \frac{1}{1.2...8}z^8 + \frac{1}{1.2..10}z^{10} + \ldots$$

$$\frac{e^z - e^{-z}}{2} = z + \frac{1}{1.2.3}z^3 + \frac{1}{1.2..5}z^5 + \frac{1}{1.2...7}z^7 + \frac{1}{1.2...9}z^9 + \frac{1}{1.2..11}z^{11} + \ldots$$

§. 60. Quoniam

$$a^z = 1 + Az + \frac{1}{1.2}A^2z^2 + \frac{1}{1.2.3}A^3z^3 + \frac{1}{1.2...4}A^4z^4 + \frac{1}{1.2...5}A^5z^5 + \ldots .$$

Exponentibus differentialibus ſumendis erit

$$\frac{d\,a^z}{dz} = A + A^2z + \frac{1}{1.2}A^3z^2 + \frac{1}{1.2.3}A^4z^3 + \frac{1}{1.2...4}A^5z^4 + \frac{1}{1.2...5}A^6z^5 + \ldots .$$

$$= A(1 + Az + \frac{1}{1.2}A^2z^2 + \frac{1}{1.2.3}A^3z^3 + \frac{1}{1.2...4}A^4z^4 + \frac{1}{1.2...5}A^5z^5 + \ldots .$$

$= A \times a^z$. Hinc

$$\frac{d^2 a^z}{dz^2} = A^2 \times a^z$$

$$\frac{d^3 a^z}{dz^3} = A^3 \times a^z$$

$$\frac{d^4 a^z}{dz^4} = A^4 \times a^z$$

$$\frac{d^5 a^z}{dz^5} = A^5 \times a^z$$

$$\vdots \qquad \vdots$$

Pro-

Proinde exponentes differentiales omnium ordinum quantitatis exponentialis a^z exponentisque ejus z sequuntur progressionem geometricam, cujus exponens est ipsi A æqualis.

§. 61. Quoniam $a = 1 + A + \frac{A^2}{1.2} + \frac{A^3}{1.2.3} + \frac{A^4}{1.2...4} + \frac{A^5}{1.2...5} + \frac{A^6}{1.2...6} + \ldots$

seu $a - 1 = A + \frac{A^2}{1.2} + \frac{A^3}{1.2.3} + \frac{A^4}{1.2...4} + \frac{A^5}{1.2...5} + \frac{A^6}{1.2...6} + \ldots$

Adhibita methodo reversionis serierum invenitur modulus A per basim a expressus. Potior autem mihi videtur usus sequentis methodi, qua simul logarithmorum ipsorum expressiones consequimur.

Qnoniam $a^z = 1 + Az + \frac{A^2}{1.2}z^2 + \frac{A^3}{1.2.3}z^3 + \frac{A^4}{1.2...4}z^4 + \frac{A^5}{1.2...5}z^5 \ldots$

fiat $z = n\Delta z$.

Erit etiam $a^{\Delta z} = 1 + A\Delta z + \frac{A^2}{1.2}\Delta z^2 + \frac{A^3}{1.2.3}\Delta z^3 + \frac{A^4}{1.2..4}\Delta z^4 + \frac{A^5}{1.2..5}\Delta z^5 + \ldots$

$$a^z = (a^{n\Delta z}) = (1 + A\Delta z + \frac{A^2}{1.2}\Delta z^2 + \frac{A^3}{1.2.3}\Delta z^3 + \frac{A^4}{1.2..4}\Delta z^4 + \frac{A^5}{1.2..5}\Delta z^5 + \ldots)^n$$

Sit autem $(1 + A\Delta z + \frac{A^2}{1.2}\Delta z^2 + \frac{A^3}{1.2.3}\Delta z^3 + \frac{A^4}{1.2..4}\Delta z^4 + \frac{A^5}{1.2..5}\Delta z^5 + \ldots)^n = 1 + v$

erit $A\Delta z + \frac{A^2}{1.2}\Delta z^2 + \frac{A^3}{1.2.3}\Delta z^3 + \frac{A^4}{1.2..4}\Delta z^4 + \frac{A^5}{1.2..5}\Delta z^5 + \ldots) = (1+v)^{\frac{1}{n}} - 1$

et $An\Delta z(1 + \frac{A}{1.2}\Delta z + \frac{A^2}{1.2.3}\Delta z^2 + \frac{A^3}{1.2..4}\Delta z^3 + \frac{A^4}{1.2..5}\Delta z^4 + \ldots) = n\left((1+v)^{\frac{1}{n}} - 1\right)$

$$= v - \frac{1-\frac{1}{n}}{1.2}v^2 + \frac{1-\frac{1}{n}}{1.2}\cdot\frac{2-\frac{1}{n}}{3}v^3 - \frac{1-\frac{1}{n}}{1.2}\cdot\frac{2-\frac{1}{n}}{3}\cdot\frac{3-\frac{1}{n}}{4}v^4 + \frac{1-\frac{1}{n}}{1.2}\cdot\frac{2-\frac{1}{n}}{3}\cdot\frac{3-\frac{1}{n}}{4}\cdot\frac{4-\frac{1}{n}}{5}v^5 - \ldots$$

Sed $n\Delta z = z = \log. 1 + v$ (hyp.)

Ergo $A \log. 1 + v(1 + \frac{A}{1.2}\Delta z + \frac{A^2}{1.2.3}\Delta z^2 + \frac{A^3}{1.2..4}\Delta z^3 + \frac{A^4}{1.2..5}\Delta z^4 + \frac{A^5}{1.2..6}\Delta z^5 + \ldots)$

$$= v - \frac{1-\frac{\Delta z}{z}}{1.2}v^2 + \frac{1-\frac{\Delta z}{z}}{1.2}\cdot\frac{2-\frac{\Delta z}{z}}{3}v^3 - \frac{1-\frac{\Delta z}{z}}{1.2}\cdot\frac{2-\frac{\Delta z}{z}}{3}\cdot\frac{3-\frac{\Delta z}{z}}{4}v^4 + \frac{1-\frac{\Delta z}{z}}{1.2}\ldots\frac{4-\frac{\Delta z}{z}}{5}v^5 - \ldots)$$

Cum

Cum hæc æquatio ſemper locum habeat, limites etiam membrorum ejus ſunt inter ſe æquales (§. 4.) Sed hi limites ſunt $A \log. 1+v$, &

$$v - \tfrac{1}{2}v^2 + \tfrac{1}{3}v^3 - \tfrac{1}{4}v^4 + \tfrac{1}{5}v^5 - \ldots.$$

Ergo $A \log. 1+v = \quad v - \tfrac{1}{2}v^2 + \tfrac{1}{3}v^3 - \tfrac{1}{4}v^4 + \tfrac{1}{5}v^5 - \ldots..$

Hinc $A \log. 1-v = -(v + \tfrac{1}{2}v^2 + \tfrac{1}{3}v^3 + \tfrac{1}{4}v^4 + \tfrac{1}{5}v^5 + \ldots..)$

Unde $A \log. \sqrt{1-vv} = -(v^2 + \tfrac{1}{2}v^4 + \tfrac{1}{3}v^6 + \tfrac{1}{4}v^8 + \ldots..)$

et $A \log. \sqrt{\frac{1+v}{1-v}} = \quad (v + \tfrac{1}{3}v^3 + \tfrac{1}{5}v^5 + \tfrac{1}{7}v^7 + \ldots..)$

Non pertinet ad ſcopum præſentem, varias evolvere formulas variaque artificia, quibus (quamvis diu poſt inventos & computatos logarithmos) calculi logarithmorum compendia fuere ſubminiſtrata. Conſulantur hac de re ſcriptores, qui artificia illa exponere ſategerunt, quos inter ſagaciſſimus Bertrand in opere ſuo inſcripto: *Developement nouveau de la partie elementaire des Mathematiques.*

§. 62. Formulam $A \log. 1 + v = v - \tfrac{1}{2}v^2 + \tfrac{1}{3}v^3 - \tfrac{1}{4}v^4 + \tfrac{1}{5}v^5 - \ldots..$ conſequi potuiſſemus per prima calculi differentialis principia, ut ſequitur.

Quoniam $\log. a^z = z$; & $\frac{d.a^z}{dz} = A.a^z$; $\frac{d.a^z}{d.\log.z} = A.a^z$; & $\frac{d.\log.z}{d.a^z} = \frac{1}{A} \times \frac{1}{a^z}$.

Sit itaque $y = \log. 1 + v$.

Erit $\frac{dy}{dv} = \frac{1}{A} \cdot \frac{1}{1+v}$; & quoniam $y = 0$; quando $v = 0$, fit (§. 27.)

$y = \frac{1}{A}(v - \tfrac{1}{2}v^2 + \tfrac{1}{3}v^3 - \tfrac{1}{4}v^4 + \tfrac{1}{5}v^5 - \ldots.)$; quod etiam poteſt obtineri convertendo $\frac{1}{1+v}$ in ſeriem, & integrando.

Nempe: $\frac{dy}{dv} = \frac{1}{A}(1 - v + v^2 - v^3 + v^4 - v^5 + \ldots.)$

$y = \frac{1}{A}(v - \tfrac{1}{2}v^2 + \tfrac{1}{3}v^3 - \tfrac{1}{4}v^4 + \tfrac{1}{5}v^5 - \tfrac{1}{5}v^6 + \ldots.)$

ſeu $A \log. 1+v = \quad (v - \tfrac{1}{2}v^2 + \tfrac{1}{3}v^3 - \tfrac{1}{4}v^4 + \tfrac{1}{5}u^5 - \tfrac{1}{6}v^6 + \ldots.)$.

Corollarium. Quantitate v manente eadem, productum $A \log. 1 + v$ ejusdem etiam magnitudinis manet. Proinde in diverſis ſyſtematibus, logarithmi unius

unius ejusdemque numeri sunt in ratione inversa modulorum horum systematum.

Est etiam $A \log . \sqrt{\frac{1+v}{1-v}} = v + \frac{1}{3}v^3 + \frac{1}{5}v^5 + \frac{1}{7}v^7 + \ldots\ldots$

Sit $\frac{1+v}{1-v} = a^{2z}$: unde $v = \frac{a^{2z}-1}{a^{2z}+1}$; & $A \log . \sqrt{\frac{1+v}{1-v}} = A \log . a^z$.

Erit $A \log . a^z = \frac{a^{2z}-1}{a^{2z}-1} + \frac{1}{3}\left(\frac{a^{2z}-1}{a^{2z}+1}\right)^3 + \frac{1}{5}\left(\frac{a^{2z}-1}{a^{2z}+1}\right)^5 + \frac{1}{7}\left(\frac{a^{2z}-1}{a^{2z}+1}\right)^7 + \ldots\ldots$

Sit $z = 1$; erit $A \log . a = \frac{a^2-1}{a^2+1} + \frac{1}{3}\left(\frac{a^2-1}{a^2+1}\right)^3 + \frac{1}{5}\left(\frac{a^2-1}{a^2+1}\right)^5 + \frac{1}{7}\left(\frac{a^2-1}{a^2+1}\right)^7 + \ldots$

Sit b basis systematis logarithmici; unde $\log . a = \log . b = 1$;

$$A = \frac{bb-1}{bb+1} + \frac{1}{3}\left(\frac{bb-1}{bb+1}\right)^3 + \frac{1}{5}\left(\frac{bb-1}{bb+1}\right)^5 + \frac{1}{7}\left(\frac{bb-1}{bb+1}\right)^7 + \ldots\ldots$$

Observatio. Modulus cujusvis systematis est logarithmus naturalis baseos hujus systematis. Etenim quoniam modulus logarithmorum naturalium seu hyperbolicorum est unitas, erit $A \log . b = \log . \text{hyp} . b$. Sed $\log . b = 1$; ergo $A = \log . \text{hyp} . b$.

Ex. gr. Basis logarithmorum vulgarium seu Briggianorum est 10. Logarithmus hyperbolicus ipsius 10 est 2, 302 $585^{\,0\,+}_{\,1\,-}$. Hinc modulus logarithmorum vulgarium est 2, 302 $585^{\,0\,+}_{\,1\,-}$. Proinde si logarithmi hyperbolici per hunc numerum dividantur, oriuntur logarithmi vulgares; & vicissim, si logarithmi vulgares per eundem numerum multiplicentur, oriuntur logarithmi hyperbolici. Cum autem logarithmorum hyperbolicorum usus sit quam frequentissimus; optandum sane foret, ut tabulæ horum logarithmorum in promtu essent non minus extensæ quam tabulæ vulgares.

§. 63. Postquam logarithmorum theoriam ex contemplatione progressionum geometricarum primariisque earundem proprietatibus deduxi, atque ita ad calculum elementarem revocavi: non inutile censeo in gratiam tironum idem subjectum paulo aliter tractare; strictim ea exponendo, quæ in opusculo *Exposition elementaire* &c. de curva logarithmica fusius tradidi.

Definitio. Curva logarithmica ſeu logiſtica ea eſt, in qua, abſciſſis in progreſſione arithmetica ſumptis, rectæ axi ordinatim applicatæ geometricam ſequuntur progreſſionem..

Corollarium. Ex definitione immediate ſequitur: partem axis abſciſſarum, inter duas rectas axi ordinatim applicatas comprehenſam, menſuram eſſe rationis harum ordinatarum; ita ut, differentia abſciſſarum manente eadem, ratio ordinatarum pariter eadem ſit; & viciſſim, ratione ordinatarum manente eadem, differentia abſciſſarum pariter eadem maneat. Si vero pars axis inter duas ordinatas ſit dupla, tripla, quadrupla, quintupla, - - - n^tupla; etiam ratio duarum ordinatarum eſt duplicata, triplicata, quadruplicata, quintuplicata, - - - n^plicata.

Quibus poſitis, ſint duæ ordinatæ, quarum ratio datur; & per puncta extrema harum ordinatarum ducatur recta, quæ axi occurrat, ſeu linea ſecans. Dico: ſubſecantem eſſe etiam datæ magnitudinis.

Fig. 16. Sint nimirum MP, $M'P'$ duæ rectæ axi ordinatim applicatæ. Ducatur MM', quæ axi occurrat in S. Sit ratio $MP : M'P'$ æqualis alicui rationi datæ; dico, lineam PS etiam eſſe magnitudine datam.

Ducatur $M'm$ axi parallela, quæ ordinatæ MP in m occurrat.

Quoniam ratio $MP : M'P'$ datur; ratio $MP : MP - M'P'$ ſeu $MP : Mm$ pariter datur. Sed $MP : Mm = PS : M'm$; ergo & ratio $PS : M'm$ datur. Sed propter rationem datam $MP : M'P'$, PP' ſeu $M'm$ datur magnitudine; ergo etiam PS datur magnitudine.

Atqui (§. 40.) ſubtangens cujusvis curvæ eſt limes ſubſecantis; ergo etiam, ſi a quocunque puncto logarithmicæ ducatur tangens, quæ axi occurrat, ſubtangens eſt datæ magnitudinis. Quæ propoſitio (ſi opus foret) immediate etiam poſſet demonſtrari, evincendo (per abſurdum) impoſſibilitatem inæqualitatis ſubtangentium diverſis logarithmicæ punctis reſpondentium.

Sit ideo $MP = y$, $AP = x$; & ſit t ſubtangens conſtans: erit $t = y\frac{dx}{dy}$, & $\frac{dx}{dy} = \frac{t}{y}$. Unde ſi $y = 1 + v$, fit

$$\frac{dx}{dy} = t . \frac{1}{1+v} = t(1 - v + v^2 - v^3 + v^4 - v^5 + v^6 - \ldots .)$$

$$\& \; x \qquad = t(v - \tfrac{1}{2}v^2 + \tfrac{1}{3}v^3 - \tfrac{1}{4}v^4 + \tfrac{1}{5}v^5 - \tfrac{1}{6}v^6 + \tfrac{1}{7}v^7 - \ldots .)$$

Pro-

Proinde in diverſis curvis logarithmicis logarithmi ejusdem rationis ſunt in ratione ſubtangentium harum curvarum; & ſubtangentes inverſe ſunt ut moduli ſyſtematum logarithmicorum hisce curvis reſpondentium.

§. 64. Ex æquatione differentiali curvæ logarithmicæ $t = y\frac{dx}{dy}$; ſeu $\frac{dy}{dx} = \frac{y}{t}$ ſequitur

$$\frac{ddy}{dx^2} = \frac{\frac{dy}{dx}}{t} = \frac{y}{t^2}$$

$$\frac{d^3y}{dx^3} = \frac{\frac{dy}{dx}}{t^2} = \frac{y}{t^3}$$

$$\frac{d^4y}{dx^4} = \frac{\frac{dy}{dx}}{t^3} = \frac{y}{t^4}$$

$$\frac{d^5y}{dx^5} = \frac{\frac{dy}{dx}}{t^4} = \frac{y}{t^5}$$

$$\cdot \qquad \cdot \qquad \cdot$$

$$\cdot \qquad \cdot \qquad \cdot$$

Fiat $x = x + \Delta x$

& proinde $y = y + \Delta y$.

Erit $y + \Delta y = y + \frac{\Delta x}{1}\frac{dy}{dx} + \frac{\Delta x^2}{1.2}\frac{ddy}{dx^2} + \frac{\Delta x^3}{1.2.3}\frac{d^3y}{dx^3} + \frac{\Delta x^4}{1.2 \ldots 4}\frac{d^4y}{dx^4} + \ldots$ (§. 32.)

$$= y + \frac{\Delta x}{1}\frac{y}{t} + \frac{\Delta x^2}{1.2}\frac{y}{t^2} + \frac{\Delta x^3}{1.2.3}\frac{y}{t^3} + \frac{\Delta x^4}{1.2 \ldots 4}\frac{y}{t^4} + \ldots$$

Atqui quoniam $y = e^x$; & $y + \Delta y = e^{x + \Delta x} = y \times e^{\Delta x}$,

erit $e^{\Delta x} = 1 + \frac{\Delta x}{1} \cdot \frac{1}{t} + \frac{\Delta x^2}{1.2} \cdot \frac{1}{t^2} + \frac{\Delta x^3}{1.2.3} \cdot \frac{1}{t^3} + \frac{\Delta x^4}{1.2 \ldots 4} \cdot \frac{1}{t^4} + \frac{\Delta x^5}{1.2 \ldots 5} \cdot \frac{1}{t^5} + \ldots$

Univerſim igitur

$$e^z = 1 + \frac{z}{1} \cdot \frac{1}{t} + \frac{z^2}{1.2} \cdot \frac{1}{t^2} + \frac{z^3}{1.2.3} \cdot \frac{1}{t^3} + \frac{z^4}{1.2 \ldots 4} \cdot \frac{1}{t^4} + \frac{z^5}{1.2 \ldots 5} \cdot \frac{1}{t^5} + \ldots$$

$= 1 + \frac{1}{1} \cdot \frac{z}{t} + \frac{1}{1.2} \cdot \frac{z^2}{t^2} + \frac{1}{1.2.3} \cdot \frac{z^3}{t^3} + \frac{1}{1.2 \ldots 4} \cdot \frac{z^4}{t^4} + \frac{1}{1.2 \ldots 5} \cdot \frac{z^5}{t^5} + \ldots$ (ut prius §. 55.)

§. 65. Quemadmodum curva logarithmica apta fuit ad theoriam logarithmorum illuſtrandam; ita & aliæ fingi poſſunt curvæ, quæ eidem ſcopo reſponderent. Celebris ob inſignes ſuas proprietates apud Geometras eſt curva *ſpiralis logarithmica* dicta. In hac curva angulus inter duos radios vectores comprehenſus eſt menſura rationis horum radiorum; unde ſequitur, data duorum radiorum ratione, ſpecie etiam dari triangulum his radiis vectoribus & chorda ipſorum extrema jungente comprehenſum. Proinde data ratione duorum radiorum; ſecans ſpiralis logarithmicæ, per puncta extrema horum radiorum ducta, inclinatur ſub angulo dato alterutri horum radiorum; unde concluditur (per §. 4.) angulum, quem tangens ſpiralis logarithmicæ facit cum radio ad punctum contactus ducto, conſtantem eſſe in eadem ſpirali logarithmica. Et diverſitas hujus anguli determinat ſyſtema logarithmicum ſpirali huic reſpondens. In hyperbola conica ad aſymptotos relata, ſpatia hyperbolica inter duas ordinatas comprehenſa creſcunt etiam ut logarithmi rationum abſciſſarum. (Vide *Caput X.*)

§. 66. Ex præcedentibus determinantur rationes differentiales quarumlibet quantitatum exponentialium, quod nonnullis exemplis oſtendam.

1°. Sit $P = x^m e^x$; erit

$$\frac{dP}{dx} = e^x (m x^{m-1} + x^m)$$

$$\frac{ddP}{dx^2} = e^x (m.m\text{-}1 x^{m-2} + 2m x^{m-1} + x^m)$$

$$\frac{d^3P}{dx^3} = e^x (m.m\text{-}1.m\text{-}2 x^{m-3} + 3m.m\text{-}1 x^{m-2} + 3m.x^{m-1} + x^m)$$

$$\frac{d^4P}{dx^4} = e^x (m \ldots m\text{-}3 x^{m-4} + 4m \ldots m\text{-}2 x^{m-3} + 6m.m\text{-}1 x^{m-2} + 4m x^{m-1} + x^m)$$

$$\frac{d^5P}{dx^6} = e^x (m \ldots m\text{-}4 x^{m-5} + 5m \ldots m\text{-}3 x^{m-4} + 10m \ldots m\text{-}2 x^{m-3} + 10m.m\text{-}1 x^{m-2} + 5m x^{m-1} + x^m)$$

$$\vdots$$

$$\frac{d^nP}{dx^n} = e^x \left(m \ldots m\text{-}(n\text{-}1) x^{m-n} + \frac{n}{1}.m \ldots m\text{-}(n\text{-}2) x^{m-n-1} + \frac{n}{1}.\frac{n-1}{2} m \ldots m\text{-}(n\text{-}3) x^{m-n-2} + \ldots \frac{n}{1} m x^{m-1} + x^m\right)$$

2°. Sit

2°. Sit $P = x^y$. $\quad \text{Log.}\, P = y \log. x$

$$\frac{dP}{dx} = P\frac{dy}{dx}\log. x + Py\frac{1}{x} = x^y\frac{dy}{dx}\log. x + yx^{y-1}.$$

3°. Sit $P = \log.^m x$;

$$\frac{dP}{dx} = m\log.^{m-1}x\,.\,\frac{1}{x}$$

$$\frac{ddP}{dx^2} = m.m-1\log.^{m-2}x\,.\,\frac{1}{xx} - m\log.^{m-1}x\,.\,\frac{1}{xx}$$

$$\frac{d^3P}{dx^4} = m.m-1.m-2\log.^{m-3}x\,.\,\frac{1}{x^3} - 3m.m-1\log.^{m-2}x\,.\,\frac{1}{x^3} + 2m.\log.^{m-1}x\,.\,\frac{1}{x^3}$$

&c. &c. &c.

§. 67. Suppofita formula differentiali $\frac{d\log. x}{dx} = \frac{1}{x}$: ex illa deduci poffent aliæ formulæ differentiales quantitatum non exponentialium.

Exempla.

1°. Sit $P = xy$

$$\log. P = \log. x + \log. y$$

$$\frac{dP}{dx}.\frac{1}{P} = \frac{1}{x} + \frac{dy}{dx}.\frac{1}{y}$$

$$\frac{dP}{dx} = y + x\frac{dy}{dx}.$$

2°. Sit $P = xyzv \ldots.$

$$\log. P = \log. x + \log. y + \log. z + \log. v$$

$$\frac{dP}{dx}.\frac{1}{P} = \frac{1}{x} + \frac{dy}{dx}.\frac{1}{y} + \frac{dz}{dx}.\frac{1}{z} + \frac{dv}{dx}.\frac{1}{v}.$$

$$\frac{dP}{dx} = yzv + \frac{dy}{dx}.xzv + \frac{dz}{dx}.xyv + \frac{dv}{dx}xyz.$$

3°. Sit $P = \frac{x}{y}$.

$$\log. P = \log. x - \log. y$$

$$\frac{dP}{dx}.\frac{1}{P} = \frac{1}{x} - \frac{dy}{dx}.\frac{1}{y}$$

$$\frac{dP}{dx} = \frac{1}{y} - x\frac{dy}{dx}.\frac{1}{yy} = \frac{y - x\frac{dy}{dx}}{yy}.$$

4°. Sit $P = (a+x)^m$

$$\log.P = m\log.(a+x)$$

$$\frac{dP}{dx}\cdot\frac{1}{P} = m.\frac{1}{a+x}$$

$$\frac{dP}{dx} = m.\frac{P}{a+x} = m(a+x)^{m-1}.$$

§. 68. Ad formulas differentiales, capite hoc traditas, reducuntur plurimæ aliæ formulæ differentiales; quæ proinde integrabiles cenfentur, tabulis logarithmicis in promtu habitis: quod paucis exemplis oftendam.

Sit $\frac{dy}{dx}=\sqrt{(xx-aa)}$; fiat $xx-aa=(z-x)^2$; hinc $x=\frac{zz+aa}{2z}$; $z-x=\frac{zz-aa}{2z}$;

$\frac{dx}{dz}=\frac{1}{2}\left(\frac{zz-aa}{zz}\right)$; hinc $\frac{dy}{dz}=\frac{1}{4}\frac{(zz-aa)^2}{z^3}=\frac{1}{4}z-\frac{1}{2}aa\frac{1}{z}+\frac{1}{4}\frac{a^4}{z^3}$;

unde $y = \frac{1}{8}zz-\frac{1}{2}aa\log.z-\frac{1}{8}\frac{a^4}{zz}+C$

$$= \frac{1}{8}(x+\sqrt{(xx-aa)})^2-\frac{1}{2}aa\log.(x+\sqrt{xx-aa})-\frac{1}{8}\frac{a^4}{(x+\sqrt{(xx-aa)})^2}+C$$

$$= \frac{1}{8}(x+\sqrt{(xx-aa)})^2-\frac{1}{2}aa\log.(x+\sqrt{xx-aa})-\frac{1}{8}(x-\sqrt{xx-aa})^2+C$$

$$= \frac{1}{2}x\sqrt{(xx-aa)} \quad -\frac{1}{2}aa\log.(x+\sqrt{(xx-aa)})+C$$

Si exponens integralis evanefcat facto $x=a$, erit $C=+\frac{1}{2}aa\log.a$

$$\& \quad y = \frac{1}{2}x\sqrt{((xx-aa))}-\frac{1}{2}aa\log.\frac{x+\sqrt{(xx-aa)}}{a}$$

$$= \frac{1}{2}x\sqrt{(xx-aa)} - \frac{1}{2}aa\log.\frac{a}{x-\sqrt{(xx-aa)}}.$$

Sit $\frac{dy}{dx} = \sqrt{(xx+aa)}$;

fit eodem modo; $y = \frac{1}{2}x\sqrt{(xx+aa)} - \frac{1}{2}aa\log.\frac{a}{\sqrt{(xx+aa)}}$.

Ad has formulas reducuntur aliæ plurimæ, quales funt $\frac{dy}{dx} = x^{2m}\sqrt{(xx\pm aa)}$, quibus m eft numerus integer.

Formulæ autem $\frac{dy}{dx} = x^{2m+1}\sqrt{(xx\pm aa)}$ integratio obtineri poteft immediate; neque pendet a logarithmis.

§. 69.

§. 69. Formulæ differentiales logarithmicæ possunt etiam quantitatum in series conversionem reddere faciliorem.

Exempla. Ponamus legem, quam coëfficientes potestatum binomii sequuntur, nobis esse adhuc ignotam; & sit

$$(1+x)^m = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 + Gx^7 + \dots$$

Sumendis logarithmis erit

$$m \log. 1+x = \log.(1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 + Gx^7 + \dots$$

Sumendis exponentibus differentialibus erit

$$m.\frac{1}{1+x} = \frac{A+2Bx+3Cx^2+4Dx^3+5Ex^4+6Fx^7+7Gx^6+\dots}{1+Ax+Bx^2+Cx^3+Dx^4+Ex^5+Fx^6+Gx^7+\dots}$$

$$\text{et } m(1+Ax+Bx^2+Cx^3+Dx^4+Ex^5+Fx^6+Gx^7+\dots)$$
$$=(1+x)(A+2Bx+3Cx^2+4Dx^3+5Ex^4+6Fx^5+7Gx^6+\dots)$$

Unde æquando coëfficientes potestatum similium juxta methodum reversionis serierum fit

$$\begin{array}{lll}
A = m & \text{unde} & A = \frac{m}{1} \\
2B+A = mA & & B = \frac{m}{1}.\frac{m-1}{2} \\
3C+2B = mB & & C = \frac{m}{1}..\frac{m-2}{3} \\
4D+3C = mC & & D = \frac{m}{1}...\frac{m-3}{4} \\
5E+4D = mD & & E = \frac{m}{1}....\frac{m-4}{5} \\
6F+5E = mE & & F = \frac{m}{1}.....\frac{m-5}{6} \\
7G+6F = mF & & G = \frac{m}{1}......\frac{m-6}{7} \\
- \; - \; - \; - & & - \; - \; - \; - \\
- \; - \; - \; - & & - \; - \; - \; -
\end{array}$$

(conformiter §. 9. *Introd.*)

2°. Eodem modo fit $(1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \dots)^m$
$$= 1 + A'x + B'x^2 + C'x^3 + D'x^4 + E'x^5 + \dots$$

Erit

$$\text{Erit}\quad m\log.(1+Ax+Bx^2+Cx^3+Dx^4+Ex^5+\ldots$$
$$=\log.(1+A'x+B'x^2+C'x^3+D'x^4+E'x^5+\ldots$$
$$\text{atque}\quad m\times\frac{A+2Bx+3Cx^2+4Dx^3+5Ex^4+\ldots}{1+Ax+Bx^2+Cx^3+Dx^4+\ldots}$$
$$=\frac{A'+2B'x+3C'x^2+4D'x^3+5E'x^4+\ldots}{1+A'x+B'x^2+C'x^3+D'x^4+\ldots}$$

Unde coëfficientes A', B', C', D', $E'\ldots$ facilius quam aliis methodis determinari poſſunt

3°. Sit $a^x = 1+Ax+Bx^2+Cx^3+Dx^4+Ex^5+\ldots$

$$x\log.a=\log.1+Ax+Bx^2+Cx^3+Dx^4+Ex^5+\ldots$$
$$\log.a=\frac{A+2Bx+3Cx^2+4Dx^3+5Ex^4+\ldots}{1+Ax+Bx^2+Cx^3+Dx^4+Ex^5+\ldots}$$
$$\log.a\ (1+Ax+Bx^2+Cx^3+Dx^4+Ex^5+\ldots)$$
$$=A+2Bx+3Cx^2+4Dx^3+5Ex^4+\ldots$$

Unde $A=\log.a$ ſeu $A=\log.a$

$B=A\frac{\log.a}{2}$ $\quad B=\frac{1}{1.2}\log.^2a$

$C=B\frac{\log.a}{3}$ $\quad C=\frac{1}{1.2.3}\log.^3a$

$D=C\frac{\log.a}{4}$ $\quad D=\frac{1}{1.2\ldots4}\log.^4a$

$E=D\frac{\log.a}{5}$ $\quad E=\frac{1}{1.2\ldots5}\log.^5a$

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$$a^x=1+x\log.a+\frac{x^2}{1.2}\log.^2a+\frac{x^3}{1.2..3}\log.^3a+\frac{x^4}{1.2..4}\log.^4a+\frac{x^5}{1.2..5}\log.^5a+\ldots\ldots$$

(ut prius §. 55.)

4°. Sit $\log.1+x=Ax+Bx^2+Cx^3+Dx^4+Ex^5+\ldots$

erit $\frac{1}{1+x}=A+2Bx+3Cx^2+4Dx^3+5Ex^4+\ldots$

$$1=A+x\begin{array}{l}A\\+2B\end{array}+x^2\begin{array}{l}2B\\+3C\end{array}+x^3\begin{array}{l}3C\\+4D\end{array}+x^4\begin{array}{l}4D\\+5E\end{array}+\ldots$$

Unde

Unde $A = +1$

$B = -\frac{1}{2}$

$C = +\frac{1}{3}$

$D = -\frac{1}{4}$

$E = +\frac{1}{5}$.

Et $\log. 1+x = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \ldots$ (ut prius §. 61.)

§. 70. Dixi (§. 78.), problema inverſum tangentium ad formulas differentiales logarithmicas ſæpe etiam in caſibus ſimpliciſſimis reduci.

Exempla. 1°. Quæratur curva, cujus ſubtangens datam habet ad abſciſſam rationem.

Erit ideo $y\frac{dx}{dy} = mx.$

hinc $\frac{dy}{dx}.\frac{1}{y} = \frac{1}{mx} = \frac{1}{m}.\frac{1}{x}$

ſeu $\frac{dy}{dv}.\frac{1}{y} = \frac{1}{m}\frac{dx}{dv}.\frac{1}{x}$

hinc $\log.\frac{y}{v} = \frac{1}{m}\log.\frac{x}{v} + C$

$m\log.y = \log.x + C$

$y^m = Cx$, quæ eſt æquatio ad parabolam, ſi m ſit quantitas poſitiva ab unitate diverſa; ad lineam rectam, ſi $m=1$; & ad hyperbolas; ſi m fuerit negativa, ſeu ſi ſubtangens & abſciſſa ad diverſas ordinatæ partes ſitæ ſint.

2°. Quæritur curva, cujus ſubtangens eſt conſtans.

Erit ideo $y\frac{dx}{dy} = c$; $\frac{dy}{dx}.\frac{1}{y} = \frac{1}{c}$; $\log.y = \frac{x}{c}$, quæ eſt æquatio ad logarithmicam.

3°. Quæritur curva, cujus tangens datur magnitudine.

Sit ideo $y\sqrt{(1+\frac{dx^2}{dy^2})} = t$; $1+\frac{dx^2}{dy^2} = \frac{tt}{yy}$; $\frac{dx^2}{dy^2} = \frac{tt-yy}{yy}$.

$$\frac{dx}{dy} = \frac{\sqrt{(tt-yy)}}{y} = \frac{tt}{y\sqrt{(tt-yy)}} - \frac{y}{\sqrt{(tt-yy)}};$$

$x = C + t\log.\frac{y}{t+\sqrt{(tt-yy)}} + \sqrt{(tt-yy)}$, quæ eſt æquatio ad curvam tractoriam dictam.

 §. 71.

§. 71. Priusquam huic capiti coronidem imponam; paucissimis attingam quæstionem inter mathematicos nimium celebrem, de natura logarithmorum quantitatum negativarum: quæ quæstio nequidem mota fuisset, si algebristæ a legitima quantitatum homogenearum definitione Euclidea non recessissent. Scilicet: *rationem inter se magnitudines habere, dicuntur, quæ multiplicatæ se invicem superare possunt.* Jam vero quantitas (ita dicta) negativa, quotiescunque multiplicetur (hoc est juxta genuinam multiplicationis notionem, quotiescunque sibi ipsi addatur, seu repetatur), nunquam superabit quantitatem positivam. Ergo nulla datur ratio inter quantitates (sic dictas) positivas & negativas.

Nodus autem solutionis hic est. Algebristæ ad quantitates ipsas transtulerunt, quæ ad signa tantum pertinebant: omnes quantitates in se spectatæ sunt positivæ, sed tantum aliæ aliis sunt oppositæ: hanc oppositionem per signa +, —, designatam non licebat ad quantitates ipsas transferre, & quasi duos ordines quantitatum hoc respectu constituere. Quæstionem hanc lucide admodum enodavit, & ad genuina principia revocavit illustris KÆSTNERUS (*Leipziger Magazin für die reine und angewandte Mathematik*, 1786. *4tes Stück*): qui (contra opinionem tum JOH. BERNOULLI, tum & D. D'ALEMBERT) LEIBNITZII & EULERI sententiæ assentitur; nempe, logarithmos quantitatum (ita dictarum) negativarum esse imaginarios; seu, nullam esse rationem inter negativas (quas vocant) quantitates & positivas. Quod attinet ad computum horum logarithmorum imaginariorum, vid. EULER *Memoires de Berlin* 1749.

CAPUT SEPTIMUM.

De functionibus trigonometricis arcuum circularium.

§. 72.

Analogia, quæ inter quantitates exponentiales a^z, & functiones trigonometricas arcuum circularium, quales sunt sinus, cosinus, tangentes &c. intercedit, jam dudum fuit a mathematicis observata. Fundamenta autem analogiæ hujus methodo satis elementari & lucida (quod sciam) exposita non fuerunt; qua id me præstitisse arbitror, sinus & cosinus cujusvis arcus circularis expressio-

pressionem per hunc arcum iisdem omnino vestigiis indagando, quibus in præcedente capite insistebam.

Cum sin. z ejusmodi sit ipsius z functio, quæ evanescit posito $z=0$; & ratio æqualitatis limes sit rationis decrescentis, quæ arcum inter & sinum ejus intercedit: sinus cujusvis arcus z est functio ipsius z, talis - - - -

$$\text{sin.}\, z = z - Az^2 + Bz^3 + Cz^4 + Dz^5 + Ez^6 + \dots$$

Et quoniam cos. z est functio ipsius z talis, ut fiat 1, quando $z=0$; minor autem sit unitate, quando z crescit ab zero inde usque ad quadrantem: erit cos. z functio ipsius z talis - - - -

$$\text{cos.}\, z = 1 - A'z + B'z^2 + C'z^3 + D'z^4 + E'z^5 + \dots$$

§. 73. Sit itaque

$$\begin{array}{rl}
 & \text{sin.}\, z = z - Az^2 + Bz^3 + Cz^4 + Dz^5 + Ez^6 + \dots \\
\text{erit} & \text{sin.}\, 2z = 2z - 2^2Az^2 + 2^3Bz^3 + 2^4Cz^4 + 2^5Dz^5 + 2^6Ez^6 + \dots \\
 & \text{sin.}\, 3z = 3z - 3^2Az^2 + 3^3Bz^3 + 3^4Cz^4 + 3^5Dz^5 + 3^6Ez^6 + \dots \\
 & \text{sin.}\, 4z = 4z - 4^2Az^2 + 4^3Bz^3 + 4^4Cz^4 + 4^5Dz^5 + 4^6Ez^6 + \dots \\
 & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
 & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{array}$$

Sumtis differentiis primis erit (§. *1. Introd.*)

$$\begin{array}{l}
2\,\text{sin.}\tfrac{1}{2}z\,\text{cos.}\tfrac{3}{2}z = z - (2^2-1)Az^2 + (2^3-1)Bz^3 + (2^4-1)Cz^4 + (2^5-1)Dz^5 + (2^6-1)Ez^6 + \dots \\
2\,\text{sin.}\tfrac{1}{2}z\,\text{cos.}\tfrac{5}{2}z = z - (3^2-2^2)Az^2 + (3^3-2^3)Bz^3 + (3^4-2^4)Cz^4 + (3^5-2^5)Dz^5 + (3^6-2^6)Ez^6 + \dots \\
2\,\text{sin.}\tfrac{1}{2}z\,\text{cos.}\tfrac{7}{2}z = z - (4^2-3^2)Az^2 + (4^3-3^3)Bz^3 + (4^4-3^4)Cz^4 + (4^5-3^5)Dz^5 + (4^6-3^6)Ez^6 + \dots \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{array}$$

Jam vero in his æquationibus primus terminus omnium membrorum priorum, si juxta suppositionem §. 72. in series explicentur, est $1z$; pariter atque primus terminus omnium membrorum posteriorum, qui igitur per se congruunt.

Sumtis differentiis secundis erit (§. *1. Introd.*)

$$\begin{array}{l}
-2^2\,\text{sin.}^2\tfrac{1}{2}z\,\text{sin.}\,2z = -\Delta''(3^2\dots1^2)Az^2 + \Delta''(3^3\dots1^3)Bz^3 + \Delta''(3^4\dots1^4)Cz^4 + \dots \\
-2^2\,\text{sin.}^2\tfrac{1}{2}z\,\text{sin.}\,3z = -\Delta''(4^2\dots2^2)Az^2 + \Delta''(4^3\dots2^3)Bz^3 + \Delta''(4^4\dots2^4)Cz^4 + \dots \\
-2^2\,\text{sin.}^2\tfrac{1}{2}z\,\text{sin.}\,4z = -\Delta''(5^2\dots3^2)Az^2 + \Delta''(5^3\dots3^3)Bz^3 + \Delta''(5^4\dots3^4)Cz^4 + \dots \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{array}$$

 Atqui

Atqui primus terminus omnium membrorum priorum, ſi juxta §. 72. in ſeries explicentur, continet tertiam poteſtatem z^3 arcus mutabilis z; ita ut membra hæc evoluta ſecundam arcus hujus poteſtatem z^2 non contineant: proinde poſteriora etiam æquationum harum membra poteſtatem z^2 non continebunt. Sed in poſterioribus membris coëfficiens poteſtatis ſecundæ z^2 eſt $\Delta'' n^2 A = 1.2A$: proinde $1.2A = 0$, & $A = 0$.

Eodem modo demonſtratur in ſerie $\text{ſin.}\, z = z - Az^2 + Bz^3 + Cz^4 + Dz^5 + \ldots$ omnes deeſſe arcus mutabilis poteſtates exponentis paris. Scilicet: ſumtis differentiis ordinis $2m$, quæ ſunt $\pm\, 2^{2m}\text{ſin.}^{2m}\tfrac{1}{2}z\,\text{ſin.}\,pz$ (§. *t. Introd.*), primus terminus omnium membrorum priorum continet poteſtatem z^{2m+1}; ita ut in his membris deſit poteſtas par z^{2m}.

At vero primi termini omnium membrorum poſteriorum continent hanc poteſtatem parem, affectam coëfficiente, cujus unus factor eſt conſtans $\Delta^{2M} n^{2m} = 1.2.3\ldots 2m$ (§. *q. Introd.*). Quare cum potentia hæc exponentis paris ex ſecundis etiam membris debeat exulare, alter coëfficientis illius factor evaneſcat oportet; & proinde ſeries, qua ſinus per arcum exprimitur, hujus eſt formæ $\text{ſin.}\, z = z - Az^3 + Bz^5 + Cz^7 + Dz^9 + \ldots$

Sumtis differentiis tertiis erit (§.)

$$2^3\text{ſin.}^3\tfrac{1}{2}z\,\text{coſ.}\tfrac{5}{2}z = \Delta'''(4^3\ldots1^3)Az^3 - \Delta'''(4^5\ldots1^5)Bz^5 - \Delta'''(4^7\ldots1^7)Cz^7 - \ldots$$
$$2^3\text{ſin.}^3\tfrac{1}{2}z\,\text{coſ.}\tfrac{7}{2}z = \Delta'''(5^3\ldots2^3)Az^3 - \Delta'''(5^5\ldots2^5)Bz^5 - \Delta'''(5^7\ldots2^7)Cz^7 - \ldots$$
$$2^3\text{ſin.}^3\tfrac{1}{2}z\,\text{coſ.}\tfrac{9}{2}z = \Delta'''(6^3\ldots3^3)Az^3 - \Delta'''(6^5\ldots3^5)Bz^5 - \Delta'''(6^7\ldots3^7)Cz^7 - \ldots$$

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Quoniam primus terminus omnium membrorum priorum juxta §. 72. in ſeries evolutorum eſt $1z^3$, erit etiam primus terminus omnium membrorum poſteriorum $1z^3$. Sed (§. *q. Introd.*) $\Delta''' n^3 = 1.2.3$; ergo $1 = 1.2.3A$, & $A = \dfrac{1}{1.2.3}$.

Sumtis differentiis quartis & deinceps quintis erit

$$2^5\text{ſin.}^5\tfrac{1}{2}z\,\text{coſ.}\tfrac{7}{2}z = \Delta^{V}(6^5\ldots1^5)Bz^5 + \Delta^{V}(6^7\ldots1^7)Cz^7 + \Delta^{V}(6^9\ldots1^9)Dz^9 + \ldots$$
$$2^5\text{ſin.}^5\tfrac{1}{2}z\,\text{coſ.}\tfrac{9}{2}z = \Delta^{V}(7^5\ldots2^5)Bz^5 + \Delta^{V}(7^7\ldots2^7)Cz^7 + \Delta^{V}(5^9\ldots2^9)Dz^9 + \ldots$$
$$2^5\text{ſin.}^5\tfrac{1}{2}z\,\text{coſ.}\tfrac{11}{2}z = \Delta^{V}(8^5\ldots3^5)Bz^5 + \Delta^{V}(8^7\ldots3^7)Cz^7 + \Delta^{V}(8^9\ldots3^9)Dz^9 + \ldots$$

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Primus

Primus terminus omnium membrorum priorum est $1z^5$ (§. 72.); ergo & primus terminus omnium membrorum posteriorum est $1z^5$. Sed $\Delta^{v} n^5 = 1.2\ldots 5$ (§. q. Introd.); ergo $1.2\ldots 5B = 1$, & $B = \frac{1}{1.2\ldots 5}$.

Sumtis differentiis sextis & deinceps septimis erit

$$-2^7 \sin.^7 \tfrac{1}{2}z \cos.\tfrac{9}{2}z = \Delta^{vii}(8^7\ldots 1^7)Cz^7 + \Delta^{vii}(8^9\ldots 1^9)Dz^9 + \Delta^{vii}(8^{11}\ldots 1^{11})Ez^{11} + \ldots$$
$$-2^7 \sin.^7 \tfrac{1}{2}z \cos.\tfrac{11}{2}z = \Delta^{vii}(9^7\ldots 2^7)Cz^7 + \Delta^{vii}(9^5\ldots 2^5)Dz^9 + \Delta^{vii}(9^{11}\ldots 2^{11})Ez^{11} + \ldots$$
$$-2^7 \sin.^7 \tfrac{1}{2}z \cos.\tfrac{13}{2}z = \Delta^{vii}(10^7..3^7)Cz^7 + \Delta^{vii}(10^9..3^9)Dz^9 + \Delta^{vii}(10^{11}..3^{11})Ez^{11} + \ldots$$

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Primus terminus omnium priorum membrorum evolutorum est $-1z^7$; ergo & primus terminus omnium membrorum posteriorum est $-1z^7$. Sed hic primus terminus est $\Delta^{vii} n^7 Cz^7 = 1.2\ldots 7Cz^7$; proinde $-1 = 1.2\ldots 7C$, & $C = -\frac{1}{1.2\ldots 7}$.

Eodem modo sumtis differentiis octavis, & deinceps nonis, erit $1z^9 = \Delta^{ix} n^9 Dz^9 = 1.2\ldots 9Dz^9$; unde $D = \frac{1}{1.2\ldots 9}$;

pariter $E = -\frac{1}{1.2\ldots 11}$

$F = \frac{1}{1.2\ldots 13}$; & generatim

$$\sin. z = z - \frac{1}{1.2.3}z^3 + \frac{1}{1.2\ldots 5}z^5 - \frac{1}{1.2\ldots 7}z^7 + \frac{1}{1.2\ldots 9}z^9 - \frac{1}{1.2\ldots 11}z^{11} + \ldots.$$

§. 74. Investigatio cosinus eodem modo peragitur.

Sit nempe $\cos. z = 1 - Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \ldots.$

hinc $\cos. 2z = 1 - 2Az + 2^2Bz^2 + 2^3Cz^3 + 2^4Dz^4 + 2^5Ez^5 + \ldots.$

$\cos. 3z = 1 - 3Az + 3^2Bz^2 + 3^3Cz^3 + 3^4Dz^4 + 3^5Ez^5 + \ldots.$

$\cos. 4z = 1 - 4Az + 4^2Bz^2 + 4^3Cz^3 + 4^4Dz^4 + 4^5Ez^5 + \ldots.$

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Hinc sumtis differentiis primis erit

$$2\sin.\tfrac{1}{2}z \sin.\tfrac{3}{2}z = Az - (2^2-1)Bz^2 - (2^3-1)Cz^3 - (2^4-1)Dz^4 - (2^5-1)Ez^5 - \ldots.$$
$$2\sin.\tfrac{1}{2}z \sin.\tfrac{5}{2}z = Az - (3^2-2^2)Bz^2 - (3^3-2^3)Cz^3 - (3^4-2^4)Dz^4 - (3^5-2^5)Ez^5 - \ldots.$$
$$2\sin.\tfrac{1}{2}z \sin.\tfrac{7}{2}z = Az - (4^2-3^2)Bz^2 - (4^3-3^3)Cz^3 - (4^4-3^4)Dz^4 - (4^5-3^5)Ez^5 - \ldots.$$

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 Quo-

Quoniam autem (§. 72.) primus terminus omnium priorum membrorum non continet primam poteſtatem z arcus mutabilis z, prima hæc poteſtas deerit etiam in poſterioribus membris; unde erit $A = 0$.

Eodem autem modo, quo prius pro ſinu factum fuit, demonſtratur omnes poteſtates impares quantitatis mutabilis z evaneſcere. Scilicet differentiæ ordinis imparis $2m+1$ membrorum priorum ſunt $2^{2m+1}\,\text{ſin.}^{2m+1}\tfrac{1}{2}z\,\text{ſin.}pz$; & proinde primi termini horum membrorum evolutorum continent poteſtatem parem z^{2m+2} arcus mutabilis z: membra ergo priora harum æquationum non continent poteſtatem imparem z^{2m+1} arcus z; unde & membris poſterioribus poteſtates hæ impares debent exulare. Sed primi termini horum membrorum ſunt $\Delta^{2m+1}n^{2m+1}Mz^{2m+1}$, ſeu $1.2\ldots 2m+1\,Mz^{2m+1}$; proinde $0 = 1.2\ldots 2m+1\,M$, & $M = 0$. Coſinus itaque per arcum exprimitur ſerie hujus formæ, quæ pares tantum poteſtates ipſius z continet.

$$\text{coſ.}\,z = 1 - Az^2 + Bz^4 + Cz^6 + Dz^8 + Ez^{10} + \ldots .$$

Sumtis differentiis ſecundis erit

$$2^2\text{ſin.}^2\tfrac{1}{2}z\,\text{coſ.}2z = \Delta''(3^2\ldots1^2)Az^2 - \Delta''(3^4\ldots1^4)Bz^4 - \Delta''(3^6\ldots1^6)Cz^6 - \ldots$$
$$2^2\text{ſin.}^2\tfrac{1}{2}z\,\text{coſ.}3z = \Delta''(4^2\ldots2^2)Az^2 - \Delta''(4^4\ldots2^4)Bz^4 - \Delta''(4^6\ldots2^6)Cz^6 - \ldots$$
$$2^2\text{ſin.}^2\tfrac{1}{2}z\,\text{coſ.}4z = \Delta''(5^2\ldots3^2)Az^2 - \Delta''(5^4\ldots3^4)Bz^4 - \Delta''(5^6\ldots3^6)Cz^6 - \ldots$$

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Et quoniam primus terminus omnium priorum membrorum evolutorum eſt $1z^2$, erit etiam primus terminus omnium poſteriorum membrorum $1z^2$. Sed $\Delta''n^2 = 1.2$ (§. q. *Introd.*); ergo $1.2A = 1$, & $A = \frac{1}{1.2}$.

Tum ſumtis differentiis tertiis & deinceps quartis, erit

$$2^4\text{ſin.}^4\tfrac{1}{2}z\,\text{coſ.}3z = \Delta^{\text{IV}}(5^4\ldots1^4)Bz^4 + \Delta^{\text{IV}}(5^6\ldots1^6)Cz^6 + \Delta^{\text{IV}}(5^8\ldots1^8)Dz^8 + \ldots$$
$$2^4\text{ſin.}^4\tfrac{1}{2}z\,\text{coſ.}4z = \Delta^{\text{IV}}(6^4\ldots2^4)Bz^4 + \Delta^{\text{IV}}(6^6\ldots2^6)Cz^6 + \Delta^{\text{IV}}(6^8\ldots2^8)Dz^8 + \ldots$$
$$2^4\text{ſin.}^4\tfrac{1}{2}z\,\text{coſ.}5z = \Delta^{\text{IV}}(7^4\ldots3^4)Bz^4 + \Delta^{\text{IV}}(7^6\ldots3^6)Cz^6 + \Delta^{\text{IV}}(7^8\ldots3^8)Dz^8 + \ldots$$

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Unde erit $1z^4 = \Delta^{\text{IV}}n^4Bz^4 = 1.2\ldots4Bz^4$; & $B = \frac{1}{1.2\ldots4}$.

Tum

Tum sumtis differentiis quintis & deinceps sextis, erit

$$-2^6 \sin.^6 \tfrac{1}{2}z \cos. 4z = \Delta^{VI}(7^6 \ldots 1^6)Cz^6 + \Delta^{VI}(7^8 \ldots 1^8)Dz^8 + \Delta^{VI}(7^{10} \ldots 1^{10})Ez^{10} + \ldots$$
$$-2^6 \sin.^6 \tfrac{1}{2}z \cos. 5z = \Delta^{IV}(8^6 \ldots 2^6)Cz^6 + \Delta^{VI}(8^8 \ldots 2^8)Dz^8 + \Delta^{VI}(8^{10} \ldots 2^{10})Ez^{10} + \ldots$$
$$-2^6 \sin.^6 \tfrac{1}{2}z \cos. 6z = \Delta^{IV}(9^6 \ldots 3^6)Cz^6 + \Delta^{VI}(9^8 \ldots 3^8)Dz^8 + \Delta^{VI}(9^{10} \ldots 3^{10})Ez^{10} + \ldots$$

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Unde erit $-1z^6 = \Delta^{VI} n^6 Cz^6 = 1.2\ldots6Cz^6 : C = -\frac{1}{1.2\ldots6}$.

Sumtis differentiis septimis & deinceps octavis, erit pariter $1z^8 = \Delta^{VIII} n^8 Dz^8$, & $D = \frac{1}{1.2\ldots8}$.

Sumtis differentiis nonis & deinceps decimis, erit $-1z^{10} = \Delta^{X} n^{10} Ez^{10}$; & $E = -\frac{1}{1.2\ldots10}$. Unde tandem

$$\cos. z = 1 - \frac{1}{1.2}z^2 + \frac{1}{1.2\ldots4}z^4 - \frac{1}{1.2\ldots6}z^6 + \frac{1}{1.2\ldots8}z^8 - \frac{1}{1.2\ldots10}z^{10} + \ldots.$$

§. 75. *Corollarium.* Tang. $z = \dfrac{z - \frac{1}{1.2.3}z^3 + \frac{1}{1.2..5}z^5 - \frac{1}{1.2..7}z^7 + \frac{1}{1.2..9}z^9 + \ldots.}{1 - \frac{1}{1.2}z^2 + \frac{1}{1.2..4}z^4 - \frac{1}{1.2..6}z^6 + \frac{1}{1.2..8}z^8 - \ldots.}$

Ad scopum præsentem non pertinet expressionem hanc fractam in seriem convertere.

Observare sufficiat, tang. z exprimi etiam per arcum z serie hujus formæ tang. $z = z + Az^3 + Bz^5 + Cz^7 + Dz^9 + \ldots.$, in qua $A = \frac{1}{3}$; $B = \frac{2}{15}$; ....

§. 76. Quoniam $\sin. z = z - \frac{1}{1.2.3}z^3 + \frac{1}{1.2..5}z^5 - \frac{1}{1.2..7}z^7 + \frac{1}{1.2..9}z^9 - \ldots.$

$$\text{est} \quad \frac{d \sin. z}{dz} = 1 - \frac{1}{1.2}z^2 + \frac{1}{1.2..4}z^4 - \frac{1}{1.2..6}z^6 + \frac{1}{1.2..8}z^8 - \ldots =$$

$= \cos. z$ (conformiter §. 49.)

$$\text{et quoniam} \quad \cos. z = 1 - \frac{1}{1.2}z^2 + \frac{1}{1.2..4}z^4 - \frac{1}{1.2..6}z^6 + \frac{1}{1.2..8}z^8 - \ldots.$$

$$\text{est} \quad \frac{d \cos. z}{dz} = -z + \frac{1}{1.2.3}z^3 - \frac{1}{1.2..5}z^5 + \frac{1}{1.2..7}z^7 - \ldots =$$

$= -\sin. z$ (conformiter §. 49.)

Et

Et quoniam $\frac{\text{ſin.}z}{\text{coſ.}z} = \text{tang.}z$; $\frac{\text{coſ.}z\frac{d\,\text{ſin.}z}{dz} - \text{ſin.}z\frac{d\,\text{coſ.}z}{dz}}{\text{coſ.}^2 z} = \frac{d\,\text{tang.}z}{dz}$;

ſeu $\frac{\text{coſ.}^2 z + \text{ſin.}^2 z}{\text{coſ.}^2 z} = \frac{d\,\text{tang.}z}{dz}$;

unde $\frac{d\,\text{tang.}z}{dz} = \frac{1}{\text{coſ.}^2 z} = \text{ſec.}^2 z$,

ſeu $\frac{dt}{dz} = \text{ſec.}^2 z$; et $\frac{dz}{dt} = \frac{1}{1+tt}$, quod geometrice etiam ſic demonſtratur.

Fig. 17. Sit C centrum & CA radius circuli, cujus arcus mutabilis ſit AXX'. Sit ab A ducta tangens hujus arcus; & ducantur CX, CX', quæ tangenti occurrant in Y & Y': tum centro C & radio CY deſcribatur arcus Yy, occurrens ipſi CX' in y.

$$\begin{array}{lll} \text{Erit} & \lim.YY' : Yy = 1 : \text{ſin.}Y' = 1 : \text{ſin.}CYA & = CY : CA \\ & Yy : XX' & = CY : CA \\ \text{ergo} & \lim.YY' : XX' & = CY^2 : CA^2 \quad (\S.\ 14.) \end{array}$$

Sed eſt $\lim.YY' : XX' = \frac{dt}{ds}$; ergo $\frac{dt}{ds} = \frac{rr+tt}{rr}$; et $\frac{ds}{dt} = \frac{rr}{rr+tt}$.

§. 77. Quoniam ſinus & coſinus functiones ſunt arcus, expreſſiones eorum potuiſſent per theorema Taylorianum obtineri modo ſequenti:

$$\text{ſin.}z \pm \Delta z = \text{ſin.}z \pm \frac{\Delta z}{1}\frac{d\,\text{ſin.}z}{dz} + \frac{\Delta z^2}{1.2}\frac{d^2\,\text{ſin.}z}{dz^2} \pm \frac{\Delta z^3}{1.2.3}\frac{d^3\,\text{ſin.}z}{dz^3} + \frac{\Delta z^4}{1.2..4}\frac{d^4\,\text{ſin.}z}{dz^4} \pm \ldots$$

Atqui eſt (§. 76.) $\frac{d\,\text{ſin.}z}{dz} = +\,\text{coſ.}z$, et $\frac{d\,\text{coſ.}z}{dz} = -\,\text{ſin.}z$

unde eſt
$$\frac{d^2\,\text{ſin.}z}{dz^2} = -\,\text{ſin.}z$$
$$\frac{d^3\,\text{ſin.}z}{dz^3} = -\,\text{coſ.}z$$
$$\frac{d^4\,\text{ſin.}z}{dz^4} = +\,\text{ſin.}z$$
$$\frac{d^5\,\text{ſin.}z}{dz^5} = +\,\text{coſ.}z$$
$$\frac{d^6\,\text{ſin.}z}{dz^6} = -\,\text{ſin.}z$$
$$\vdots \qquad \vdots$$

hinc

Hinc

$$\sin.z\pm\Delta z=\sin.z\pm\frac{\Delta z}{1}\cos.z-\frac{\Delta z^2}{1.2}\sin.z\mp\frac{\Delta z^3}{1.2.3}\cos.z+\frac{\Delta z^4}{1...4}\sin.z\pm\frac{\Delta z^5}{1..5}\cos.z-\frac{\Delta z^6}{1..6}\sin.z\mp..$$

Unde est $\frac{\sin.(z+\Delta z)+\sin.(z-\Delta z)}{2}=\sin.z-\frac{\Delta z^2}{1.2}\sin.z+\frac{\Delta z^4}{1...4}\sin.z-\frac{\Delta z^6}{1...6}\sin.z+...$

$$\frac{\sin.(z+\Delta)z-\sin.(z-\Delta z)}{2}=\frac{\Delta z}{1}\cos.z-\frac{\Delta z^3}{1..3}\cos.z+\frac{\Delta z^5}{1..5}\cos.z-\frac{\Delta z^4}{1..7}\cos.z+..$$

seu $\sin.z\cos.\Delta z=\sin.z-\frac{\Delta z^2}{1.2}\sin.z+\frac{\Delta z^4}{1..4}\sin.z-\frac{\Delta z^6}{1..6}\sin.z+...$

$$\cos.z\sin.\Delta z=\frac{\Delta z}{1}\cos.z-\frac{\Delta z^3}{1..3}\cos.z+\frac{\Delta z^5}{1..5}\cos.z-\frac{\Delta z^7}{1..7}\cos.z+...$$

unde $\cos.\Delta z=1-\frac{\Delta z^2}{1.2}+\frac{\Delta z^4}{1...4}-\frac{\Delta z^6}{1...6}+....$

$$\sin.\Delta z=\frac{\Delta z}{1}-\frac{\Delta z^3}{1.2.3}+\frac{\Delta z^5}{1...5}-\frac{\Delta z^7}{1...7}+....$$

Proinde generatim $\cos.z=1-\frac{z^2}{1.2}+\frac{z^4}{1.2...4}-\frac{z^6}{1.2...6}+....$

$$\sin.z=z-\frac{z^3}{1.2.3}+\frac{z^5}{1.2...5}-\frac{z^7}{1.2...7}+....$$

§. 78. Vidimus (§. 59.), quod

$$\frac{e^z+e^{-z}}{2}=1+\frac{z^2}{1.2}+\frac{z^4}{1.2...4}+\frac{z^6}{1.2...6}+\frac{z^8}{1.2...8}+....$$

$$\frac{e^z-e^{-z}}{2}=z+\frac{z^3}{1.2.3}+\frac{z^5}{1.2...5}+\frac{z^7}{1.2...7}+\frac{z^9}{1.2...9}+....$$

Series hæ magnam respective habent affinitatem cum expressionibus cos.z & sin.z. Etenim si in priori serie pro $+z^2$ substituatur $-z^2$, seu si loco z substituatur $z\sqrt{-1}$; prodit $1-\frac{z^2}{1.2}+\frac{z^4}{1.2...4}-\frac{z^6}{1.2...6}+\frac{z^8}{1.2...8}-....$ seu cos.z: unde eadem substitutione priori æquationis membro applicata, infertur

$$\cos.z=\frac{e^{+z\sqrt{-1}}+e^{-z\sqrt{-1}}}{2}.$$

 Pariter

Pariter in ſerie poſteriori, loco z ſubſtituto ſigno $z\sqrt{-1}$, prodit $\sqrt{-1}\left(z-\frac{z^3}{1.2.3}+\frac{z^5}{1.2\ldots5}-\frac{z^7}{1.2\ldots7}+\frac{z^9}{1.2\ldots9}-\ldots\right)$, ſeu $\sqrt{-1}\,\text{ſin.}\,z$; unde ſimiliter infertur $\sqrt{-1}\,\text{ſin.}\,z=\frac{e^{+z\sqrt{-1}}-e^{-z\sqrt{-1}}}{2}$, ſeu $\text{ſin.}\,z=\frac{e^{+z\sqrt{-1}}-e^{-z\sqrt{-1}}}{2\sqrt{-1}}$.

Hinc $\text{tang.}\,z=\frac{1}{\sqrt{-1}}\times\frac{e^{z\sqrt{-1}}-e^{-z\sqrt{-1}}}{e^{z\sqrt{-1}}+e^{-z\sqrt{-1}}}$, ſeu $\text{tang.}\,z\sqrt{-1}=\frac{e^{z\sqrt{-1}}-e^{-z\sqrt{-1}}}{e^{z\sqrt{-1}}+e^{-z\sqrt{-1}}}$; unde $1+\text{tang.}\,z\sqrt{-1}:1-\text{tang.}\,z\sqrt{-1}=e^{z\sqrt{-1}}:e^{-z\sqrt{-1}}=e^{2z\sqrt{-1}}:1$;

$e^{2z\sqrt{-1}}=\frac{1+\text{tang.}\,z\sqrt{-1}}{1-\text{tang.}\,z\sqrt{-1}}$; $2z\sqrt{-1}=\log.\frac{1+\text{tang.}\,z\sqrt{-1}}{1-\text{tang.}\,z\sqrt{-1}}$;

$z=\frac{1}{2\sqrt{-1}}\log.\frac{1+\text{tang.}\,z\sqrt{-1}}{1-\text{tang.}\,z\sqrt{-1}}$, qua formula continetur fundamentum calculi logarithmorum quantitatum (ita dictarum) tam negativarum quam imaginariarum (vid. §. 71.).

Obſervatio. Formulæ exponentiales imaginariæ $\text{coſ.}\,z=\frac{e^{z\sqrt{-1}}+e^{-z\sqrt{-1}}}{2}$, $\text{ſin.}\,z=\frac{e^{z\sqrt{-1}}-e^{-z\sqrt{-1}}}{2\sqrt{-1}}$, $\text{tang.}\,z=\frac{1}{\sqrt{-1}}\,\frac{e^{z\sqrt{-1}}-e^{-z\sqrt{-1}}}{e^{z\sqrt{-1}}+e^{-z\sqrt{-1}}}$, $z=\frac{1}{2\sqrt{-1}}\log.\frac{1+\text{tang.}\,z\sqrt{-1}}{1-\text{tang.}\,z\sqrt{-1}}$, frequentiſſime in calculo integrali uſurpantur, & plurimas inveſtigationes mirifice juvant. Spectari autem debent tanquam mera ſigna majoris facilitatis cauſa a mathematicis introducta; atque irrita foret inveſtigatio ſignificationis ſymbolorum $e^{z\sqrt{-1}}$, $e^{-z\sqrt{-1}}$, quibus ſeorſim ſumtis nulla idea reſpondet; nec operationes in ipſis inſtitutæ ad reale quid conducere poſſunt, niſi quatenus impoſſibilitatis ſigna, quæ involvunt, ſeſe mutuo deſtruunt.

§. 79. Sumto $z\sqrt{-1}=\frac{1}{2}\log.\frac{1+t\sqrt{-1}}{1-t\sqrt{-1}}=\log.\sqrt{\frac{1+t\sqrt{-1}}{1-t\sqrt{-1}}}$:

quoniam eſt $\log.\sqrt{\frac{1+v}{1-v}}=v+\frac{1}{3}v^3+\frac{1}{5}v^5+\frac{1}{7}v^7+\frac{1}{9}v^9+\ldots$ (§. 61.)

erit $z\sqrt{-1}=t\sqrt{-1}\left(1-\frac{1}{3}tt+\frac{1}{5}t^4-\frac{1}{7}t^6+\frac{1}{9}t^8-\ldots\right)$

et $z=t-\frac{1}{3}t^3+\frac{1}{5}t^5-\frac{1}{7}t^7+\frac{1}{9}t^9-\ldots$

quæ deductio exempli loco ſit utilitatis harum expreſſionum imaginariarum.

Ad

Ad hanc expressionem etiam conducit exponens differentialis $\frac{dz}{dt} = \frac{1}{1+u}$ (§. 16.)

Inde enim est $\frac{dz}{dt} = 1 - tt + t^4 - t^6 + t^8 - t^{10} + \ldots\ldots$

unde $z = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \frac{1}{9}t^9 - \frac{1}{11}t^{11} + \ldots\ldots$

Porro hæc expressio immediate etiam ex primis limitum principiis deducitur modo sequenti:

Quoniam (§. ac. *Introd.*) $\text{sin.}\,\varphi = \frac{(\text{cos.}\,n\varphi + \text{sin.}\,n\varphi\sqrt{-1})^{\frac{1}{n}} - (\text{cos.}\,n\varphi - \text{sin.}\,n\varphi\sqrt{-1})^{\frac{1}{n}}}{2\sqrt{-1}}$

$$n\,\text{sin.}\,\varphi = n\frac{(\text{cos.}\,n\varphi + \text{sin.}\,n\varphi\sqrt{-1})^{\frac{1}{n}} - (\text{cos.}\,n\varphi - \text{sin.}\,n\varphi\sqrt{-1})^{\frac{1}{n}}}{2\sqrt{-1}}$$

$$= n\Big(\frac{1}{n}\,\text{cos.}\,n\varphi^{\frac{1}{n}-1}\,\text{sin.}\,n\varphi\Big)$$

$$- \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)}{1.2.3}\,\text{cos.}\,n\varphi^{\frac{1}{n}-3}\,\text{sin.}^3 n\varphi$$

$$+ \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\ldots\left(\frac{1}{n}-4\right)}{1.2\ldots4}\,\text{cos.}\,n\varphi^{\frac{1}{n}-5}\,\text{sin.}^5 n\varphi$$

$$- \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\ldots\left(\frac{1}{n}-6\right)}{1.2\ldots6}\,\text{cos.}\,n\varphi^{\frac{1}{n}-7}\,\text{sin.}^7 n\varphi$$

$$+ \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\ldots\left(\frac{1}{n}-8\right)}{1.2\ldots8}\,\text{cos.}\,n\varphi^{\frac{1}{n}-9}\,\text{sin.}^9 n\varphi$$

$$- \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\ldots\left(\frac{1}{n}-10\right)}{1.2\ldots10}\,\text{cos.}\,n\varphi^{\frac{1}{n}-11}\,\text{sin.}^{11} n\varphi$$

$$+ \quad - \quad - \quad - \quad -$$

$$- \quad - \quad - \quad - \quad -$$

= cos.

$$= \cos. n\varphi^{\frac{1}{n}} \Big(\text{tang.}\, n\varphi$$

$$- \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{1.2.3} \text{tang.}^3 n\varphi$$

$$+ \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\ldots\left(4-\frac{1}{n}\right)}{1.2\ldots5} \text{tang.}^5 n\varphi$$

$$- \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\ldots\left(6-\frac{1}{n}\right)}{1.2\ldots7} \text{tang.}^7 n\varphi$$

$$+ \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\ldots\left(8-\frac{1}{n}\right)}{1.2\ldots9} \text{tang.}^9 n\varphi$$

$$- \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\ldots\left(10-\frac{1}{n}\right)}{1.2\ldots11} \text{tang.}^{11} n\varphi$$

$$\pm \quad - \quad - \quad - \quad -$$

unde facto $n\varphi = z$, seu $\varphi = \frac{1}{n} z$ est

$$n \sin. \frac{1}{n} z = \cos. z^{\frac{1}{n}} \Big(\text{tang.}\, z$$

$$- \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{1.2.3} \text{tang.}^3 z$$

$$+ \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\ldots\left(4-\frac{1}{n}\right)}{1.2\ldots5} \text{tang.}^5 z$$

$$- \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\ldots\left(6-\frac{1}{n}\right)}{1.2\ldots7} \text{tang.}^7 z$$

$$+ \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\ldots\left(8-\frac{1}{n}\right)}{1.2\ldots9} \text{tang.}^9 z$$

$$- \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\ldots\left(10-\frac{1}{n}\right)}{1.2\ldots11} \text{tang.}^{11} z$$

$$\pm \quad - \quad - \quad - \quad -$$

Quo-

Quoniam æquatio hæc semper locum habet, etiam limites membrorum ejus sunt inter se æquales; sed crescente n hi limites sunt respective

$$z \ \& \ \text{tang.}z - \tfrac{1}{3}\text{tang.}^3 z + \tfrac{1}{5}\text{tang.}^5 z - \tfrac{1}{7}\text{tang.}^7 z + \tfrac{1}{9}\text{tang.}^9 z - \ldots .$$

ergo $z = \text{tang.}z - \tfrac{1}{3}\text{tang.}^3 z + \tfrac{1}{5}\text{tang.}^5 z - \tfrac{1}{7}\text{tang.}^7 z + \tfrac{1}{9}\text{tang.}^9 z - \ldots .$

Exempla. Sit p quadrans circumferentiæ, & $z = \tfrac{1}{2}p = 45°$: erit tang. $z = 1$; unde $\tfrac{1}{2}p = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \ldots .$

Sit $z = \tfrac{1}{3}p = 30°$: $t = \frac{1}{\sqrt{3}}$;

$$\tfrac{1}{3}p = \frac{1}{\sqrt{3}}\left(1 - \tfrac{1}{3}\cdot\tfrac{1}{3} + \tfrac{1}{5}\cdot\frac{1}{3^2} - \tfrac{1}{7}\cdot\frac{1}{3^3} + \tfrac{1}{9}\cdot\frac{1}{3^4} - \tfrac{1}{11}\cdot\frac{1}{3^5} + \ldots .\right.$$

Observatio. Posterior series, quæ priore longe promtius convergit, inprimis fuit computationi circumferentiæ circuli applicata. Ea usus Cl. MACHINUS peripheriam circuli diametri = 1 ad centum usque figuras decimales computavit; & post illum celeb. LAGNY ad 127 figuras usque eundem calculum produxit (adjutus tamen aliis etiam compendiis, quæ non satis explicuit. Vid. *Memoires de l'Academie des Sciences de Paris*, 1719.). Bibliothecam Oxoniensem Bodleianam manuscriptum possidere ex epistola D. HORNSBY refert ill. KÆSTNER (*Anfangsgründe der Arithm. und Geometrie*, 5te *Aufl.* 1792.): quod ad 156 usque figuras decimales continuatum rationis peripheriæ ad diametrum exponentem exhibeat; & in 113tia cyphra D. LAGNY 8 loco 7 substitui debere moneat.

§. 80. Series $z = t - \tfrac{1}{3}t^3 + \tfrac{1}{5}t^5 - \tfrac{1}{7}t^7 + \tfrac{1}{9}t^9 - \ldots$ eo citius convergit, ideoque usui eo magis fit accommodata, quo minor est t. Cum autem divergat, quando t est unitate major: prima specie non videtur omnibus arcubus posse applicari; quod tamen locum habet, uti ratiocinio sequenti patebit.

Et primum quidem series ista applicatur dimidio circumferentiæ quadranti, seu arcui 45°; ubi $t = 1$ (quamvis fatendum sit, eam tunc lente admodum convergere.)

Quando autem arcus major est dimidio quadrante seu 45°: tangens ejus major est unitate; sed tangens complementi ipsius est unitate minor, ac proinde complementum prioris arcus computari potest per seriem convergentem.

Clarifs. EULERUS artificium tradit, quo prædicta feries fit ufui quam maxime accommodata; & quod e re effe cenfeo hic breviter exponere.

Lemma. Efto arcus, cujus tangens fit pars quæpiam aliquota radii. Arcus hic dividi poteft in duos alios arcus, quorum tangentes pariter erunt radii partes aliquotæ, eæque minores.

Sit $\frac{1}{T}$ tangens alicujus arcus, exiftente T numero integro pofitivo. Arcus hic dividi poteft in duos alios, quorum tangentes $\frac{1}{t}$, $\frac{1}{t'}$ etiam erunt partes aliquotæ radii, eæque minores.

Etenim quoniam $\frac{1}{T}$ eft tangens arcus, qui eft fumma arcuum, quorum tangentes funt $\frac{1}{t}$ & $\frac{1}{t'}$; erit $\frac{1}{T} = \frac{\frac{1}{t}+\frac{1}{t'}}{1-\frac{1}{tt'}}$: unde (facili calculo) deducitur $(t-T)(t'-T) = TT+1$; hinc $t-T = \frac{TT+1}{t'-T}$. Ut t fit numerus integer, fiat denominator $t'-T$ æqualis diviforibus numeratoris $TT+1$; erit factum.

Exempla. Sit $T=1$, feu arcus propofitus fit dimidius quadrans. $TT+1=2$; $\begin{matrix} t'-T=1,2 \\ t-T=2,1 \end{matrix}$; $\begin{matrix} t'=2,3 \\ t=2,2 \end{matrix}$. Dimidius igitur quadrans dividitur in duos arcus, quorum tangentes funt $\frac{1}{2}$ & $\frac{1}{3}$ radii. Summa horum arcuum, ferie præcedenti promtius convergente computatorum, erit dimidius quadrans.

Sit nunc $T=2$; $TT+1=5$; $\begin{matrix} t'-T=1,5 \\ t-T=5,1 \end{matrix}$; $\begin{matrix} t'=3,7 \\ t=7,3 \end{matrix}$. Quadrans ideo dimidiæ circumferentiæ dividitur in tres arcus, quorum duorum mutuo æqualium tangens eft $\frac{1}{3}$, & tertii tangens eft $\frac{1}{7}$.

Sit tandem $T=3$; $TT+1=10$; $\begin{matrix} t'-T= 1,2,5,10 \\ t-T=10,5,2,1 \end{matrix}$; $\begin{matrix} t'= 4,5,8,13 \\ t=13,8,5,4 \end{matrix}$. Proinde quadrans dimidiæ circumferentiæ dividitur v. gr. in quinque arcus, ut fequitur: $\frac{1}{2}p = 2\,\text{arc. tang.}\,\frac{1}{5} + 1\,\text{arc. tang.}\,\frac{1}{7} + 2\,\text{arc. tang.}\,\frac{1}{8}$.

Sufficiat, hujus methodi principium breviter expofuiffe. Qui plura voluerit, adeat *Commentarios Acad. imp. Petrop. ad ann.* 1737. & BERTRAND *Developement de la partie elementaire des Mathematiques.*

§. 81.

§. 81. Formulæ differentiales capite hoc traditæ (§. 76.) felici ſucceſſu ad inveſtigandas proprietates linearum transcendentium, functiones circulares involventium, adhibentur. Sufficiat, exempli loco de cycloïde dicere, cujus tam frequentes & eximiæ in matheſi mixta applicationes occurrunt.

Sit AB axis, & BD baſis cycloïdis AMD. Sit ANB ſemi-circulus ſuper axe AB tanquam diametro deſcriptus; & recta MP axi ordinatim applicata circumferentiæ hujus in N occurrat. Eſt $MN =$ arc. AN. Sit $AB = 2r$, $AP = x$, $MP = y$; erit itaque $y = \sqrt{(2rx - xx)} + r\,\text{arc. ſin. v.}\frac{x}{r}$: unde Fig. 18.a

$$\frac{dy}{dx} = \frac{r-x}{\sqrt{(2rx-xx)}} + \frac{r}{\sqrt{(2rx-xx)}} = \frac{2r-x}{\sqrt{(2rx-xx)}} = \sqrt{\frac{2r-x}{x}} = \frac{\sqrt{(2rx-xx)}}{x} =$$

$\frac{NP}{AP} =$ cot. ANP. Sit MT tangens cycloidis in M; eſt $\frac{dy}{dx} =$ cot. TMP (§. 41): quare cot. $TMP =$ cot. ANP; $TMP = ANP$; & proinde MT eſt ipſi NA parallela.

Si eſſet $y = \sqrt{(2rx-xx)} + R\,\text{arc. ſin. v.}\frac{x}{r}$; eſſet

$\frac{dy}{dx} = \frac{r-x}{\sqrt{(2rx-xx)}} + \frac{R}{\sqrt{(2rx-xx)}} = \frac{R+r-x}{\sqrt{(2rx-xx)}}$: unde cycloïdum contractarum & protractarum tangentes determinantur.

Caput Octavum.

De ſumma et differentia duarum quantitatum exponentialium in factores reſolvenda.

§. 82.

Summæ & differentiæ $e^x \pm e^y$ duarum quantitatum exponentialium e^x, e^y, in factores reſolutio tanti eſt in calculis ſuperioribus momenti, & adeo fœcundas ſuppeditat applicationes, ut ei evolvendæ merito mathematici ſtuduerint; quos inter eminet celeb. Eulerus in *Introductione ad Analyſin infinitorum* aliasque. Cum autem Euleriana methodus notione infiniti innixa minus mihi arrideret; optabam, ut reſolutio hæc ad genuina & elementaria limitum principia poſſet reduci. Quod me præſtitiſſe opinor in diſſertatione inſcripta: *Sur la Decompoſition*

tion de la somme & de la difference de deux puissances à exposans quelconques de la base des logarithmes hyperboliques, dans le but de degager cette decomposition de toute idée de l'infini. (*Memoires de l'Acad. de Berlin*, 1787—1788.) Et cum dissertationis hujus objectum cum calculorum superiorum principiis ita arcte cohæreat, strictim illud hic exponere e re esse censui.

§. 83. Per theorema Cotesianum (§. *ad Introd.*) est

$$\left(1+\frac{x}{2n}\right)^{2n}+\left(1-\frac{x}{2n}\right)^{2n}=\left(1+\frac{x}{2n}\right)^2-2\left(1+\frac{x}{2n}\right)\left(1-\frac{x}{2n}\right)\operatorname{cos}.\frac{1}{2n}\pi+\left(1-\frac{x}{2n}\right)^2$$

$$\times\left(1+\frac{x}{2n}\right)^2-2\left(1+\frac{x}{2n}\right)\left(1-\frac{x}{2n}\right)\operatorname{cos}.\frac{3}{2n}\pi+\left(1-\frac{x}{2n}\right)^2$$

$$\times\left(1+\frac{x}{2n}\right)^2-2\left(1+\frac{x}{2n}\right)\left(1-\frac{x}{2n}\right)\operatorname{cos}.\frac{5}{2n}\pi+\left(1-\frac{x}{2n}\right)^2$$

$$\vdots$$

$$\times\left(1+\frac{x}{2n}\right)^2-2\left(1+\frac{x}{2n}\right)\left(1-\frac{x}{2n}\right)\operatorname{cos}.\frac{2n-1}{2n}\pi+\left(1-\frac{x}{2n}\right)^2$$

$$=2\left(\left(1-\operatorname{cos}.\frac{1}{2n}\pi\right)+\frac{xx}{4nn}\left(1+\operatorname{cos}.\frac{1}{2n}\pi\right)\right)$$

$$\times\, 2\left(\left(1-\operatorname{cos}.\frac{3}{2n}\pi\right)+\frac{xx}{4nn}\left(1+\operatorname{cos}.\frac{3}{2n}\pi\right)\right)$$

$$\times\, 2\left(\left(1-\operatorname{cos}.\frac{5}{2n}\pi\right)+\frac{xx}{4nn}\left(1+\operatorname{cos}.\frac{5}{2n}\pi\right)\right)$$

$$\vdots$$

$$\times\left(\left(1-\operatorname{cos}.\frac{2n-1}{2n}\pi\right)+\frac{xx}{4nn}\left(1+\operatorname{cos}.\frac{2n-1}{2n}\pi\right)\right)$$

$$=4^n \operatorname{fin}.^2\frac{1}{2n}p.\operatorname{fin}.^2.\frac{3}{2n}p.\operatorname{fin}.^2.\frac{5}{2n}p\ldots\operatorname{fin}.^2\frac{2n-1}{2n}p\times\left(\left(1+\frac{xx}{4nn}\operatorname{cot}.^2\frac{1}{2n}p\right)\right.$$

$$\times\left(1+\frac{xx}{4nn}\operatorname{cot}.^2\frac{3}{2n}p\right)$$

$$\times\left(1+\frac{xx}{4nn}\operatorname{cot}.^2\frac{5}{2n}p\right)$$

$$\vdots$$

$$\left.\times\left(1+\frac{xx}{4nn}\operatorname{cot}.^2\frac{2n-1}{2n}p\right)\right)$$

Hinc

Hinc (§. *ae. Introd.*) $\dfrac{\left(1+\frac{x}{2n}\right)^{2n}+\left(1-\frac{x}{2n}\right)^{2n}}{2}=\Big(\left(1+\frac{xx}{4nn}\cot.^2\frac{1}{2n}p\right)$

$$\times\left(1+\frac{xx}{4nn}\cot.^2\frac{3}{2n}p\right)$$

$$\times\left(1+\frac{xx}{4nn}\cot.^2\frac{5}{2n}p\right)$$

$$\vdots$$

$$\times\left(1+\frac{xx}{4nn}\cot.^2\frac{2n-1}{2n}p\right)\Big)$$

Quoniam autem hæc æquatio ſemper obtinet, limites etiam membrorum ejus ſunt inter ſe æquales. Atqui aucto n (§. 57.) limes prioris membri eſt $\frac{e^x+e^{-x}}{2}$; & (§. 75.) limites factorum $1+\frac{xx}{4nn}\cot.^2\frac{1}{2n}p$ ſunt reſpective $1+\frac{xx}{pp}$

$$\begin{array}{lcl} 1+\frac{xx}{4nn}\cot.^2\frac{3}{2n}p & & 1+\frac{xx}{9pp} \\ 1+\frac{xx}{4nn}\cot.^2\frac{5}{2n}p & & 1+\frac{xx}{25pp} \\ 1+\frac{xx}{4nn}\cot.^2\frac{7}{2n}p & & 1+\frac{xx}{49pp} \\ \vdots\qquad\quad\vdots & & \vdots\qquad\vdots \end{array}$$

Ergo (§. 21.) limes poſterioris membri eſt

$$\left(1+\frac{xx}{pp}\right)\left(1+\frac{xx}{9pp}\right)\left(1+\frac{xx}{25pp}\right)\left(1+\frac{xx}{49pp}\right)\ldots\ldots$$

Proinde $\dfrac{e^x+e^{-x}}{2}=\left(1+\frac{xx}{pp}\right)\left(1+\frac{xx}{9pp}\right)\left(1+\frac{xx}{25pp}\right)\left(1+\frac{xx}{49pp}\right)\ldots\ldots$

§. 84. Per theorema Coteſianum (§. *ad. Introd.*) eſt

$$\left(1+\frac{x}{2n}\right)^{2n}-\left(1-\frac{x}{2n}\right)^{2n}=\left(\left(1+\frac{x}{2n}\right)^2-\left(1-\frac{x}{2n}\right)^2\right)\left(\left(1+\frac{x}{2n}\right)^2-2\left(1+\frac{x}{2n}\right)\left(1-\frac{x}{2n}\right)\text{coſ.}\frac{2}{2n}\pi+\left(1-\frac{x}{2n}\right)^2\right)$$

$$\times\left(1+\frac{x}{2n}\right)^2-2\left(1+\frac{x}{2n}\right)\left(1-\frac{x}{2n}\right)\text{coſ.}\frac{4}{2n}\pi+\left(1-\frac{x}{2n}\right)^2$$

$$
\begin{aligned}
&\times\left(1+\frac{x}{2n}\right)^2-2\left(1+\frac{x}{2n}\right)\left(1-\frac{x}{2n}\right)\operatorname{cos}.\frac{6}{2n}\pi+\left(1-\frac{x}{2n}\right)^2\\
&\quad\vdots\\
&\times\left(1+\frac{x}{2n}\right)^2-2\left(1+\frac{x}{2n}\right)\left(1-\frac{x}{2n}\right)\operatorname{cos}.\frac{2n-2}{2n}\pi+\left(1-\frac{x}{2n}\right)^2\Big).\\
=&\ \frac{2x}{n}\times 2\Big(\Big(1-\operatorname{cos}.\frac{2}{2n}\pi+\frac{xx}{4nn}\Big(1+\operatorname{cos}.\frac{2}{2n}\pi\Big)\Big)\\
&\times 2\Big(1-\operatorname{cos}.\frac{4}{2n}\pi+\frac{xx}{4nn}\Big(1+\operatorname{cos}.\frac{4}{2n}\pi\Big)\Big)\\
&\times 2\Big(1-\operatorname{cos}.\frac{6}{2n}\pi+\frac{xx}{4nn}\Big(1+\operatorname{cos}.\frac{6}{2n}\pi\Big)\Big)\\
&\quad\vdots\\
&\times 2\Big(1-\operatorname{cos}.\frac{2n-2}{2n}\pi+\frac{xx}{4nn}\Big(1+\operatorname{cos}.\frac{2n-2}{2n}\pi\Big)\Big)\\
=&\ \frac{2x}{n}4^{n-1}\operatorname{sin}.^2\frac{1}{n}p\operatorname{sin}.^2\frac{2}{n}p\operatorname{sin}.^2\frac{3}{n}p\ldots\operatorname{sin}.^2\frac{n-1}{n}p\Big(\Big(1+\frac{xx}{4nn}\operatorname{cot}.^2\frac{1}{n}p\Big)\\
&\Big(1+\frac{xx}{4nn}\operatorname{cot}.^2\frac{2}{n}p\Big)\\
&\times\Big(1+\frac{xx}{4nn}\operatorname{cot}.^2\frac{3}{n}p\Big)\\
&\quad\vdots\\
&\times\Big(1+\frac{xx}{4nn}\operatorname{cot}.^2\frac{n-1}{n}p\Big)\Big)\\
=&\ (\S.\ ae.\ \textit{Introd.})\ 2x\Big(\Big(1+\frac{xx}{4nn}\operatorname{cot}.^2\frac{1}{n}p\Big)\\
&\Big(1+\frac{xx}{4nn}\operatorname{cot}.^2\frac{2}{n}p\Big)\\
&\Big(1+\frac{xx}{4nn}\operatorname{cot}.^2\frac{3}{n}p\Big)\\
&\quad\vdots\\
&\Big(1+\frac{xx}{4nn}\operatorname{cot}.^2\frac{n-1}{n}p\Big)\Big).
\end{aligned}
$$

Cum hæc æquatio ſemper obtineat, limites etiam membrorum ejus ſunt inter ſe æquales (§. 4.). Atqui creſcente n limes differentiæ

(1 +

$\dfrac{\left(1+\frac{x}{2n}\right)^{2n}-\left(1-\frac{x}{2n}\right)^{2n}}{2}$ est $\dfrac{e^x - e^{-x}}{2}$ (§. 57.) Et limites factorum

$$\begin{array}{ll}
\left(1+\frac{xx}{4nn}\cot.^2\frac{1}{n}p\right) \text{ sunt respective (§. 75.)} & 1+\frac{xx}{pp} \\
1+\frac{xx}{4nn}\cot.^2\frac{2}{n}p & 1+\frac{xx}{4pp} \\
1+\frac{xx}{4nn}\cot.^2\frac{3}{n}p & 1+\frac{xx}{9pp} \\
1+\frac{xx}{4nn}\cot.^2\frac{4}{n}p & 1+\frac{xx}{16pp}. \\
\vdots & \vdots
\end{array}$$

Ergo (§. 21.) limes posterioris membri est

$$2x\left(1+\frac{xx}{pp}\right)\left(1+\frac{xx}{4pp}\right)\left(1+\frac{xx}{9pp}\right)\left(1+\frac{xx}{16pp}\right)\ldots.$$

ergo $\dfrac{e^x - e^{-x}}{2} = x\left(1+\dfrac{xx}{pp}\right)\left(1+\dfrac{xx}{4pp}\right)\left(1+\dfrac{xx}{9pp}\right)\left(1+\dfrac{xx}{16pp}\right)\ldots.$

§. 85. Duarum formularum $\dfrac{e^x \pm e^{-x}}{2}$ in factores resolutarum ope aliæ etiam formulæ in factores resolvuntur, quales sunt $\dfrac{e^{+x} \pm e^{\pm y}}{2}$.

Etenim 1°. $\dfrac{e^x + e^y}{2}\left[= e^{\frac{x+y}{2}} \times \dfrac{e^{\frac{x-y}{2}} + e^{-\frac{x-y}{2}}}{2}\right] =$

$$= e^{\frac{x+y}{2}}\left(1+\frac{(x-y)^2}{\pi\pi}\right)\left(1+\frac{(x-y)^2}{9\pi\pi}\right)\left(1+\frac{(x-y)^2}{25\pi\pi}\right)\ldots \text{ (§. 83.)}$$

2°. Hinc $\dfrac{e^x + e^{-y}}{2} = e^{\frac{x-y}{2}}\times\left(\left(1+\dfrac{(x+y)^2}{\pi\pi}\right)\left(1+\dfrac{(x+y)^2}{9\pi\pi}\right)\left(1+\dfrac{(x+y)^2}{25\pi\pi}\right)\ldots\right.$

3°. $\dfrac{e^x - e^y}{2}\left[= e^{\frac{x+y}{2}} \times \dfrac{e^{\frac{x-y}{2}} - e^{-\frac{x-y}{2}}}{2}\right] =$

$$= \frac{x-y}{2}\cdot e^{\frac{x+y}{2}}\left(\left(1+\frac{(x-y)^2}{4\pi\pi}\right)\left(1+\frac{(x-y)^2}{16\pi\pi}\right)\left(1+\frac{(x-y)^2}{36\pi\pi}\right)\ldots\right. \text{ (§. 84.)}$$

4°. Hinc $\dfrac{e^x - e^{-y}}{2} = \dfrac{x+y}{2}e^{\frac{x-y}{2}}\left(\left(1+\dfrac{(x+y)^2}{4\pi\pi}\right)\left(1+\dfrac{(x+y)^2}{16\pi\pi}\right)\left(1+\dfrac{(x+y)^2}{36\pi\pi}\right)\ldots\right.$

 Hinc

Hinc etiam $\frac{e^x+1}{2} = e^{\frac{1}{2}x}\left(\left(1+\frac{xx}{\pi\pi}\right)\left(1+\frac{xx}{9\pi\pi}\right)\left(1+\frac{xx}{25\pi\pi}\right)\ldots\right.$,

$$\frac{e^x-1}{2} = \tfrac{1}{2}x \,.\, e^{\frac{1}{2}x}\left(1+\frac{xx}{4\pi\pi}\right)\left(1+\frac{xx}{16\pi\pi}\right)\left(1+\frac{xx}{36\pi\pi}\right)\ldots$$

§. 86. Formulæ $e^x \pm e^{-x}$ & $e^y \pm e^{-y}$ multiplicentur in se invicem respective.

1°. $(e^x+e^{-x})(e^y+e^{-y}) = e^{x+y}+e^{x-y}+e^{-(x-y)}+e^{-(x+y)}$; unde facto $\begin{matrix} x+y=v \\ x-y=z \end{matrix}$ formula $e^v+e^{-v}+e^z+e^{-z}$ in factores resolvitur.

2°. $(e^x+e^{-x})(e^y-e^{-y}) = e^{x+y}-e^{-(x+y)}-e^{x-y}+e^{-(x-y)}$; unde formula $e^v-e^{-v}-e^z+e^{-z}$ etiam in factores resolvitur.

3°. $(e^x-e^{-x})(e^y-e^{-y}) = e^{x+y}+e^{-(x+y)}-e^{x-y}-e^{-(x+y)}$; unde formula $e^v+e^{-v}-e^z-e^{-z}$ etiam in factores resolvitur.

Aucto factorum numero; hæc resolutio ad longe plures alias formulas extendi potest.

§. 87. Sit etiam formula $e^x - 2\text{cos}.2\varphi + e^{-x}$ in factores resolvenda. Juxta §. *af. Introd.* est

$$\left(1+\frac{x}{4n+2}\right)^{4n+2} - 2\left(1+\frac{x}{4n+2}\right)^{2n+1}\left(1-\frac{x}{4n+2}\right)^{2n+1}\text{cos}.2\varphi + \left(1-\frac{x}{4n+2}\right)^{4n+2}$$

$$= \left(1+\frac{x}{4n+2}\right)^2 - 2\left(1+\frac{x}{4n+2}\right)\left(1-\frac{x}{4n+2}\right)\text{cos}.\frac{2\varphi}{2n+1} + \left(1-\frac{x}{4n+2}\right)^2$$

$$\times\left(\left(1+\frac{x}{4n+2}\right)^2 - 2\left(1+\frac{x}{4n+2}\right)\left(1-\frac{x}{4n+2}\right)\text{cos}.\frac{2\pi-2\varphi}{2n+1} + \left(1-\frac{x}{4n+2}\right)^2\right)$$

$$\times\left(\left(1+\frac{x}{4n+2}\right)^2 - 2\left(1+\frac{x}{4n+2}\right)\left(1-\frac{x}{4n+2}\right)\text{cos}.\frac{2\pi+2\varphi}{2n+1} + \left(1-\frac{x}{4n+2}\right)^2\right)$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\times\left(\left(1+\frac{x}{4n+2}\right)^2 - 2\left(1+\frac{x}{4n+2}\right)\left(1-\frac{x}{4n+2}\right)\text{cos}.\frac{2n\pi-2\varphi}{2n+1} + \left(1-\frac{x}{4n+2}\right)^2\right)$$

$$\times\left(\left(1+\frac{x}{4n+2}\right)^2 - 2\left(1+\frac{x}{4n+2}\right)\left(1-\frac{x}{4n+2}\right)\text{cos}.\frac{2n\pi+2\varphi}{2n+1} + \left(1-\frac{x}{4n+2}\right)^2\right)$$

= 2(1

$$
\begin{aligned}
&= 2\left(1-\cos.\frac{2\varphi}{2n+1}+\frac{xx}{(4n+2)^2}\left(1+\cos.\frac{2\varphi}{4n+1}\right)\right.\\
&\times 2\left(1-\cos.\frac{2\pi-2\varphi}{2n+1}+\frac{xx}{(4n+2)^2}\left(1+\cos.\frac{2\pi-2\varphi}{2n+1}\right)\right.\\
&\times 2\left(1-\cos.\frac{2\pi+2\varphi}{2n+1}+\frac{xx}{(4n+2)^2}\left(1+\cos.\frac{2\pi+2\varphi}{2n+1}\right)\right.\\
&\quad\vdots\qquad\qquad\qquad\vdots\qquad\qquad\qquad\vdots\\
&\times 2\left(1-\cos.\frac{2n\pi-2\varphi}{2n+1}+\frac{xx}{(4n+2)^2}\left(1+\cos.\frac{2n\pi-2\varphi}{2n+1}\right)\right.\\
&\times 2\left(1-\cos.\frac{2n\pi+2\varphi}{2n+1}+\frac{xx}{(4n+2)^2}\left(1+\cos.\frac{2n\pi+2\varphi}{2n+1}\right)\right).
\end{aligned}
$$

$$
\begin{aligned}
= 4^{2n+1}\sin.\frac{\varphi}{2n+1}\sin.^2\frac{\pi-\varphi}{2n+1}\sin.^2\frac{\pi+\varphi}{2n+1}\ldots\sin.^2\frac{n\pi-\varphi}{2n+1}\cdot\sin.^2\frac{n\pi+\varphi}{2n+2}
&\times\left(1+\frac{xx}{(4n+2)^2}\cot.^2\frac{\varphi}{2n+1}\right)\\
&\times\left(1+\frac{xx}{(4n+2)^2}\cot.^2\frac{\pi-\varphi}{2n+1}\right)\\
&\times\left(1+\frac{xx}{(4n+2)^2}\cot.^2\frac{\pi+\varphi}{2n+1}\right)\\
&\times\left(1+\frac{xx}{(4n+2)^2}\cot.^2\frac{2\pi-\varphi}{2n+1}\right)\\
&\quad\vdots\qquad\qquad\qquad\qquad\vdots\\
&\times\left(1+\frac{xx}{(4n+2)^2}\cot.^2\frac{n\pi-\varphi}{2n+1}\right)\\
&\times\left(1+\frac{xx}{(4n+2)^2}\cot.^2\frac{n\pi+\varphi}{2n+1}\right)
\end{aligned}
$$

$$
\begin{aligned}
= (\text{§. ag. *Introd.*})\ 4\sin.^2\varphi
&\times\left(1+\frac{xx}{(4n+2)^2}\cot.^2\frac{\varphi}{2n+1}\right)\\
&\times\left(1+\frac{xx}{(4n+2)^2}\cot.^2\frac{\pi-\varphi}{2n+2}\right)\\
&\times\left(1+\frac{xx}{(4n+2)^2}\cot.^2\frac{\pi+\varphi}{2n+1}\right)\\
&\quad\vdots\qquad\qquad\qquad\qquad\vdots\\
&\times\left(1+\frac{xx}{(4n+2)^2}\cot.^2\frac{n\pi-\varphi}{2n+1}\right)\\
&\times\left(1+\frac{xx}{(4n+2)^2}\cot.^2\frac{n\pi+\varphi}{2n+1}\right).
\end{aligned}
$$

 Cum

Cum hæc æquatio ſemper obtineat, etiam limites membrorum ejus ſunt inter ſe æquales (§. 4.). Unde (§§. 57. 75. 21.)

$$e^x - 2\cos.2\varphi + e^{-x} = 4\sin.\varphi \times \left(\left(1 + \frac{xx}{4\varphi\varphi}\right)\right.$$
$$\times \left(1 + \frac{xx}{4(\pi - \varphi)^2}\right)$$
$$\times \left(1 + \frac{xx}{4(\pi + \varphi)^2}\right)$$
$$\times \left(1 + \frac{xx}{4(2\pi - \varphi)^2}\right)$$
$$\times \; - \; - \; -$$
$$\times \; - \; - \; -$$

Eadem expreſſio eodem modo deducitur ex reſolutione in factores formulæ $a^{4n} - 2a^{2n}b^{2n}\cos.2\varphi + b^{2n}$.

Scholium. Quoniam applicationes formularum hoc capite demonſtratarum (tam ad ſummationes plurimarum ſerierum, quam ad varias expreſſiones functionum circularium & logarithmicarum) ab Eulero aliisque mathematicis ita traditæ ſunt, ut nihil, quod deſiderari poſſit, relinquant: mihi ſufficiat, eas ad vera ſua principia revocaſſe, & deductionem ipſarum ab indeterminata & obſcura infiniti notione liberaſſe. Vid. inter alios Euleri *Introductio.* Cap. IX. X. XI.

Caput Nonum.

De infinito, quod vocant, mathematico.

§. 88.

Fig. 11. Sit elipſis conica, cujus æquatio eſt $yy = \frac{bb}{aa}(aa - xx)$, abſciſſis axis ex centro ſumtis. Ducta tangente MT, quæ axi in T occurrat, fit $PT = -\left(\frac{aa}{x} - x\right)$ (§. 42.). Fiat $x = 0 = 0 \times a$; erit $PT = -\left(\frac{aa}{a \times 0} - 0\right) = -\frac{a}{0} = -\frac{1}{0} \times a$.

Gene-

Generatim. Sit ellipsis cujusvis ordinis, cujus æquatio $y^m = \frac{b^m}{a^m}(a^m - x^m)$: erit $PT = -(a \times \frac{a^{m-1}}{x^{m-1}})$; & facto $x = 0 \times a$, erit $PT = -a \times \frac{1}{0^{m-1}}$.

Cum expressiones $\frac{1}{0}$, $\frac{1}{0^m}$ frequentissime a mathematicis usurpentur, & in calculis præsertim superioribus sæpius occurrant: e re esse censeo (priusquam ulterius progrediar) veram horum symbolorum significationem expendere, & meam de illis sententiam profiteri; eo magis quod de indole ipsorum inter mathematicos non omnino convenit, pluresque etiam insignes & aliunde summopere commendandi scriptores opinionem de illis amplexi sunt, quæ cum sensu communi pugnare videtur. Quod ut faciam, quantum potero, accurate & luculenter, a familiari exemplo sequente ordiar.

§. 89. Duo viatores A & B, dato intervallo D a se invicem distantes, in eadem recta, datis velocitatibus, versus eandem plagam progrediantur; ita ut unus eorum A alterum B persequatur. Tres heic, quod ad rationem velocitatum viatorum attinet, distinguendi sunt casus.

1°. Viator A celerius progrediatur quam viator B. Hoc casu intervallum, quo duo viatores a se invicem primum distabant, continue minuitur, donec A ipsum B attingat.

2°. Ambo viatores æqualibus progrediantur velocitatibus. Hoc casu intervallum, quo duo viatores primum distabant, idem semper manet; utcunque diu progrediantur, & utcunque magnam viam ambo emetiantur.

3°. Viator A lentius progrediatur altero B. Hoc casu intervallum, quo duo viatores primum distabant, continue increscit; & tanto majus fit, quo diutius incedunt. Iidem viatores aut potuerunt anterius proficisci ex uno eodemque puncto, ad plagam sito oppositam ei, juxta quam progrediuntur; aut illic sibi jam mutuo occurrisse; aut denique in eo puncto sibi mutuo occurrerent, si uterque retrogrederetur.

Hinc dato intervallo, quo duo viatores a se invicem distant, datisque velocitatibus, quibus progrediuntur; si quæratur locus occursus ipsorum mutui: prius quam quæstionis propositæ investigatio suscipiatur, inquirendum est, utrum possi-

poſſibilis ſit, nec ne; ſeu inquirendum eſt in rationem velocitatum, qua prædicta poſſibilitas aut impoſſibilitas determinatur. Algebriſta autem non ſemper ita caute procedit, neque ſemper prævium hoc examen inſtituit. Univerſalitatis (nimio quandoque) ſtudio impulſus formulas quærit generales, quas ad omnes omnino caſus flectere ſatagit, & iis etiam applicare, ad quos non quadrant; quod hoc ipſo exemplo patebit.

Sint nimirum V, v, velocitates datæ duorum viatorum A, B, intervallo $= D$ primitus invicem diſtantium, & juxta eandem plagam progredientium, ita ut A ipſum B perſequatur. Velocitate V poſita majore quam v; quando viator A alterum B attingit, erit

via ab B percurſa $D \times \frac{v}{V-v}$

- - A - - $D \times \frac{V}{V-v}$; & formulæ hæ apte omnino repræſentant caſum, quo $V > v$.

Algebriſta autem, omnes univerſim caſus una eademque formula complecti ſatagens, quærit etiam: quid juxta eas fiat, ſi $V \equiv v$? $V \lessgtr v$?

Caſu autem, quo $V = v$, expreſſiones ſpatiorum ab utroque viatore percurſorum transformantur in hanc $D \times \frac{1}{0}$; cujus ſignificationem definire oportet.

Ut cum ſenſu communi reſponſum hoc algebræ conveniat, aliter illud interpretari non poſſum, niſi dicendo: illud indicare excluſionem ſeu impoſſibilitatem occurſus mutui viatorum. Algebraiſta autem de occurſu mutuo loqui pergens, dum de illo agi amplius non poteſt, viatores ſibi mutuo occurrere in *diſtantia infinita* pronuntiat. Quæ loquendi forma rite explicata nihil aliud ſignificare poteſt, quam locum occurſus mutui viatorum hoc caſu aſſignari non poſſe; nec ullibi unquam viatores ſibi mutuo occurrere. Symbolum igitur $\frac{1}{0}$, quod formulis præcedentibus caſui huic applicatis introducitur, de impoſſibilitate quæſtionis nunc propoſitæ nos monet, eodem prorſus modo, quo ſigna $\sqrt{-1}$, $\sqrt[2n]{-1}$ impoſſibilitatem docent ſolutionis & repugnantiam mutuam conditionum problematum, quorum tanquam poſſibilium tractatio algebraica ſigna illa introducere coëgit.

Quam-

Quamdiu $V > v$; diſtantia occurſus viatorum a locis, e quibus egredi ponuntur, pendet ab intervallo D horum locorum. Facto autem $V = v$, impoſſibilitas occurſus mutui eadem manet, quodcunque ſit intervallum D, quo primitus ſejungebantur; & abſurdum eſſet dicere: impoſſibilitatem hanc fieri duplam aut triplam, prouti intervallum illud ſit duplo aut triplo majus.

Quod ad tertium caſum attinet, quo $V < v$; formulæ præcedentes fiunt $D \times \frac{v}{-(v-V)}$, $D \times \frac{V}{-(v-V)}$, quæ transformantur in has $-D \times \frac{v}{v-V}$, $-D \times \frac{V}{v-V}$, & indicant: punctum occurſus mutui nunc jacere ad plagam oppoſitam ei, verſus quam viatores progrediuntur; neque in parte anteriori, ſed in parte poſteriori eſſe quærendum. Quoniam autem caſus hic ad præcipuum tractationis præſentis objectum non pertinet, breviter de eo in ſequentibus agetur.

Exemplum allatum adeo luculentum eſt, ut ſupervacaneum foret diutius rei huic immorari, niſi ea, utut per ſe ſimplex, pluribus inanium loquendi modorum tricis involuta fuiſſet. Liceat itaque idem exemplum ſub forma tantillum diverſa, eaque geometrica, exponere.

Problema generale hoc eſt: datam rectam in ratione data producere.

Sit AB recta magnitudine data, producenda ad X usque, in directione quæ tendit ab A verſus B; ita ut rectæ AX, BX ſint inter ſe in ratione data, quam habent lineæ a & b. Fig. 19. 1°.

Conſtructio. Ex punctis A, B ad easdem rectæ AB partes ducantur duæ rectæ Aa, Bb, ſibi invicem parallelæ, & in data ratione linearum a, b. Jungatur ab recta, quæ (ſi fieri poſſit) datæ AB productæ occurrat in X. Hoc erit punctum quæſitum.

Ut punctum X jaceat verſus plagam propoſitam, oportet, ſit $Aa > Bb$. Sed ſi $Aa < Bb$, recta ba occurrit datæ AB verſus plagam priori oppoſitam productæ.

Sit autem $Aa = Bb$. Quadrilateri $AabB$ lateribus Aa, Bb poſitis ſibi invicem æqualibus & parallelis, quadrilaterum hoc eſt parallelogrammum (El. I. 33.); proinde lineæ AB, ab ſibi mutuo occurrere non poſſunt. Quod ſi parallelæ eſſe Fig. 19. 2°.

dicantur rectæ, quæ ſibi mutuo in diſtantia infinita occurrunt, idem erit, ac ſi dicatur: eas ſibi mutuo nullibi, ſeu non occurrere.

Fig. 19. 1°. Ut calculus conſtructioni præcedenti applicetur, datæ AB per punctum b parallela agatur, quæ rectæ Aa in a' occurrat. Triangula $aa'b$, aAX, bBX ſunt invicem ſimilia; proinde $\begin{matrix} aa' : aA = ba' : AX \\ aa' : bB = ba' : BX \end{matrix}$; ſeu $\begin{matrix} a-b : a = AB : AX \\ a-b : b = AB : BX \end{matrix}$;

unde $\begin{matrix} AX = AB \times \dfrac{a}{a-b} \\ BX = AB \times \dfrac{b}{a-b}. \end{matrix}$

Cum calculus hic nitatur triangulorum $aa'b$, aAX, bBX ſimilitudine, omnino exiſtentiam eorum triangulorum ſupponit; quæ cum non exiſtant, quando punctum a' incidit in punctum a, ſeu quando $Aa = Bb$; mirum videri non debet, coactam calculi ad hunc caſum applicationem, utpote fundamento ſuo deſtitutam, ad impoſſibile deducere.

Si eſſet punctum occurſus eo caſu, quo Aa & Bb inter ſe ſunt æquales; eſſent etiam AX & BX invicem æquales: cum tamen quantitate data AB differre ſupponantur. Abſurdum hoc non moratur infiniti fautores, qui duas magnitudines infinitas invicem æquales manere ſtatuunt, ſi una illarum (vel utraque) finitam patiatur mutationem. Genuinus formulæ hujus loquendi ſenſus alius eſſe non poteſt, quam hic: *Sint duæ quantitates mutabiles, quarum utraque major reddi poteſt quacunque quantitate propoſita, & data ſit differentia prædictarum quantitatum; ratio æqualitatis limes eſt rationis decreſcentis majoris ad minorem.* Hoc patet ex ipſa conſtructione præcedente. Data enim ratione quacunque $Aa : Bb$ majoris ad minorem, utut parum ab ratione æqualitatis differat; fiat $Bb' \begin{smallmatrix} > Bb \\ < Aa \end{smallmatrix}$, ſeu fiat $Aa : Bb' \begin{smallmatrix} > 1 : 1 \\ < Aa : Bb \end{smallmatrix}$; ducta ab' occurret ipſi AB productæ in X'; & fiet $AX' : BX' = Aa : Bb' \begin{smallmatrix} > 1 : 1 \\ < Aa : Bb \end{smallmatrix}$. Idemque facile methodo mere algebraica demonſtratur.

Fig. 19. 2°. Negatio occurſus linearum AB, ab nititur æqualitate linearum Aa, Bb, ſeu valore (180°) ſummæ angulorum BAa, baA; nec ad ejus excluſionem quicquam confert magnitudo lineæ AB: & quemadmodum idea parallelismi duarum

rum rectarum unica est, ita etiam dici nequit impossibilitatem occursus mutui duarum linearum AB, ab majorem esse aut minorem, prouti linea AB est major aut minor; seu signum impossibilitatis $\frac{1}{0}$ est signum unicum, quæcunque sit quantitas a, quacum per multiplicationis signum conjungatur; eodem prorsus modo, quo omnia signa imaginaria $a\sqrt{-1}$ ad unicum $\sqrt{-1}$ communi mathematicorum consensu reducuntur.

§. 90. Geometricam problematis propositi (utut elementaris) evolutionem eo lubentius exposui, quod hæc ipsa quæstio, aut aliæ ipsi affines, sub falso obtutu consideratæ, variis loquendi modis (meo quidem judicio sensu cassis) ansam præbuerunt. Dicunt v. gr. algebristæ, lineam rectam circulum esse, cujus radius est infinitus. Uno aut altero exemplo indicare sufficiet, quid sibi velit ejusmodi loquendi modus.

Sint duo puncta positione data. Notum est: rectam, quæ bifariam & ad angulos rectos secat lineam rectam puncta data jungentem, locum esse omnium punctorum ab his punctis æqui-distantium. Nempe non modo quodvis punctum in perpendiculo hoc situm æqui-distat a duobus punctis datis; sed etiam quodvis punctum extra hoc perpendiculum (in eodem plano) situm inæqualiter distat ab his punctis.

Notum pariter est (vid. APOLLONII PERGÆI *Loca plana* L.II. Prop. 2.): circumferentiam circuli locum esse omnium punctorum, quorum distantiæ a duobus punctis datis sunt inter se in ratione data a ratione æqualitatis diversa.

Duos hosce casus, utut inter se diversos, & ab antiquis geometris sedulo sejunctos, sub uno complecti satagunt algebristæ. Cum radius circuli, cujus circumferentia est locus propositus, eo major fiat, quo ratio data, a ratione æqualitatis diversa, ad eam propius accedit; & quemadmodum ratio data ad rationem æqualitatis propius accedere potest quam ratio quævis proposita a ratione æqualitatis diversa, ita etiam radius prædicti circuli quacunque linea data major fieri possit: concludunt inde, quod, si ratio data fuerit ipsa ratio æqualitatis, radius circuli fiat infinitus. Sed hoc casu locus propositus est linea recta; ergo (dicunt,) linea recta est circulus, cujus radius infinitus. Cum autem hoc casu idea circuli necessario connectatur cum discrepantia rationis datæ a ra-

tione æqualitatis, ſublata hac diſcrepantia tollitur etiam idea circuli. Qui dicit, lineam rectam eſſe circulum, cujus radius eſt infinitus, de circulo loquitur, cujus neque centrum neque radius poſſunt aſſignari; dicit itaque, lineam rectam circulum eſſe non circularem.

Fig. 19. 10. Scilicet: ſit diſtantia AB punctorum A, $B = D$; & ſit ratio data æqualis rationi $a : b$, a ratione æqualitatis diverſæ. Secetur AB in puncto x, eademque producatur ad X usque in ratione data. Biſecetur Xx in Z; erit Z centrum, & Zx radius circuli propoſiti. Fit $Zx = D \times \frac{ab}{\overline{a-b}.\overline{a+b}}$

$$AZ = D \times \frac{aa}{\overline{a-b}.\overline{a+b}}$$

$$BZ = D \times \frac{bb}{\overline{a-b}.\overline{a+b}};$$ unde inferunt algebriſtæ, Zx, AZ, BZ infinitas fieri, quando $a = b$; cum dicere debuiſſent, nullum eſſe radium, nullum centrum.

Idem valet de aliis quibusdam locis ad circumferentiam circuli. Sint v. gr. duo puncta poſitione data, ad quæ ex puncto quocunque agantur rectæ; & data ſit differentia ſpatiorum, quæ ad quadrata harum rectarum datas habent rationes, inter ſe inæquales. Locus prædicti puncti etiam eſt circumferentia quædam circuli (Apollonii pergæi *Loca plana* L. II. Prop. 4.). Sed ſi rationes datæ ſint inter ſe æquales, locus puncti illius eſt recta poſitione data. Unde iterum concludunt algebriſtæ, lineam rectam circulum eſſe, cujus radius eſt infinitus; cum antiqui geometræ caſum hunc ab altero diverſum ſeorſim pariter tractaverint.

Atque hic rurſus ſe offert ſimplicitas ideæ impoſſibilitatis. Quemadmodum enim linea recta ita eſt unica, ut omnes lineæ rectæ mutuo congruant; ita etiam impoſſibilitas curvaturæ ſit unica, dum contra nullus eſt limes variorum graduum curvaturæ.

§. 91. Quam de vera ſymboli $\frac{1}{0}$ ſignificatione profeſſus ſum ſententiam, altero exemplo, eoque mere algebraico, confirmabo.

Sit

Sit fractio $\frac{1}{x-a.x-b} = \frac{1}{a-b}.\frac{1}{x-a}$
$+ \frac{1}{b-a}.\frac{1}{x-b}.$

Hinc per fractionem $\frac{1}{x-a}$ utrinque multiplicando, fit

$$\frac{1}{(x-a)^2.x-b} = \frac{1}{a-b}.\frac{1}{x-a^2}$$
$$+ \frac{1}{b-a}.\frac{1}{x-a.x-b}$$
$$= \frac{1}{a-b}.\frac{1}{x-a^2}$$
$$- \frac{1}{a-b^2}.\frac{1}{x-a}$$
$$+ \frac{1}{a-b^2}.\frac{1}{x-b}.$$

Fractio igitur $\frac{1}{x-a^2.x-b}$, cujus denominator factorem duplicem $x-a^2$ continet, nequit unice resolvi in fractiones, quarum denominatores factoribus simplicibus $x-a$, $x-b$ æquales sint. Sed resolutio fractionis illius necessario continet fractionem, cujus denominator est factor duplex $x-a^2$.

Proponatur autem fractio $\frac{1}{x-a.x-a'.x-b}$, cujus factores denominatoris $x-a$, $x-a'$, aut reipsa inæquales sint, aut tanquam tales tractentur. Erit

$$\frac{1}{x-a.x-a'.x-b} = \frac{1}{a-a'.a-b}.\frac{1}{x-a}$$
$$+ \frac{1}{a'-a.a'-b}.\frac{1}{x-a'}$$
$$+ \frac{1}{b-a.b-a'}.\frac{1}{x-b}.$$

Tum fiat $a=a'$; erit $\frac{1}{x-a.x-a'.x-b} = \frac{1}{a-a'.a-b}.\frac{1}{x-a} = \frac{1}{0}.\frac{1}{a.a-b}.\frac{1}{x-a}$

$$+ \frac{1}{a'-a.a'-b}.\frac{1}{x-a'} \quad \pm \frac{1}{0}.\frac{1}{a.a-b}.\frac{1}{x-a}$$
$$+ \frac{1}{b-a.b-a'}.\frac{1}{x-b} \quad + \quad \frac{1}{b-a^2}.\frac{1}{x-b}$$

Pro-

Proinde introductione symboli $\frac{1}{0}$ monemur, resolutionem functionis $\frac{1}{(x-a)^2}\cdot\frac{1}{x-b}$ in fractiones, quarum denominatores sint factores simplices $x-a$, $x-b$, esse impossibilem, uti jam vidimus.

Hoc casu, ut ad veram resolutionem perveniamus, conjungendæ sunt duæ fractiones $\begin{array}{c}\frac{1}{\overline{a-a'}\cdot\overline{a-b}}\cdot\frac{1}{x-a}\\ \frac{1}{\overline{a'-a}\cdot\overline{a'-b}}\cdot\frac{1}{x-a'}\end{array}$, considerando factores $\frac{\overline{a-b}\cdot\overline{a'-b}}{\overline{x-a}\cdot\overline{x-a'}}$, tanquam respective diversos. Erit

$$\frac{1}{a-a'}\left\{\begin{array}{l}\frac{1}{a-b}\cdot\frac{1}{x-a}\\ -\frac{1}{a'-b}\cdot\frac{1}{x-a'}\end{array}\right\} = \frac{1}{a-a'}\times\frac{aa-a'a'-b(a-a')-x(a-a')}{\overline{a-b}\cdot\overline{a'-b}\cdot\overline{x-a}\cdot\overline{x-a'}} = \frac{a+a'-b-x}{\overline{a-b}\cdot\overline{a'-b}\cdot\overline{x-a}\cdot\overline{x-a'}}$$

$$= \frac{a-b-(x-a')}{\overline{a-b}\cdot\overline{a'-b}\cdot\overline{x-a}\cdot\overline{x-a'}} = \begin{array}{l}\frac{1}{a'-b}\cdot\frac{1}{\overline{x-a}\cdot\overline{x-a'}}\\ -\frac{1}{\overline{a-b}\cdot\overline{a'-b}}\cdot\frac{1}{x-a}\end{array} = \begin{array}{l}\frac{1}{a-b}\cdot\frac{1}{\overline{x-a}^2}\\ -\frac{1}{\overline{a-b}^2}\cdot\frac{1}{x-a}\end{array},\ \text{casu, quo}\ a = a'.$$

Idem dicatur de aliis fractionum resolutionibus.

§. 92. Transeo ad exemplum geometricum, & rei præsenti omnino accommodatum.

Cum in omni triangulo (rectilineo) angulorum internorum summa æqualis sit duobus rectis: si detur summa duorum angulorum internorum, ad trianguli possibilitatem requiritur, ut summa minor sit duobus rectis. Proinde omnes calculi in triangulis, quorum summa duorum angulorum datur, instituti cum hac propositione fundamentali debent consentire.

Sit trianguium, cujus latera sint A, B, C;
& anguli ipsis respective oppositi a, b, c.

Anguli b & c, & latus A dentur magnitudine; erit

$$B = A\frac{\text{sin.}\,b}{\text{sin.}\,a} = A\frac{\text{sin.}\,b}{\text{sin.}\,180^\circ-(b+c)} = A\frac{\text{sin.}\,b}{\text{sin.}\,\overline{b+c}}$$

$$C = \quad - \quad\quad - \quad = A\frac{\text{sin.}\,c}{\text{sin.}\,\overline{b+c}},$$ quæ expressiones veræ sunt, quamdiu $b+c < 180^\circ$.

Sit

Sit $b+c=180°$; lineæ B & C fiunt parallelæ: ſublato linearum B, C mutuo occurſu, ſimul evaneſcit omnis idea tam anguli a, quam trianguli & laterum illius; & ſimul omnia ruunt fundamenta propoſitionum, quæ nonniſi de triangulis actu exiſtentibus poſſunt prædicari. Algebriſta autem nimio univerſalitatis ſtudio formulas præcedentes etiam nunc conſervare, & ad eum quoque caſum flectere aggreditur, in quem quadrare non poſſunt. Cum ſcilicet poſito $b+c=180°$, & proinde $\text{ſin.}\, b+c=0$; formulæ præcedentes fiant

$$B = A\,\frac{\text{ſin.}\,b}{0},\quad C = A\,\frac{\text{ſin.}\,c}{0}$$

pronuntiat: verticem trianguli etiamnum ficti ab latere A infinite diſtare, ipſaque latera B & C infinita fieri. Hic autem loquendi modus (meo quidem judicio) aliter tolerari non poteſt, niſi quatenus impoſſibilitatem afferat trianguli, cujus duorum angulorum ſumma ſit duobus rectis æqualis. Hæc autem impoſſibilitas a valore ſummæ angulorum b, c unice pendet, quæcunque ſit rectæ A ipſis adjacentis magnitudo, & quicunque ſint anguli b, c ſeorſim ſumti. Attamen ſi formulæ, quibus trianguli latera determinantur, huic etiam caſui applicari poſſent, quo deeſt punctum occurſus; impoſſibilitas determinationis laterum trianguli etiamnum ficti diverſa appareret, pro diverſa magnitudine tam lineæ A, quam angulorum b & c, juxta formulas $B=A\frac{\text{ſin.}\,b}{0}$, $C=A\frac{\text{ſin.}\,c}{0}$. Unde ſequeretur, lineas ab uno eodemque puncto iuxta datam directionem ductas eſſe inter ſe magnitudine utlibet diverſas, eo momento, quo infinite eſſe dicuntur.

Idem luculentius etiam patet, quando formulæ, quæ veræ ſunt de ſuperficiebus triangulorum, ſpatiis applicantur, quæ jam non ſunt triangula.

Superficies trianguli, cujus latera & anguli ſunt ut prius $\begin{matrix} A, & B, & C \\ a, & b, & c \end{matrix}$, eſt $\frac{1}{2}A^2 \times \frac{\text{ſin.}\,b\ \text{ſin.}\,c}{\text{ſin.}\,b+c}$; quæ formula caſu, quo $b+c=180°$, & proinde $\text{ſin.}\,b=\text{ſin.}\,c$, transformatur in hanc $\frac{1}{2}A^2\,\frac{\text{ſin.}^2\,b}{0}$. Proinde ſi formulas, quæ de ſpatiis finitis veræ ſunt, ſpatiis etiam illimitatis applicare liceret; ſpatia illimitata inter duas

rectas

rectas parallelas extensa forent in ratione duplicata distantiarum harum parallelarum, cum spatia hæc inter se esse in ratione simplici harum distantiarum tradantur.

§. 93. Ut hactenus dicta, quantum fieri potest, distincte curvarum doctrinæ possint applicari; a curva in elementis considerata, circulo nempe, ordiar.

Fig. 20. Sit circulus, cujus centrum C, radius $CA = r$. Ex puncto M ducatur tangens MT, quæ radio CA producto in T occurrat; seu sit MT ipsi CM perpendicularis, atque ex eodem puncto M demittatur MP perpendicularis radio CA. Sit $CP = x$; erit $CT = \frac{rr}{x}$. Expressio hæc (ex similitudine triangulorum CMP, CTM deducta) valorem rectæ CT determinat, quamdiu ipsa CT existit. Ut autem proportio $CP : CM = CM : CT$, seu $x : r = r : CT$, institui possit, necesse est, ut CP ad CM seu x ad r aliquam habeat rationem; seu necesse est, (triangulo CMP existente) linea CP non sit zero. Posita igitur $CP = 0 = 0 \times r$, symbolum $CT = \frac{rr}{0 \times r} = \frac{r}{0}$ monet non amplius locum dari subtangenti. Hoc etiam ex antedictis fluit (§. 92.), cum hoc casu anguli ACM, CMT ambo sint recti; ac proinde lineæ CA, MT invicem parallelæ.

Idem valet de ellipsi, cujus æquatio est $y = \frac{b}{a}\sqrt{(aa - xx)}$. Itaque subtangens $PT = \frac{aa}{x} - x$; unde cot.$PMT = \frac{b}{a} \frac{x}{\sqrt{aa - xx}}$. Facto $x = 0 = a \times 0$, fit $PT = \frac{a}{0}$, cot.$PMT = \frac{b}{a} \times 0$; unde angulus $PMT = 90°$, & PT determinari nequit.

Ut hæc clarius etiam, si fieri possit, ob oculos ponantur, regrediamur ad definitionem subtangentis, & ad modum, quo determinavimus, eam esse limitem (si quis sit) subsecantis. Posuimus (§. 39.) tangentem curvæ diametro, ad quam refertur, occurrere. Intervallum quoddam datum limes est datus (juxta definitionem §. 1.), ad quem subsecans perpetuo accedit, sed quem nunquam attingit. Sublato tangentis & diametri occursu mutuo, fundamenta omnia, quibus rationis, quam subsecans habet ad lineam diametro ordinatim applicatam, limitem determinavimus, per se labuntur. Linea MQ, diametro & proinde tangenti parallela, tangenti non amplius occurrit. Sublato itaque triangulo

Fig. 21.

gulo MQQ', cujus contemplatione nituntur ratiocinia §. 39.; ipſa etiam hæc ratiocinia & conſequentiæ ex illis deductæ corruunt. Nominatim cadit conſequentia: rectas PT, MT, non amplius determinatas, limites eſſe rectarum PS, MS.

Sit $AP = x$; $PM = y$; $Pp = \Delta x$;

$$mp = y \pm \frac{\Delta x}{1} \cdot \frac{dy}{dx} + \frac{\Delta x^2}{1.2} \cdot \frac{ddy}{dx^2} \pm \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} + \ldots .$$

$$\& \quad \Delta y = \pm \frac{\Delta x}{1} \cdot \frac{dy}{dx} + \frac{\Delta x^2}{1.2} \cdot \frac{ddy}{dx^2} \pm \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} + \ldots .$$

Itaque $PS\left(= y \cdot \frac{\Delta x}{\Delta y}\right) = y \times \frac{1}{\pm \frac{dy}{dx} + \frac{\Delta x}{1.2} \cdot \frac{ddy}{dx^2} \pm \frac{\Delta x^2}{1.2.3} \cdot \frac{d^3y}{dx^3} + \ldots}$

Ut aliquis ſit limes rectæ PS, oportet, ſit limes aliquis expreſſionis ipſius; proindeque exponens differentialis $\frac{dy}{dx}$ non ſit zero. Quodſi autem $\frac{dy}{dx} = 0$: ſymbolo $PS = y \times \frac{1}{0}$ monemur, ſubſecantem PS ultra omnem limitem datum creſcere poſſe; de ſubtangente igitur non eſſe loquendum.

Caſus proinde, quo tangens MT & diameter PA invicem ſunt parallelæ, tanquam ſingularis & unicus eſt conſiderandus. Scilicet ex puncto M ducatur recta quæcunque MS, quæ cum tangente MT angulum faciat TMS utcunque exiguum. Recta hæc MS occurrit diametro AP, cum ſumma angulorum SMP, SPM (minor ſumma angulorum TMP, MPA) minor ſit duobus rectis; eademque MS curvæ etiam in altero puncto M' occurrit (cum jam recta MT curvam in puncto M contingere ſupponatur). Imminuto angulo TMS, lineæ MS, PS creſcunt quidem; veruntamen magnitudo illarum definitur per hunc angulum. Neque dici poteſt (juxta definitionem §. 1.) prædictas lineas limitum eſſe capaces, easdemque a lineis magnitudine indeterminatis (ſeu neque datis neque dabilibus) minus differre poſſe, quam ulla quantitate propoſita.

§. 94. Quoniam frequentiſſime a mathematicis uſurpatur formula: parabolam eſſe ellipſin infinitam, ſeu, cujus axis eſt infinitus; e re erit genuinam locutionis hujus ſignificationem declarare: quod pluribus præſtabo modis, ut, quæ de hoc exemplo monuero, facilius poſſint ad alia ſimilia applicari.

Cum nempe modos inter geneſeos ſectionum conicarum tres ſint præcipui, ad quos reliqui facile poſſunt reduci; unumquemque illorum ſeorſim expendam.

1°. Sectio conica referatur ad focum & ad directricem.

Fig. 21. Sit AB recta poſitione data, & ſit F punctum poſitione datum; & deſcribenda ſit ellipſis, cujus focus F & directrix AB, data ratione diſtantiarum ſingulorum ellipſis punctorum a foco F & a directrice AB.

Ex puncto F demittatur in AB perpendicularis FD, quæ ſecetur in S in ratione data; erit S vertex axis transverſi ſectionis, foco F vicinior.

Quoniam ſectio conica propoſita debet eſſe elliptica: ut vertex alter ſectionis ſeu ejusdem axis determinetur, producenda eſt recta DF in S', ita ut $DS' : FS' = DS : SF$; & quoniam DS' debet eſſe major quam FS', oportet, ſit $DS > SF$. Seu ut ſectio propoſita ſit elliptica, ratio data diſtantiarum puncti cujusvis ſectionis a directrice & a foco debet eſſe ratio majoris ad minorem.

Quo poſito erit SS' axis transverſus ellipſis; biſecta autem SS' in C, erit CS dimidius axis transverſus, & CF excentricitas.

Sit $FD = d$; ſitque ratio data $SF : SD = a : b$:

$$\text{erit } SF = d \times \frac{a}{a+b}; \quad SD = \frac{b}{a+b}$$

$$S'F = d \times \frac{a}{a-b}; \quad S'D = d \times \frac{b}{a-b}$$

$$SS' = d \times \frac{2aa}{a-b.a+b}$$

$$CS = d \times \frac{aa}{a-b.a+b}$$

$$CF = d \times \frac{ab}{a-b.a+b}$$

quadratum dimidii axis ſecundi $SF \times S'F = dd \times \dfrac{aa}{a-b.a+b}$.

Quando ratio data $a : b$ eſt ratio æqualitatis: puncta S', C, F' non amplius poſſunt determinari; ſeu punctorum horum poſitiones ſunt impoſſibiles, & rectarum SS', SC, SF' magnitudines pariter fiunt impoſſibiles: ſcilicet curva deſcribenda non amplius in ſe recurrit, ac proinde præcipuum cnrvarum ellipticarum caracterem amittit. Curva itaque hoc caſu deſinit eſſe elliptica; ſed eſt curva ſui

ſui generis, parabola nempe, quæ neque centrum, neque alterum focum, neque alterum verticem habet. Qui dicit, parabolam eſſe ellipſin, cujus centrum a foco infinite diſtat; dicit, parabolam eſſe ellipſin, cujus centrum aſſignari nequit, ſeu cujus centrum nullibi exiſtit, ſeu quæ centrum non habet. Parabola igitur eſſet ellipſis non elliptica; quod ſenſu caret. Parabola igitur curva eſt non elliptica, & ſui generis.

Impoſſibilitatem poſitionis punctorum S', C, F', & magnitudinis rectarum FS', FC, FF', indicant formulæ præcedentes; quæ, poſito $a=b$, impoſſibilitatis ſigno $\frac{1}{0}$ afficiuntur.

Quemadmodum autem ratio æqualitatis limes eſt rationis decreſcentis majoris ad minorem, ita etiam ratio æqualitatis limes eſt rationis diſtantiarum cujusvis puncti ellipſis a directrice & a foco huic proximo. Unde parabola limes eſt ellipſium eandem directricem eundemque focum poſitione datos habentium, eodem prorſus modo, quo circulus limes eſt figurarum ipſi inſcriptarum aut circumſcriptarum.

2°. Sectionum conicarum originem in ipſo cono contemplabor.

Sit ACA' ſectio coni recti (v. gr.), plano per axem transeunte facta. Sit SMS' coni hujus ſectio plano priori perpendiculari facta, quodque lateribus CA, CA' in punctis S & S' occurrat. Erit igitur ſectio communis utriusque plani SS' axis transverſus ellipſis, & expreſſio axis hujus eſt Fig. 22.

$$CS \times \frac{\text{ſin.}C}{\text{ſin.}S'} = CS \times \frac{\text{ſin.}C}{\text{ſin.}(S'SA-C)} = SS'.$$

Quamdiu angulus $S'SA$ major eſt angulo C: planum SMS' lateri CA' occurrit; proinde tam puncti S' poſitio, quam rectæ SS' magnitudo determinatur, & ſectio eſt elliptica; utcunque parum angulus ASS' (major angulo C) ab hoc angulo differat. Atque uti angulus C limes eſt anguli decreſcentis ASS' (quatenus angulus hic major ponitur angulo C); ita etiam parabola, quæ ſectio eſt coni, quando angulus ASS' deſinit eſſe major angulo C, ſeu fit ipſi æqualis, limes eſt ſectionis, quæ reſpondet inæqualitati angulorum ASS' & C: ac ceſſatio ſectionis mutuæ rectarum SS', CA', proindeque impoſſibilitas magnitudinis rectæ SS' denotatur introductione ſigni $\frac{1}{0}$ in expreſſionem lineæ SS', quæ fit

$$SS' = CS \times \frac{\text{ſin.}C}{\text{ſin.}0} = CS \times \frac{\text{ſin.}C}{0}.$$

Sit MP recta ipsi SS' perpendicularis, seu axi SS' ordinatim applicata; & per MP agatur planum basi parallelum, quod lateribus coni CA, CA' in punctis B, B', occurrat. Erit $MP^2 = BP \times PB'$. Atqui

$$BP : SP = \text{sin.}S : \text{sin.}B$$
$$PB' : S'P = \text{sin.}S' : \text{sin.}B'$$

hinc $BP \times PB' : SP \times PS' = \text{sin.}S \times \text{sin.}S' : \text{sin.}B \times \text{sin.}B'$.

Proportio hæc locum habet, quamdiu punctum S' assignatur. Deficiente autem puncto S', & proinde deficiente simul triangulo $PB'S'$, proportio $PB' : S'P = \text{sin.}S' : \text{sin.}B'$ (quæ trianguli $PS'B'$ existentia nititur) institui non potest. Sed casus hic singularis peculiari etiam modo erui debet. Tum scilicet recta PB' datur magnitudine; & ex priori proportione $BP : SP = \text{sin.}S : \text{sin.}B$ consequitur $BP \times PB' : SP \times PB' = \text{sin.}S : \text{sin.}B = \text{sin.}C : \text{sin.}B$ seu $MP^2 : SP \times PB' = \text{sin.}C : \text{sin.}B$, quæ est proportio ad parabolam.

Altera proportio $PB' : S'P = \text{sin.}S' : \text{sin.}B'$ efficit $S'P = B'P\dfrac{\text{sin.}B'}{\text{sin.}S'}$; & impossibilitas rectæ $S'P$ denotatur introductione signi $\frac{1}{0}$, quando $S' = 0$.

3°. Consideremus denique sectionum ellipticarum & parabolicarum æquationes a parametro ipsarum pendentes.

Fig. 23. Sit SS' axis ellipseos, cujus parameter sit SA. Sit MP recta axi ordinatim applicata. Ducatur $S'A$, cui MP occurrat in N.

Per naturam ellipsis est $MP^2 : SP \times PS' = AS : SS' = NP : PS' = SP \times PN : SP \times PS'$; proinde $MP^2 = SP \times PN$. Ducatur per A recta axi SS' parallela, cui MP occurrat in Q; erit $MP^2 = SP(SA - NQ)$.

Quamdiu igitur curva proposita est elliptica, seu quamdiu punctum S' datur positione; quadratum ordinatæ MP minus est rectangulo sub parametro SA & abscissa SP. Atqui puncto P manente eodem; quo major est recta SS', eo minor fit recta $NQ\left(= SP \times \dfrac{SA}{SS'}\right)$: & quemadmodum nullus est limes magnitudinis rectæ SS', ita etiam nullus est limes parvitatis rectæ NQ; seu parameter SA limes est rectæ $PN(= SA - NQ)$, & rectangulum $SA \times SP$ limes est magnitudinis rectanguli $SP \times PN$. Ordinatæ igitur parabolæ limites sunt ordinatarum ellipsium per eadem axis puncta ductarum; denique ipsa parabola limes est elli-

ellipſium eadem parametro deſcriptarum; ad quem curvæ hæ eo propius accedunt, quo major fit earum axis transverſus.

Hinc intelligitur, quomodo æquatio elliptica $yy = x(p - \frac{p}{a}x)$, parabolicam $yy = px$ contineat: quatenus ſcilicet recta AS' ipſi SP non amplius occurrente; ſeu facta ipſi SS' parallela, recta PN fit ipſi SA æqualis, & proinde NQ ſeu $\frac{p}{a}x$ fit zero.

Quæcunque de relatione, quæ curvas inter parabolicas & ellipticas locum habet, dicta ſunt, applicari pariter debent ad relationem curvas inter parabolicas & hyperbolicas intercedentem: & quemadmodum v. gr. ordinatæ parabolæ limites ſunt ordinatarum creſcentium ellipſium eundem verticem eandemque parametrum habentium; ita etiam ordinatæ parabolæ limites ſunt ordinatarum decreſcentium hyperbolarum eundem verticem eandemque parametrum habentium. Scilicet in æquatione hyperbolæ, $yy = x(p + \frac{p}{a}x)$, parameter data p limes eſt parvitatis quantitatis mutabilis $p + \frac{p}{a}x$, & rectangulum px limes eſt parvitatis rectanguli decreſcentis $x(p + \frac{p}{a}x)$; denique rectæ in parabola axi ordinatim applicatæ limites ſunt ordinatarum decreſcentium hyperbolarum, iisdem axeos punctis reſpondentium.

Porro quæ de ſectionibus conicis dicta fuerunt, facile (mutatis mutandis) ad alias curvas ellipticas, parabolicas, & hyperbolicas ſuperiorum generum applicentur; quare diutius his immorari ſupervacaneum eſſe cenſeo.

§. 95. Alia exempla, ſententiam de vera ſymboli $\frac{1}{0}$ ſignificatione hactenus expoſitam illuſtrantia, brevius perſtringam.

1°. Sit æquatio hyperbolæ conicæ ad aſymptoton relatæ $xy = ab$; & quæratur punctum, ubi aſymptoto curva occurrit, ſeu ubi fit $y = 0 = b \times 0$. Quoniam eſt $x = \frac{ab}{y}$; facto $y = 0 = b \times 0$, erit $x = \frac{a}{0}$; proinde occurſus mutuus curvæ hyperbolicæ & aſymptoti eſt impoſſibilis.

Æquatione $x = \frac{ab}{0}$ monemur: ſuperficie rectanguli alicujus magnitudine data, non poſſe unum ex lateribus ejus ita magnum fieri, ut alterum evaneſcat.

 2°. Sit

2°. Sit æquatio hyperbolæ conicæ ad axem transversum relatæ, $yy = \frac{bb}{aa}(xx - aa)$; & quæratur tangens, quæ axi occurrat in centro C.

Quoniam est $y = \frac{b}{a}\sqrt{(xx\text{-}aa)}$; fit $\frac{dy}{dx} = \frac{b}{a} \times \frac{x}{\sqrt{(xx\text{-}aa)}}$, $\frac{dx}{dy} = \frac{a}{b} \times \frac{\sqrt{(xx\text{-}aa)}}{x}$, & $y\frac{dx}{dy} = \frac{xx\text{-}aa}{x} = x - \frac{aa}{x}$, unde distantia centri a puncto, ubi tangens axi occurrit, est $\frac{aa}{x}$: quare distantia hac magnitudine data d, fit $\frac{aa}{x} = d$; & $x = \frac{aa}{d}$. Posita autem $d = 0 = a \times 0$, fit $x = \frac{a}{0}$, & proinde impossibilis. Nulla igitur est tangens hyperbolæ, quæ axi in centro occurrat; sed punctum, ubi tangens axi occurrit, eo propius ad centrum accedit, quo punctum, ex quo tangens ducitur, a centro longius removetur.

3°. Sit parabola conica, cujus æquatio est $y = \sqrt{2px}$; & quæratur punctum parabolæ, ex quo ducta tangens fiat axi parallela.

Quoniam $y = \sqrt{2px}$; fit $\frac{dy}{dx} = \frac{p}{\sqrt{2px}} = \frac{p}{y}$. Jam vero (§. 41.) $\frac{dy}{dx}$ est tangens trigonometrica (quæ dicatur t) anguli, sub quo tangens parabolæ axi occurrit. Est ideo $\frac{p}{y} = t$; & $y = \frac{p}{t}$. Fiat autem tangens axi parallela, & proinde nullum faciat angulum cum axe: erit etiam $t = 0$, & $y = \frac{p}{0}$; quod impossibilitatem arguit ducendi rectam axi parabolæ parallelam, quæ hanc curvam tangat.

§. 96. Ne quis autem hactenus dicta male interpretetur, quasi signa $\frac{1}{0}$, $(\frac{1}{0})^n$ ab algebristis introducta ex mathesi proscribere velim. Quemadmodum alia impossibilitatis signa $\sqrt{-1}$, $\sqrt[2n]{-1}$, a mathematicis usurpata, plurimas investigationes mirifice juvant, quæ absque illorum subsidio (pro intellectus humani debilitate) aut impossibiles fuissent aut longe difficiliores; ita etiam hæc impossibilitatis signa, $\frac{1}{0}$, $(\frac{1}{0})^n$, adminicula esse possunt & instrumenta, quibus intellectus humanus juvatur, ut ad investigationes reales possit eniti. Dicam ideo cum omnibus algebristis: generalem sectionum conicarum (v. gr.) æquationem

nem esse $yy = x(p \pm \frac{p}{a}x)$; quæ casu, quo $a = \frac{1}{0} = \infty$, sit pro parabola $yy = px$. Spero autem, tirones verum illationis hujus sensum & nexum ejus cum theoria limitum ex præcedentibus habituros esse perspectum; nec ob introductionem forte necessariam symbolorum $\frac{1}{0}$, $(\frac{1}{0})^n$, seu ∞, ∞^n procliviores fore ad magnitudines reales signis his substernendas, quam proni fuerint ad admittendam quantitatum (quas vocant) imaginariarum existentiam, quarum symbola in calculos quasi necessario irrepunt, aut majoris saltem facilitatis causa adhibentur.

Superfluum autem censeo, ostendendis introductionis signorum horum commodis immorari, quæ frequens recentiorum mathematicorum usus abunde comprobat.

§. 97. Sententiam pluribus hic expositam, quod scilicet symbolum $\frac{1}{0}$ signum sit impossibilitatis, breviter jam attigeram in dissertatione: *Exposition elementaire* &c. inprimis Cap. XI. pag. 158. 159. 165. Eandem profitetur ill. KÆSTNER in dissertatione inscripta *De translatis in dictione geometrarum*, his verbis: „ Ex vulgato tang. $n = \frac{\text{sin.}\, n}{\text{cos.}\, n}$ eruitur tangens anguli recti infinita — — — „ Revera autem angulus rectus nec cosinum habet, nec tangentem. Cosinum „ non habere utique dico, cum dico, ejus cosinum esse = 0; non enim habet, „ qui nihil habet. Hinc facile intelligitur, tangentem etiam habere non posse. „ Tangentem anguli recti infinitam esse nihil aliud dicit, quam anguli ad rectum crescentis tangentem crescere ultra omnes limites, ita ut cosinus infra „ omnes limites decrescit. Eo momento, quo angulus sit rectus, cessant notio „ tangentis & cosinus; hoc discrimine, quod in eum statum, ut cosinus notio „ cesset, res deducta sit perpetuis decrementis cosinus; in eum vero, ut cesset „ notio tangentis, perpetuis incrementis tangentis." Ac nuperrime Celeb. Abbas CALUSO in *novis Commentariis Academiæ Turinensis* T. II. idem tradit judicium his aliisque verbis: „ Que l'on demande *t* tang. de *z*. Dans ce probleme „ ce n'est point la position de *t*, c'est la grandeur de *t*, que l'on veut. Cette „ grandeur est la distance du point d'attouchement à l'intersection de la tan- „ gente & de la secante, qui fait l'angle *z* avec le raïon perpendiculaire à la „ tan-

„ tangente. Ainſi, le point d'attouchement demeurant, à meſure que j'en eloi-„ gne celui de l'interſection, je fais croitre t & z. Mais je ne puis jamais eloi-„ gner l'interſection pour avoir $z = 90°$, parceque l'interſection de deux paral-„ leles eſt impoſſible. Or on ne peut concevoir de diſtance d'une interſection, „ que l'on conçoit impoſſible. Il faut donc concevoir t impoſſible, lorsque „ $z = 90°$.

„ Il faut diſtinguer trois cas d'impoſſibilité.

„ 1°. Celui que nous venons de remarquer dans t, impoſſible parce qu'on „ ne peut aſſés eloigner les extrèmes, qui doivent terminer la grandeur.

„ 2°. Celui où elle eſt impoſſible, parce qu'on ne peut aſſés les aprocher, „ dont il ſuffira de donner pour exemple la cotangente de 90°.

„ 3°. Celui où un des extrèmes devroit être en même tems des deux co-„ tés oppoſés de l'autre; c'eſt le cas des imaginaires, dont l'impoſſibilité a tou-„ jours eté reconnuë.„

§. 98. Admiſſo, ſymbolum $\frac{1}{0}$ ſignum eſſe impoſſibilis; mirum non eſt, mathematicorum conatus rem ſigno huic reſpondentem exprimendi eſſe inanes. Methodos inter, quibus ſigni hujus indolem definire annixi ſunt, ſequens notari meretur reductio. Poſito $0 = 1 - 1$, & $\frac{1}{0} = \frac{1}{1-1}$, in ſeriem convertatur expreſſio $\frac{1}{1-1}$; erit $\frac{1}{1-1} = 1 + 1 + 1 + 1 + 1 + \ldots$ Quæ ſeries cum ſine limite poſſit continuari, inferunt: ſignum $\frac{1}{0}$ ſymbolum eſſe quantitatis realis, ſeriei nempe infinitæ terminis invicem æqualibus conſtantis. Circa hanc reductionem hæc mihi obſervanda videntur.

1°. Quoniam 0 eſt differentia (ſic dicta) duarum quantitatum invicem æqualium; eodem omnino jure fiet $0 = n - n$, & $\frac{1}{0} = \frac{1}{n-n} = \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \ldots$ Jam vero ponatur $n = \frac{1}{0}$: erit quoque $\frac{1}{0} = 0 + 0 + 0 + 0 + \ldots$; quæ ſeries (ſi nihilorum congeriem ita nuncupare licet) quantitatem infinitam dictam $\frac{1}{0}$ nunquam efficere poteſt.

2°. Sem-

2°. Semper eſt $\frac{1}{a-b}=\frac{1}{a}+\frac{b}{a^2}+\frac{b^2}{a^3}+\frac{b^3}{a^4}+\ldots+\frac{b^{n-2}}{a^{n-1}}+\frac{b^{n-1}}{a^n}+\frac{\frac{b^n}{a^n}}{a-b}$. Proinde ſeries ex evolutione fractionis $\frac{1}{a-b}$ orta ad valorem hujus fractionis eo tantum caſu propius continue accedit, quo $b<a$; ita ut ſupplementum $\frac{\frac{b^n}{a^n}}{a-b}$ eo minus fiat, quo major eſt terminorum evolutorum numerus. Contra, quando $b>a$, ſupplementum $\frac{\frac{b^n}{a^n}}{a-b}$ eo majus fit, quo major eſt terminorum evolutorum numerus; proindeque ſeries evoluta a fractione $\frac{1}{a-b}$ eo magis diſcrepat, quo major eſt terminorum evolutorum numerus. Caſu autem, quo $a=b$, ſupplementum $\frac{\frac{b^n}{a^n}}{a-b}$ idem ſemper manet; & proinde ſeries evoluta neque accedit ad valorem fractionis primariæ $\frac{1}{a-b}$, neque ab illo recedit. Quin cum hoc caſu, quo $a=b$, ſupplementum $\frac{\frac{a^n}{b^n}}{a-b}$ ſeriei, quousque libeat continuatæ, ſit ipſi fractioni $\frac{1}{a-a}$ in ſeriem explicandæ æqualis; perpetuo hoc ipſiusmet recurſu docemur, converſionem fractionis $\frac{1}{a-a}$ in ſeriem ineptam eſſe ad determinandam rem, quæ ipſi reſpondeat.

Quoniam ideo $\frac{1}{0}=1+1+1+\ldots+\frac{1}{0}$; cur non liceret concludere ſummam $1+1+1+\ldots=0$? Sed revera nihil aliud ex hac æquatione concludi poteſt, niſi quod $1=0+0+0+\ldots+0+1$, ſeu $1=1$.

Scholium. Fractionis $\frac{1}{a-b}$ in ſeriem converſio caſu, quo $b>a$ (& proinde $\frac{1}{a-b}=-\frac{1}{b-a}$) mathematicos quosdam eo deduxit, ut quantitates (quas vocant) negativas eſſe plusquam infinitas (*a*) pronuntiarent. Cum fractionis $\frac{1}{a-b}$ in

(*a*) Ipſe Wallisius, qui ad matheſeos progreſſus tantopere contulit, nonniſi timide hunc vocem primus (quod ſciam) emiſit. De rationibus plus quam infinitis locutus addit: *ſi id ſine ſolecismo dici poſſit.* (Vid. *Arithmetica infinitorum.* Prop. CI. Scholium.)

in feriem converfio cafu, quo $b > a$, nihil doceat, quod ad valorem iftius fractionis, nifi ratio habeatur fupplementi $\frac{\frac{b^n}{a^n}}{a-b}$ eo majoris, quo major eft terminorum evolutorum numerus; patet, infolens illud effatum omni hoc refpectu fundamento carere.

Cafu, quo $b > a$, fractio $\frac{1}{a-b}$ eft $-\frac{1}{b-a}$, quæ in feriem convergentem evoluta fit $-\left(\frac{1}{b}+\frac{a}{bb}+\frac{a^2}{b^3}+\frac{a^3}{b^4}+\frac{a^4}{b^5}+\ldots\right)$; & abfurdum foret dicere, feriem hanc eandem effe cum altera $\frac{1}{a}+\frac{b}{aa}+\frac{b^2}{a^3}+\frac{b^3}{a^4}+\frac{b^4}{a^5}+\ldots$.

Si ad exemplum familiare, quo hoc caput exorfi fumus (§. 89.), redeamus; luculenter apparebit, quantum hæc fententia a fenfu communi abhorreat. Expreffio $D \times \frac{V}{V-v}$ cafu, quo $v > V$, fit $-D \times \frac{V}{v-V}$; quæ indicat, punctnm occurfus duorum viatorum jacere ad partes oppofitas iis, verfus quas viatores progrediuntur. Neque ullo modo concipi poteft, fpatium ab illis percurrendum majus effe fpatio (ficto) $D \times \frac{V}{V-v} = D \times \frac{1}{0}$, quod velocitatum æqualitati refpondeat.

Nec magis folidum eft, quod nonnulli mathematici dicunt: hyperbolas effe parabolas plus quam infinitas. Parabola curva eft fui generis, ad quam, tanquam limitem, propius propiusque accedere poffunt reliquæ fectiones conicæ. Et quemadmodum polygona circulo infcripta aut circumfcripta femper a circulo differunt; ita etiam ellipfes & hyperbolæ a parabola femper difcrepant.

Huc etiam pertinent, quæ (§. 71.) de fignorum operationum ad quantitates ipfas translatione obfervavi.

§. 99. Admiffo, fymbolum $\frac{1}{0}$ fignum effe impoffibilitatis; facile intelligemus, quid fibi velint figna $\frac{1}{0^n}$, ∞^n, a mathematicis ufurpata, quibus varios infinitorum ordines diftinguere autumarunt. Quemadmodum omnes quantitates imaginariæ, utut exponentibus diverfis affectæ, ad fignum unicum impoffibilitatis $\sqrt{-1}$ reducuntur; ita etiam omnia fymbola $\frac{1}{0^n}$, ∞^n, unicam indicare impoffi-

possibilitatis ideam asserere non dubito. Id quod uno aut altero exemplo liquebit.

Sit æquatio ellipsium $y^m = \frac{b^m}{a^m}(a^m - x^m)$; $y = \frac{b}{a}(a^m - x^m)^{\frac{1}{m}}$ Fig. 11.

unde subtang. $(= y\frac{dx}{dy}) = \frac{a^m}{x^{m-1}} - x = PT.$

Quo minor est x, eo major est fractio $\frac{a^m}{x^{m-1}}$; eademque tanto rapidius crescit, quo major est exponens m. Ipso autem momento, quo fit $x = 0$, symbolum $\frac{a^m}{0^{m-1}}$ non majorem indicat occursus mutui tangentis & axis impossibilitatem, quam signum simplex impossibilitatis $\frac{1}{0}$, quod obtinet, quando $m = 2$, uti in ellipsi conica. In ellipsibus variorum ordinum subtangentes inæqualiter tendunt versus impossibilitatem, quamdiu actu existunt; sed unico modo eam attingunt, aut in ea persistunt. Recta ex puncto positione dato ducta secundum innumeras leges ad eum tendere potest situm, quo fiat rectæ positione datæ parallela; sed situs hic est unicus, quicunque sit modus, quo ad illum pervenerit.

Sit æquatio hyperbolarum ad axem transversum relatarum $y^m = \frac{b^m}{a^m}(x^m - a^m)$; subtang. $(= y\frac{dx}{dy}) = x - \frac{a^m}{x^{m-1}}$. Puncti igitur, ubi tangens axi occurrit, a centro hyperbolarum distantia est $a\frac{a^{m-1}}{x^{m-1}}$: & si requiratur, ut distantia hæc evanescat; erit $a\frac{a^{m-1}}{x^{m-1}} = 0$, seu $x^{m-1} = a^{m-1} \times \frac{a}{0}$, $x = a\sqrt[m-1]{\frac{a}{0}}$. Quo symbolo monemur, impossibile esse, ut tangentes hyperbolarum cujuscunque ordinis axi in centro occurrant; neque impossibilitas hæc major minorve esse dici potest pro vario ordine hyperbolarum.

Quæcunque dixi de symboli $\frac{1}{1-1}(= \frac{1}{0})$ in seriem conversione, a fortiori debent ad series ex symboli $\frac{1}{(1-1)^n}(= \frac{1}{0^n})$ in series conversione ortas applicari, qua varios infinitorum ordines declarare fuit tentatum.

Omitto alia argumenta ex contemplatione serierum & spatiorum curvilineorum deducta: quæ plerumque contradictoria nituntur suppositione, dari aliquam quantitatem infinitam; & de quibus, occasione data, strictim dicere sufficiet.

Caput Decimum.

De quadratura curvarum.

§. 100.

Curva quævis referatur ad aliquem axem per rectas huic axi ordinatim applicatas, & eidem v. gr. perpendiculares. (*a*) Superficies curvæ denotet aream trianguli mixtilinei, quod arcu curvæ & abscissa atque ordinatim applicata axis terminatur; vel aream quadrilateri mixtilinei, arcu curvæ, abscissa axis, & duabus ejusdem ordinatis comprehensi. Sit rectangulum, cujus unum latus detur magnitudine, & cujus latus alterum crescat uti abscissa axis. Dico: rationem differentialem superficierum curvæ & rectanguli æqualem esse rationi rectæ axi ordinatim applicatæ, quæ est basis superficiei curvæ, ad rectam magnitudine datam.

Fig. 24. Esto SMM' curva aliqua, ad axem SPP' relata per rectas MP, $M'P'$ huic ordinatim & perpendiculariter applicatas. Ad punctum S constituatur recta indefinita axi perpendicularis. Fiant rectangula $SARP$, $SAR'P'$, quorum latus commune SA æquale sit rectæ magnitudine datæ, & altera latera sint abscissæ axis SP, SP'. Dico, rationem differentialem superficiei curvæ & rectanguli æqualem esse rationi $MP : SA$.

Etenim dum abscissa SP crescit quantitate PP', mutationes simultaneæ superficierum curvæ & rectanguli sunt spatia $MPP'M'$ & $RPP'R'$. Compleantur rectangula $PP'M'm'$, $PP'mM$, quorum prius est curvæ circumscriptum, posterius inscriptum (ad normam §. 1. Ex. 3.).

Ratio mutationum simultanearum superficiei curvæ SMP & rectanguli SR minor est ratione rectangulorum PM', PR', seu ratione $PM' : SA$; major autem ratione rectangulorum Pm, PR', seu ratione $PM : SA$. Quoniam autem ratio æqualitatis limes est rationis rectarum $M'P'$, MP, seu rectanguiorum PM', Pm; ratio rectangulorum PM', PR' minor fieri potest quacunque ratione proposita, quæ

(*a*) Si coordinatarum angulus non sit rectus; quæcunque de rectangulis dicentur, transferenda sunt ad parallelogramma æqui-angula, quorum anguli sunt angulo coordinatarum æquales.

quæ major ſit ratione $PM : SA$; & a fortiori ratio ſpatii $PMM'P'$ ad rectangulum PR' minor fieri poteſt quacunque ratione propoſita, quæ major eſt ratione $PM : SA$. Proinde ratio $PM : SA$ limes eſt rationis mutationum ſimultanearum ſpatiorum SMP & $SARP$, ſeu eſt ratio differentialis horum ſpatiorum.

Sit ideo $SP = x$, $MP = y$, $SA = a$, & ſuperficies curvæ dicatur S; erit $\lim.\frac{\Delta S}{a\Delta x} = \frac{y}{a}$, ſeu $\lim.\frac{\Delta S}{\Delta x} = y$, & $\frac{dS}{dx} = y$.

Data igitur (ex æquatione curvæ) relatione x & y, determinatio propoſita ſuperficiei curvæ ad calculum integralem reducta eſt.

Exemplum. Sit $a^{m-1}y = x^m$; erit $\frac{dS}{dx} = y = \frac{x^m}{a^{m-1}}$, & $S = C + \frac{1}{m+1}\cdot\frac{x^{m+1}}{a^{m-1}} = C + \frac{1}{m+1}xy$.

1. *Caſus.* Sit m numerus poſitivus, & initium curvæ ſit in vertice axis. Curva propoſita eſt parabolica. Et quoniam ſuperficies (per hyp.) evaneſcit facto $x = 0$, eſt $C = 0$; unde $S = \frac{1}{m+1}xy$. Area igitur ſemi-ſegmenti parabolici, cujus æquatio eſt $y = \frac{x^m}{a^{m-1}}$, eſt ad rectangulum circumſcriptum, uti $1 : m+1$.

2. *Caſus.* Sit m numerus negativus, ſeu ſit $yx^m = a^{m+1}$, quæ eſt æquatio hyperbolarum ad aſymptotos relatarum. Quoniam $\frac{dS}{dx} = y = a^{m+1}x^{-m}$; fit $S = C - \frac{1}{m-1}a^{m+1}x^{-m+1} = C - \frac{1}{m-1}xy$. Ut caſus hujus examen fiat facilius, ſit $AB = b$ ordinata, cui reſpondet abſciſſa $SB = a$, a cujus extremo B ſumatur initium abſciſſarum; erit ideo $y(a+x)^m = ba^m$, $\frac{dS}{dx} = a^m b(a+x)^{-m}$, $S = C - \frac{1}{m-1}a^m b(a+x)^{-m+1} = C - \frac{1}{m-1}\cdot\frac{a^m b}{(a+x)^{m-1}} = C - \frac{1}{m-1}y(a+x)$. Jam vero ſuperficies propoſita $ABPM$ evaneſcit, facto $BP = x = 0$, & $y = AB = b$; proinde $C = + \frac{1}{m-1}ab$, & $S = \frac{1}{m-1}(ab - y(a+x))$. Fig. 25.

1°. Sit $m > 1$, $y(a+x) = \frac{a^m b}{(a+x)^{m-1}} = ab\left(\frac{a}{a+x}\right)^{m-1}$. Quoniam autem $a+x$ ſeu SP major fieri poteſt quacunque quantitate propoſita; contra $\frac{a}{a+x}$

 &

& $\left(\frac{a}{a+x}\right)^{m-1}$ minor fieri poteſt quacunque quantitate propoſita: proinde ab limes eſt quantitatis creſcentis $ab - y(a+x)$; adeoque ſpatium $\frac{1}{m-1} ABSD$ limes eſt ſpatii aſymptotici creſcentis $ABPM$, quod æquare, nedum ſuperare nequit, utcunque magna ſumatur abſciſſa BP.

In eodem caſu fiat x negativa; unde $y(a-x)^m = a^m b$, & $y = a^m b(a-x)^{-m}$.

$$\text{Erit } S = \frac{1}{m-1} a^m b\left((a-x)^{-m+1} - C\right)$$

$$= \frac{1}{m-1} a^m b\left((a-x)^{-m+1} - a^{-m+1}\right)$$

$$= \frac{1}{m-1} a^m b\left(\frac{1}{\overline{a-x}^{m-1}} - \frac{1}{a^{m-1}}\right)$$

$$= \frac{1}{m-1} \quad \left(y(a-x) - ab\right).$$

Quoniam autem nullus eſt limes parvitatis quantitatis $a-x$, inde ab a ad zero usque decreſcentis: contra nullus eſt limes magnitudinis quantitatum creſcentium $\frac{a}{a-x}$, $\frac{a^{m-1}}{(a-x)^{m-1}}$; & proinde nullus eſt limes magnitudinis ſpatii aſymptotici ad alteras partes ordinatæ AB.

Hoc caſu poſito $x = a$, fit $S = \frac{1}{m-1} ab \cdot \frac{1}{0^{m-1}}$; quod ſymbolum indicat, tam contradictionem ſuppoſitionis, fieri poſſe $x = a$, ſeu hyperbolam aſymptoto occurrere, quam impoſſibilitatem ſpatii aſymptotici limitem determinandi.

2°. Sit $m < 1$: erit $y(a+x) = \frac{a^m b}{(a+x)^{m-1}} = ab \times \left(\frac{a+x}{a}\right)^{1-m}$, & $S = \frac{1}{1-m} ab\left(\left(\frac{a+x}{a}\right)^{1-m} - 1\right) = \frac{1}{1-m}\left(y(a+x) - ab\right)$. Quoniam autem nullus eſt limes magnitudinis ipſius $a+x$, nullus etiam eſt limes magnitudinis quantitatum $\frac{a+x}{a}$, $\left(\frac{a+x}{a}\right)^{1-m}$; & proinde, creſcente abſciſſa BP, ſpatium aſymptoticum creſcit, & majus fieri poteſt quocunque ſpatio dato.

Sit autem x negativa. Quoniam $y(a-x)^m = a^m b$, erit ut prius $S = \frac{1}{1-m}\left(ab - y(a-x)\right) = \frac{1}{1-m} ab\left(1 - \left(\frac{a-x}{a}\right)^{1-m}\right)$: & quoniam, creſcente x a zero inde

inde usque ad a, nullus eſt limes parvitatis quantitatum $\frac{a-x}{a}$, $\left(\frac{a-x}{a}\right)^{1-m}$; ſpatium aſymptoticum ex altera parte ordinatæ AB obtinet limitem magnitudinis, nempe $\frac{1}{1-m}ab$, ſeu $\frac{1}{1-m}AB \times SB$.

Hoc caſu eſt etiam $S = \frac{1}{1-m}ab\left(\left(\frac{b}{y}\right)^{\frac{1-m}{m}} - 1\right)$. Facto $y = 0$, fit $S = \frac{1}{1-m}ab\left(\left(\frac{b}{0}\right)^{\frac{1-m}{m}} - 1\right)$; quod indicat tam contradictionem ſuppoſitionis, fieri poſſe $y = 0$, quam impoſſibilitatem ſpatii aſymptotici limitem determinandi.

3°. Sit $m = 1$. Hoc caſu $\frac{dS}{dx} = y = \frac{ab}{a+x}$, quæ eſt æquatio differentialis logarithmica. Proinde $S = ab(C + \log. a + x)$. Et quoniam S evaneſcit, facto $x = 0$; fit $C = -\log. a$, & $S = ab \log. \frac{a+x}{a}$. Proinde ſpatla hyerbolica $ABPM$ creſcunt ut logarithmi rationum $SP : SB$ abſciſſarum.

Geometricam proprietatis hujus hyperbolæ conicæ demonſtrationem ad calcem hujus capitis differre ſatius mihi eſſe videtur, ne principalium propoſitionum ſeries nimium abrumpatur.

Quando $m \gtrless 1$; alterutrum ſpatiorum aſymptoticorum, ordinata aliqua AB ſeparatorum, limitem habet magnitudinis. Caſu autem, quo $m = 1$, utrumque horum ſpatiorum majus fieri poteſt quocunque ſpatio dato.

§. 101. Quoniam curvæ parabolicæ & hyperbolicæ magni ſunt in theoria curvarum momenti; haud abs re erit earum quadraturam paulo aliter inveſtigare, & ad prima limitum rationum principia reducere.

1°. Curva quadranda ſit parabola SMM', cujus æquatio $a^{m-1}y = x^m$. Ad verticem S ducatur tangens SA; ducatur etiam ad M tangens curvæ, quæ axi SP occurrat in T. Ducatur ordinata $M'P'$; per M & M' ducantur ipſi SA perpendiculares MQ, $M'Q'$, quæ ipſis $M'P'$, MP (productis, ſi neceſſe fuerit) in m & m' occurrant. Fig. 24.

Rect.

$$\begin{array}{lllllll}
 & \text{Rect.}\,MPP'm & : \text{rect.}\,MQQ'm' & = MP\times Mm & : & MQ\times M'm. & \\
\text{Sed} & \text{lim.}\,Mm & : M'm & = PT & : & MP & (\text{§. } 40.) \\
\text{et} & MP & : MQ & = MP & : & MQ & \\
\text{Ergo} & \text{lim.}(MP\times Mm & : MQ\times M'm) & = PT & : & MQ & (\text{§. } 14.) \\
 & & & = PT & : & SP & \\
 & & & = 1 & : & m & (\text{§. } 42.) \\
\text{Proinde} & \text{lim.f.rect.}\,MPP'm & : \text{lim.f.rect.}\,MQQ'm' & = 1 & : & m & (\text{§. } 15.)
\end{array}$$

Sed ſpatium parabolicum exterius $SMM'Q'$ limes eſt ſummæ rectangulorum huic inſcriptorum $MQQ'm'$, & ſpatium parabolicum internum $SMM'P'$ limes eſt rectangulorum huic pariter inſcriptorum $MPP'm$; proinde (§. 4.)

$$\begin{array}{llllll}
 & SMM'P' & : SMM'Q' & = 1 & : & m \\
\text{ſeu} & SMP & : SMQ & = 1 & : & m \\
\text{hinc} & SPMQ & : SMP & = m+1 & : & 1 \\
 & SPMQ & : SMQ & = m+1 & : & m.
\end{array}$$

2°. Curva quadranda ſit hyperbola, cujus aſymptotæ SP, SQ, & cujus æquatio eſt $y(a+x)^m = a^m b$. Per M ducatur tangens, quæ aſymptotis in T &

Fig. 25. T' occurrat. Sint MP, MQ, $M'P'$, $M'Q'$ rectæ aſymptotis ordinatim applicatæ. Rectæ MP, $M'Q'$ ſibi mutuo occurrant in m', & rectæ MQ, $M'P'$ ſibi mutuo occurrant in m.

$$\begin{array}{lllllll}
 & \text{Rect.}\,MPP'm & : \text{rect.}\,mQQ'M' & = MP\times M'm' & : & MQ\times Mm'. & \\
\text{Sed} & \text{lim.}\,M'm' & : Mm' & = PT & : & MP & (\text{§. } 40.) \\
 & \text{lim.}\,MP & : mQ & = MP & : & MQ & \\
\text{ergo} & \text{lim.}\,MP\times M'm' & : mQ\times Mm' & = PT & : & MQ & (\text{§. } 14.) \\
 & & & = PT & : & SP & \\
 & & & = 1 & : & m & (\text{§. } 42.) \\
\text{Ergo} & \text{lim.(f.rect.}\,PMmP' & : \text{f.rect.}\,mQQ'M') & = 1 & : & m & (\text{§. } 15.)
\end{array}$$

Sed ſpatium hyperbolicum $ABP'M'$ limes eſt ſummæ rectangulorum $PMmP'$ (§. 1.), & ſpatium hyperbolicum $ADQ'M'$ limes eſt ſummæ rectangulorum

... Ergo (§. 4.) $ABP'M' : ADQ'M' = 1 : m$

etiam $ABPM : ADQM = 1 : m.$

1°. Sit

1°. Sit $m < 1$; erit $ABPM > ADQM$; & $ABPM - ADQM = MRBP - ARQD = MQSP - ABSD$.

Quoniam
$$\begin{aligned} ABPM &: ADQM = 1 : m \\ ABPM &: MQSP - ABSD = 1 : 1 - m \\ \& \ ADQM &: MQSP - ABSD = m : 1 - m. \end{aligned}$$

2°. Sit $m > 1$; erit $ABPM < ADQM$, & $ADQM - ABPM = ARQD - MRSP = ABSD - MQSP$.

Quoniam
$$\begin{aligned} ABPM &: ADQM = 1 : m \\ ABPM &: ABSD - MQSP = 1 : m - 1 \\ \& \ ADQM &: ABSD - MQSP = m : m - 1. \end{aligned}$$

3°. Sit $m = 1$. Nil aliud inde concludi poteſt, niſi quod $ABPM = ADQM$.

§. 102. Quadratura parabolarum & hyperbolarum viam ſternit ad quadraturam plurimarum curvarum, quarum ſuperficies ab illarum ſuperficiebus pendent.

Sit exempli gratia $bby = xx(a-x)$

erit $\dfrac{dS}{dx} = y = \dfrac{axx - x^3}{bb}$. $S = \dfrac{\frac{1}{3}ax^3 - \frac{1}{4}x^4}{bb} = \dfrac{x^3(\frac{1}{3}a - \frac{1}{4}x)}{bb}$.

Sit $S = 0$, quando $x = 0$, & ſumatur ſuperficies reſpondens abſciſſæ $x = a$; erit $S = \frac{1}{12}\dfrac{a^4}{bb}$.

Sit in genere

$b^{m+n-1}y = x^m(a-x)^n$; erit

$$\frac{dS}{dx} = y = \frac{x^m(a-x)^n}{b^{m+n-1}}$$
$$= \frac{x^m}{b^{m+n-1}}\left(a^n - \frac{n}{1}a^{n-1}x + \frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}x^2 - \frac{n}{1}\cdots\frac{n-2}{3}a^{n-3}x^3 + \frac{n}{1}\cdots\frac{n-3}{4}a^{n-4}x^4 - \ldots\right)$$

unde

$$S = \frac{1}{b^{m+n-1}}\left(\frac{1}{m+1}a^n x^{m+1} - \frac{n}{1}\cdot\frac{1}{m+2}a^{n-1}x^{m+2} + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{1}{m+3}a^{n-2}x^{m+3} - \frac{n}{1}\cdots\frac{n-2}{3}\cdot\frac{1}{m+4}a^{n-3}x^{m+4} + \ldots\right)$$

Quæ ſeries abrumpitur, ſi fuerit n numerus integer & poſitivus.

Curva propoſita ſit circulus, cujus æquatio eſt $yy = rr - xx$; $\dfrac{dS}{dx} = y = \sqrt{rr - xx}$. Formula integranda in ſeriem convertatur; erit

$$\frac{dS}{dx}=r-\tfrac{1}{2}\frac{xx}{r}-\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\frac{x^4}{r^3}-\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3}\frac{x^6}{r^5}-\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3}\cdot\frac{\frac{5}{2}}{4}\frac{x^8}{r^7}-\tfrac{1}{2}\cdots\frac{\frac{7}{2}}{5}\cdot\frac{x^{10}}{r^9}-\ldots\,.$$

$$S=rx-\tfrac{1}{2}\cdot\tfrac{1}{3}\frac{x^3}{r}-\frac{\frac{1}{2}\cdot\frac{1}{2}}{1.2}\cdot\tfrac{1}{5}\frac{x^5}{r^3}-\frac{\frac{1}{2}}{1}\cdots\frac{\frac{3}{2}}{3}\cdot\tfrac{1}{7}\frac{x^7}{r^5}-\frac{\frac{1}{2}}{1}\cdots\frac{\frac{5}{2}}{4}\cdot\tfrac{1}{9}\frac{x^9}{r^7}-\frac{\frac{1}{2}}{1}\cdots\frac{\frac{7}{2}}{5}\cdot\tfrac{1}{11}\cdot\frac{x^{11}}{r^9}-\ldots$$

Sit $x=r$, & proinde S quadrans circuli; erit

$$S=rr(1-\tfrac{1}{2}\cdot\tfrac{1}{3}-\tfrac{1}{8}\cdot\tfrac{1}{5}-\tfrac{1}{16}\cdot\tfrac{1}{7}-\tfrac{5}{128}\cdot\tfrac{1}{9}-\tfrac{35}{1280}\cdot\tfrac{1}{11}-\ldots\,.)$$

Si curva propoſita eſſet ellipſis, cujus æquatio eſt $yy=\frac{bb}{aa}(aa-xx)$; formula eadem eſt faciendo $rr=ab$.

Curva propoſita ſit hyperbola conica ad axem transverſum relata, cujus æquatio eſt $yy=\frac{bb}{aa}(xx-aa)$.

$$\frac{dS}{dx}=y=\frac{b}{a}\sqrt{(xx-aa)}=\frac{b}{a}\left(x-\tfrac{1}{2}\frac{aa}{x}-\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\frac{a^4}{x^3}-\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3}\frac{a^6}{x^5}-\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3}\cdot\frac{\frac{5}{2}}{4}\frac{a^8}{x^7}-\ldots\,.\right)$$

$$S=C+\frac{b}{a}\left(\tfrac{1}{2}xx-\tfrac{1}{2}aa\log.x+\tfrac{1}{2}\cdot\frac{\frac{1}{2}}{2}\cdot\tfrac{1}{2}\frac{a^4}{x^2}+\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3}\cdot\tfrac{1}{4}\cdot\frac{a^6}{x^4}+\frac{\frac{1}{2}}{1}\cdots\frac{\frac{5}{2}}{4}\cdot\tfrac{1}{6}\cdot\frac{a^8}{x^6}+\ldots\,.\right)$$

Sit $S=0$, quando $x=a$;

$$S=\frac{b}{a}\left((\tfrac{1}{2}xx-aa)-\tfrac{1}{2}aa\log.\frac{x}{a}+\tfrac{1}{8}\cdot\tfrac{1}{2}\frac{aa(xx-aa)}{x^2}+\tfrac{1}{16}\cdot\tfrac{1}{4}\cdot\frac{aa(x^4-a^4)}{x^4}+\tfrac{5}{128}\cdot\tfrac{1}{6}\cdot\frac{aa(x^6-a^6)}{x^6}+\ldots\right).$$

Obſervatio 1. Hyperbolæ conicæ ad aſymptoton relatæ quadraturam ad logarithmos reduci ſuperius vidimus (§. 100.): ideo etiam quadratura curvæ hujus ad logarithmos reduci poterit, ad quemcunque axem curva referatur. Reipſa ex ratione differentiali $\frac{dS}{dx}=\frac{b}{a}\sqrt{xx-aa}$

conſequitur etiam $S=\tfrac{1}{2}ab\log.\frac{x-\sqrt{xx-aa}}{a}+\tfrac{1}{2}\frac{b}{a}x\sqrt{xx-aa}$

$$=\tfrac{1}{2}ab\log.\frac{a}{x+\sqrt{xx-aa}}+\frac{b}{a}x\sqrt{xx-aa}.$$

Si vero hyperbola ad axem ſecundum refertur, ut æquatio ſit $yy=\frac{aa}{bb}(xx+bb)$, & $\frac{dS}{dx}=y=\frac{a}{b}\sqrt{(xx+bb)}$; pariter eſt

$$S=\tfrac{1}{2}ab\log.\frac{b}{\sqrt{(xx+bb)}-x}+\tfrac{1}{2}\frac{a}{b}x\sqrt{xx+bb}.$$

Obſer-

Observatio 2. Ad duas formulas $\frac{dS}{dx} = \sqrt{aa-xx}$; $\frac{dS}{dx} = \sqrt{xx \pm aa}$, reducuntur quam plurimæ aliæ formulæ, quæ proinde in terminis finitis integrabiles cenſentur; cum in promtu ſint quadratura (appropinquata) circuli, & tabulæ logarithmicæ.

Sic exiſtente m numero integro formulæ $\frac{dS}{dx} = x^m\sqrt{(aa-xx)}$, $\frac{dS}{dx} = x^m\sqrt{(xx \pm aa)}$ aut ad duas formulas præcedentes reducuntur; aut ad formulas $\frac{x}{\sqrt{aa-xx}}$, & $\frac{x}{\sqrt{xx \pm aa}}$, quæ ſunt immediate integrabiles.

Sit æquatio curvæ logarithmicæ $\frac{b}{y} = e^{\frac{x}{t}}$; erit $\frac{dx}{dy} = -\frac{t}{y}$.

$$\text{Sed } \frac{dS}{dx} = y$$

$$\text{ergo } \frac{dS}{dx} = -t, \quad \& \quad S = C - ty.$$

Sit $S = 0$, quando $y = b$; erit $C = bt$, & $S = t(b-y)$. Quoniam y minor fieri poteſt quacunque quantitate propoſita, limes ſpatii aſymptotici curvæ logarithmicæ eſt bt. Quæ formulæ immediate poſſunt ex primis limitum principiis demonſtrari.

Sit æquatio generalis cycloidum $y = \sqrt{(2rx-xx)} + R\,\text{arc.ſin.v.}\frac{x}{r}$. Facto $V = x\,\text{arc.ſin.v.}\frac{x}{r}$, fit $\text{arc.ſin.v.}\frac{x}{r} = \frac{dV}{dx} - \frac{x}{\sqrt{(2rx-xx)}}$;

$$\begin{aligned}\text{proinde } \frac{dS}{dx} &= \sqrt{(2rx-xx)} + R\frac{dV}{dx} - R\frac{x}{\sqrt{(2rx-xx)}}\\ &= \sqrt{(2rx-xx)} + R\frac{dV}{dx} + R\frac{r-x}{\sqrt{(2rx-xx)}} - \frac{Rr}{\sqrt{(2rx-xx)}};\end{aligned}$$

$$\begin{aligned}\text{hinc } S = {} & \tfrac{1}{2}rr\,\text{arc.ſin.v.}\frac{x}{r} - \tfrac{1}{2}(r-x)\sqrt{(2rx-xx)}\\ & + Rx\,\text{arc.ſin.v.}\frac{x}{r} + R\sqrt{(2rx-xx)}\\ & - Rr\,\text{arc.ſin.v.}\frac{x}{r}\end{aligned}$$

Sit $x = 2r$; erit $S = \frac{r(2R+r)}{2}\pi$.

 Sci-

Scilicet area cycloidis eſt ad aream circuli genitoris, ut $2R+r:r$; & nominatim cycloidis vulgaris area eſt tripla area circuli genitoris.

§. 103. Quoniam $\frac{dS}{dx}=y$; erit (§. 35.) juxta ſeriem Bernoullianam

$$S = xy - \frac{x^2}{1.2}\frac{dy}{dx} + \frac{x^3}{1.2.3}\frac{ddy}{dx^2} - \frac{x^4}{1.2...4}\frac{d^3y}{dx^3} + \frac{x^5}{1.2...5}\frac{d^4y}{dx^4} - \frac{x^6}{1.2...6}\frac{d^5y}{dx^5} + \dots$$

Cum autem quadratura curvarum una ſit ex præcipuis calculi integralis applicationibus, haud alienum ab re erit formulam hanc accuratius explicare, intimumque ejus cum methodo exhauſtionis nexum oſtendere.

Sit curva quæcunque ad axem aliquem relata, cujus curvæ ſegmentum coordinatis x & y terminatum denotet S. Abſciſſa axis in numerum quemcunque n partium Δx mutuo æqualium diviſa, inſcribantur & circumſcribantur curvæ rectangula ad normam §. 1. Ex. 3.

Rectæ axi ordinatim applicatæ, quæ abſciſſis x, $x-\Delta x$, $x-2\Delta x$, $x-3\Delta x$, $x-4\Delta x \dots x-(n-1)\Delta x$ reſpondent, ſunt reſpective (§. 31.).

$$\begin{array}{l}
y \\
y - \frac{\Delta x}{1}\frac{dy}{dx} + \frac{\Delta x^2}{1.2}\frac{ddy}{dx^2} - \frac{\Delta x^3}{1.2.3}\frac{d^3y}{dx^3} + \frac{\Delta x^4}{1.2..4}\frac{d^4y}{dx^4} - \frac{\Delta x^5}{1.2..5}\frac{d^5y}{dx^5} + \dots \\
y - 2\frac{\Delta x}{1}\frac{dy}{dx} + 2^2\frac{\Delta x^2}{1.2}\frac{ddy}{dx^2} - 2^3\frac{\Delta x^3}{1.2.3}\frac{d^3y}{dx^3} + 2^4\frac{\Delta x^4}{1.2..4}\frac{d^4y}{dx^4} - 2^5\frac{\Delta x^5}{1.2..5}\frac{d^5y}{dx^5} + \dots \\
y - 3\frac{\Delta x}{1}\frac{dy}{dx} + 3^2\frac{\Delta x^2}{1.2}\frac{ddy}{dx^2} - 3^3\frac{\Delta x^3}{1.2.3}\frac{d^3y}{dx^3} + 3^4\frac{\Delta x^4}{1.2..4}\frac{d^4y}{dx^4} - 3^5\frac{\Delta x^5}{1.2..5}\frac{d^5y}{dx^5} + \dots \\
y - 4\frac{\Delta x}{1}\frac{dy}{dx} + 4^2\frac{\Delta x^2}{1.2}\frac{ddy}{dx^2} - 4^3\frac{\Delta x^3}{1.2.3}\frac{d^3y}{dx^3} + 4^4\frac{\Delta x^4}{1.2..4}\frac{d^4y}{dx^4} - 4^5\frac{\Delta x^5}{1.2..5}\frac{d^5y}{dx^5} + \dots \\
\qquad\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
y-(n-1)\frac{\Delta x}{1}\frac{dy}{dx} + (n-1)^2\frac{\Delta x^2}{1.2}\frac{ddy}{dx^2} - (n-1)^3\frac{\Delta x^3}{1.2.3}\frac{d^3y}{dx^3} + (n-1)^4\frac{\Delta x^4}{1.2..4}\frac{d^4y}{dx^4} - (n-1)^5\frac{\Delta x^5}{1.2..5}\frac{d^5y}{dx^5} + \dots
\end{array}$$

Rectangula curvæ circumſcripta ſunt reſpective producta harum expreſſionum per altitudinem communem Δx; tandemque ſumma rectangulorum circumſcriptorum eſt ſumma omnium horum productorum, nempe

$y\Delta x$.

$$\begin{aligned}
& y\Delta x.n \\
& - \frac{\Delta x^2}{1}\frac{dy}{dx}(1+2+3+4\ldots n\text{-}1) \\
& + \frac{\Delta x^3}{1.2}\frac{ddy}{dx^2}(1^2+2^2+3^2+4^2+\ldots(n\text{-}1)^2) \\
& - \frac{\Delta x^4}{1.2.3}\frac{d^3y}{dx^3}(1^3+2^3+3^3+4^3+\ldots(n\text{-}1)^3) \\
& + \frac{\Delta x^5}{1.2\ldots4}\frac{d^4y}{dx^4}(1^4+2^4+3^4+4^4+\ldots(n\text{-}1)^4) \\
& - \frac{\Delta x^6}{1.2\ldots5}\frac{d^5y}{dx^5}(1^5+2^5+3^5+4^5+\ldots(n\text{-}1)^5) \\
& + \text{ - - - - - - -}
\end{aligned}
\qquad
\begin{aligned}
& = y\Delta x.n \\
& - \frac{\Delta x^2}{1}\frac{dy}{dx}(\text{ſ}.\,1\ldots n\text{-}1) \\
& + \frac{\Delta x^3}{1.2}\frac{ddy}{dx^2}(\text{ſ}.\,1^2\ldots n\text{-}1^2) \\
& - \frac{\Delta x^4}{1\ldots3}\frac{d^3y}{dx^3}(\text{ſ}.\,1^3\ldots n\text{-}1^3) \\
& + \frac{\Delta x^5}{1\ldots.4}\frac{d^4y}{dx^4}(\text{ſ}.\,1^4\ldots n\text{-}1^4) \\
& - \frac{\Delta x^6}{1\ldots.5}\frac{d^5y}{dx^5}(\text{ſ}.\,1^5\ldots n\text{-}1^5). \\
& + \text{ - - - - -}
\end{aligned}$$

$$\begin{aligned}
& = y\Delta x.n \\
& - \frac{\Delta x^2}{1}\frac{dy}{dx}(\tfrac{1}{2}nn+Bn) \\
& + \frac{\Delta x^3}{1.2}\frac{ddy}{dx^2}(\tfrac{1}{3}n^3+Bnn+Cn) \\
& - \frac{\Delta x^4}{1\ldots3}\frac{d^3y}{dx^3}(\tfrac{1}{4}n^4+Bn^3+Cn^2+Dn) \\
& + \frac{\Delta x^5}{1\ldots4}\frac{d^4y}{dx^4}(\tfrac{1}{5}n^5+Bn^4+Cn^3+Dn^2+En) \\
& - \frac{\Delta x^6}{1\ldots5}\frac{d^5y}{dx^5}(\tfrac{1}{6}n^6+Bn^5+Cn^4+Dn^3+En^2+Fn) \\
& + \text{ - - - - - - - -}
\end{aligned}$$

$$\begin{aligned}
& = xy \\
& - \frac{1}{1}\frac{dy}{dx}(\tfrac{1}{2}x^2+Bx\Delta x) \\
& + \frac{1}{1.2}\frac{ddy}{dx^2}(\tfrac{1}{3}x^3+Bx^2\Delta x+Cx\Delta x^2) \\
& - \frac{1}{1.2.3}\frac{d^3y}{dx^3}(\tfrac{1}{4}x^4+Bx^3\Delta x+Cx^2\Delta x^2+Dx\Delta x^3) \\
& + \frac{1}{1.2\ldots4}\frac{d^4y}{dx^4}(\tfrac{1}{5}x^5+Bx^4\Delta x+Cx^3\Delta x^2+Dx^2\Delta x^3+Ex\Delta x^4) \\
& - \frac{1}{1.2\ldots5}\frac{d^5y}{dx^5}(\tfrac{1}{6}x^6+Bx^5\Delta x+Cx^4\Delta x^2+Dx^3\Delta x^2+Exx\Delta x^4+Fx\Delta x^5) \\
& + \text{ - - - - - - - - -}
\end{aligned}$$

Pro-

Proinde limitis ſummæ rectangulorum circumſcriptorum, hoc eſt, areæ ſegmenti curvæ expreſſio limes eſt ſummæ omnium horum terminorum; nempe

$$xy - \frac{x^2}{1.2}\frac{dy}{dx} + \frac{x^3}{1.2.3}\frac{ddy}{dx^2} - \frac{x^4}{1\ldots4}\frac{d^3y}{dx^3} + \frac{x^5}{1\ldots5}\frac{d^4y}{dx^4} - \frac{x^6}{1\ldots6}\frac{d^5y}{dx^5} + \ldots$$

Eademque methodo proceditur per rectangula curvæ inſcripta.

Scholium. Methodus, qua parallelogrammorum curvæ alicui circumſcriptorum ſumma determinata fuit, exempli loco eſſe poteſt methodi, qua ſerierum plurimarum ſumma, juvante theoremate Tayloriano, ad poteſtatum numerorum naturalium ſummas poteſt reduci.

§. 104. Si curva referatur ad aliquem focum, ut coordinatæ ſint radii vectores, & anguli, quos hi radii cum aliquo radio vectore poſitione dato comprehendunt: quadratura curvæ ſic determinatæ iisdem nititur principiis.

Fig. 26. Sit SMM' curva ad focum F relata per radios vectores FM, FM', & angulos SFM, SFM', quos radii vectores FM, FM' cum radio FS poſitione dato comprehendunt.

Centro F, radiis FM, FM', deſcribantur arcus Mm, $M'm'$, ut curvæ inſcribantur atque circumſcribantur ſectores circulares MFm, $M'Fm'$. Sector curvæ MFM' major eſt ſectore circulari inſcripto MFm; minor autem circumſcripto $M'Fm'$. Cum autem ratio æqualitatis limes ſit rationis radiorum FM, FM'; ratio æqualitatis limes etiam eſt rationis ſectorum circularium $M'Fm'$, MFm; & a fortiori limes rationis ſectoris curvæ MFM', & ſectoris circularis MFm. Radius magnitudine datus FS ſit r, & centro F radio FS deſcribatur arcus circularis, qui radiis FM & FM' in punctis X & X' occurrat. Sit etiam π circumferentia circuli, cujus radius eſt unitas; ſit angulus $SFM = x$; angulus $MFM' = \Delta x$, & ſectorem curvæ SFM denotet S; unde $MFM' = \Delta S$.

$$\begin{array}{lrcl}
 & \lim. MFM' : MFm & = & 1 : 1 \\
 & MFm : XFX' & = & MF^2 : FS^2 \\
 & & = & yy : rr \\
\text{ergo} & \lim. MFM' : XFX' & = & yy : rr \\
\text{ſeu} & \lim. MFM' : \frac{1}{2}rr\frac{\Delta x}{\pi} & = & yy : rr
\end{array}$$

lim.

$$\lim. \Delta S \;:\; \tfrac{1}{2} rr \frac{\Delta x}{\pi} = yy \;:\; rr$$

$$\lim. \frac{\Delta S}{\Delta x} \;:\; \tfrac{1}{2} rr \times \frac{1}{\pi} = yy \;:\; rr$$

seu $$\frac{dS}{dx} \;:\; \tfrac{1}{2} rr \times \frac{1}{\pi} = yy \;:\; rr$$

et $\dfrac{dS}{dx} \quad - \quad = \dfrac{1}{2\pi} yy$, quæ eſt æquatio differentialis ſuperficierum curvarum ad focum per radios vectores relatarum.

Exempla. Sit $y = r\frac{x}{\pi}$, quæ eſt æquatio ſpiralis Archimedeæ.

$yy = rr\frac{xx}{\pi\pi}$, $\frac{1}{2\pi}yy = rr \times \frac{xx}{2\pi^3}$, $\frac{dS}{dx} = rr \times \frac{xx}{2\pi^3}$, $S = C + \frac{1}{3} rr \frac{x^3}{2\pi^3}$. Sit $S = 0$, quando $x = 0$; erit $S = \frac{1}{6} rr \frac{x^3}{\pi^3} = \frac{1}{6} yy \times \frac{x}{\pi} = \frac{1}{6} \frac{y^3}{r}$. Nempe ſuperficies ſpiralis Archimedeæ creſcit in ratione triplicata angulorum *SFM*, ſeu in ratione compoſita ex duplicata ratione radii vectoris *FM* & ratione ſimplici anguli *SFM*, ſeu tandem in ratione triplicata radii vectoris *FM*.

Sit $y = r\left(\frac{x}{\pi}\right)^m$; erit $yy = rr\left(\frac{x}{\pi}\right)^{2m}$, $\frac{dS}{dx} = \frac{1}{2\pi} yy = \frac{1}{2\pi} rr \left(\frac{x}{\pi}\right)^{2m}$,

$$S = C + \frac{1}{2\pi} rr . \frac{1}{2m+1} \frac{x^{2m+1}}{\pi^{2m}} = C + \tfrac{1}{2} . \frac{1}{2m+1} rr . \frac{x^{2m+1}}{\pi^{2m+1}} = C + \tfrac{1}{2} \frac{1}{2m+1} yy . \frac{x}{\pi}.$$

Superficies igitur omnium ſpiralium, quarum æquatio eſt $y = r\left(\frac{x}{\pi}\right)^m$, ſunt in ratione compoſita ex ratione angulorum *SFM* ſimplici, & ratione radiorum vectorum *FM* duplicata.

Sit $y = re^x$, quæ eſt æquatio ſpiralis logarithmicæ; erit $\frac{dS}{dx} = \frac{1}{2} yy = \frac{1}{2} rre^{2x}$; $S = \frac{1}{4} rre^{2x} = \frac{1}{4} yy$. Proinde ſuperficies ſpiralis logarithmicæ creſcit in ratione duplicata radiorum vectorum, ſeu etiam in progreſſione geometrica.

Sit $y = r\,\text{ſec.}^2 \tfrac{1}{2} x$, quæ eſt æquatio parabolæ ad focum relatæ, in qua r æqualis quadranti parametri. Erit $\frac{dS}{dx} = \frac{1}{2} yy = \frac{1}{2} rr\,\text{ſec.}^4 \tfrac{1}{2} x$; unde

$S = rr\,\text{tang.} \tfrac{1}{2} x (1 + \tfrac{1}{3} \text{tang.}^2 \tfrac{1}{2} x) = rr(\text{tang.} \tfrac{1}{2} x + \tfrac{1}{3} \text{tang.}^3 \tfrac{1}{2} x)$.

§. 105.

§. 105. Cum quadratura curvarum ad focum relatarum reducatur ad integrationem æquationis differentialis $\frac{dS}{dx} = yy$: quadratura hæc etiam poteſt obtineri per ſeriem Bernoullianam, prouti in §. 36. extenſa fuit.

Erit ſcilicet
$$
\begin{aligned}
2S = {} & xyy \\
& - 2\frac{x^2}{1.2}y\,\frac{dy}{dx} \\
& + 2\frac{x^3}{1.2.3}\left(\left(\frac{dy}{dx}\right)^2 + y\frac{ddy}{dx^2}\right) \\
& - 2\frac{x^4}{1.2\ldots4}\left(3\frac{dy}{dx}\cdot\frac{ddy}{dx^2} + y\frac{d^3y}{dx^3}\right) \\
& + 2\frac{x^5}{1.2\ldots5}\left(3\left(\frac{ddy}{dx^2}\right)^2 + 4\frac{dy}{dx}\cdot\frac{d^3y}{dx^3} + y\frac{d^4y}{dx^4}\right) \\
& - 2\frac{x^6}{1.2\ldots6}\left(10\frac{ddy}{dx^2}\cdot\frac{d^3y}{dx^3} + 5\frac{dy}{dx}\cdot\frac{d^4y}{dx^4} + y\frac{d^5y}{dx^5}\right) \\
& + \cdots
\end{aligned}
$$

Si opus foret, eandem formulam ad methodum exhauſtionis immediate reducere, & ex primis limitum notionibus inferre liceret, uti in §. 103.

§. 106. Quemadmodum data curvæ æquatio ad ſuperficiei ipſius determinationem conducit; ita viciſſim cognitio hujus ſuperficiei poteſt relationem coordinatarum curvæ ſuppeditare. Quod paucis exemplis illuſtrare ſufficiat.

Sit $S = \frac{1}{m}xy$; erit $\frac{dS}{dx} = \frac{1}{m}y + \frac{1}{m}x\frac{dy}{dx}$

atqui $\frac{dS}{dx} = y$; ergo $y = \frac{1}{m}y + \frac{1}{m}x\frac{dy}{dx}$

et $my = y + x\frac{dy}{dx}$.

1°. Sit $m = 1$; fit $0 = x\frac{dy}{dx}$: unde $\frac{dy}{dx} = 0$, & $y = C$. Qui caſus eſt rectanguli, cujus altitudo magnitudine datur.

2°. Sit $m > 1$; fit $y(m-1) = x\frac{dy}{dx}$; ſeu $\frac{dx}{dy}\cdot\frac{1}{x} = \frac{1}{m-1}\cdot\frac{1}{y}$

unde $C + \log. x = \frac{1}{m-1}\log. y$.

$\log. ax = \frac{1}{m-1}\log. y$. $y = a^{m-1}x^{m-1}$.

Si

Si $m = 2$; fit $y = ax$, quæ eſt æquatio ad lineam rectam: in reliquis caſibus curva eſt parabolica, ad axem aut ad tangentem ex vertice ductam relata.

3°. Sit $m < 1$; fit $y(1-m) + x\frac{dy}{dx} = 0$; unde $(1-m)\log.y + \log.x = \log.C$; $y^{1-m}x = C$, quæ eſt æquatio hyperbolarum ad aſymptoton relatarum.

Sit $S = t(b-y)$, $\frac{dS}{dx} = -t\frac{dy}{dx}$. Sed $\frac{dS}{dx} = y$: ergo $y = -t\frac{dy}{dx}$, & $1 = -\frac{t}{y} \times \frac{dy}{dx}$; hinc $x = C - t\log.y$. Sit $x = a$, quando $y = b$; $x = t\log.b - t\log.y$, ſeu $\frac{x}{t} = \log.\frac{b}{y}$, quæ eſt æquatio curvæ logarithmicæ.

Curva referatur ad aliquem focum; & ſit $S = yy \times \frac{x}{r}$. Erit $\frac{dS}{dx} = yy \times \frac{1}{r} + 2y\frac{dy}{dx} \times \frac{x}{r}$. Atqui $\frac{dS}{dx} = yy$ (§. 104): ergo $yy(1 - \frac{1}{r}) = 2y\frac{dy}{dx} \times \frac{x}{r}$; ſeu $y(r-1) = 2x\frac{dy}{dx}$, & $\frac{dx}{dy} \times \frac{1}{2x} = \frac{1}{y} \times \frac{1}{r-1}$; hinc $C + \frac{1}{2}\log.x = \frac{1}{r-1}\log.y$; ſeu $\frac{1}{2}\log.ax = \frac{1}{r-1}\log.y$, $ax = y^{\frac{2}{r-1}}$, quæ eſt æquatio ad ſpirales vulgares.

Sit $S = \frac{p}{q}yy$; $\frac{dS}{dx} = 2\frac{p}{q}y\frac{dy}{dx} = yy$; $2\frac{p}{q}.\frac{dy}{dx} = y$; $\frac{1}{2}\frac{q}{p}.\frac{dx}{dy} = \frac{1}{y}$; $C + \frac{1}{2}\frac{q}{p}x = \log.y$, quæ eſt æquatio ad ſpiralem logarithmicam.

Appendix.

De quadratura hyperbolæ conicæ.

Ne ſeries propoſitionum principalium, hoc capite contentarum, abrumperetur; ſatius eſſe ratus ſum particularem hyperbolæ conicæ quadraturam geometrice conſideratam ſeorſim exponere, ut ſequitur. (Vid. §. 99.)

§. 107. *Lemma.* Si curva quæcunque diametro ſit prædita (hoc eſt, ſi datur recta, quæ omnes rectas ſibi mutuo parallelas & curva utrinque terminatas bifariam ſecat): dico, hanc diametrum ſpatium quoque curvilineum in duas partes æquales dividere.

Etenim curvæ inſcribantur & circumſcribantur parallelogramma (ad normam §. 1. Ex. 3.). Parallelogramma hæc bina ſumpta ad utramque diametri partem ſunt mutuo æqualia; proinde & ſummæ horum parallelogrammorum ad utramque diametri partem ſunt invicem æquales: unde & limites harum ſummarum, nempe duæ partes ſpatii curvilinei ad utramque diametri partem ſitæ, æquales ſunt.

Fig. 27. *Theorema.* Super aſymptoto hyperbolæ conicæ ſumantur a centro C rectæ CA, CA', CB, CB' in proportione geometrica. Per puncta A, A', B, B' agantur rectæ alteri aſymptoto parallelæ, quæ hyperbolæ in punctis D, D', E, E' occurrant. Ex centro C agantur rectæ CD, CD', CE, CE'; dico, ſectores DCD', ECE', & trapezia $ADD'A'$, $BEE'B'$, eſſe invicem æqualia.

Jungantur rectæ ED', $E'D$, quæ aſymptotis in G, H, G', H' reſpective occurrant. Notum eſt, fore $GE = D'H$, $G'E' = DH'$.

Propter parallelas EB, CH eſt $GB : GE = A'C : D'H$

atqui $GE = D'H$

ergo $GB = A'C$; & pariter $B'G' = AC$.

Atqui (hyp.) $CA : CA' = CB : CB' = E'B' : EB$ (quoniam $CB \times BE = CB' \times B'E'$);

ergo $G'B' : GB = E'B' : EB$; proinde triangula $G'B'E'$, GBE ſunt invicem ſimilia, & rectæ $G'E'$, GE ſunt invicem parallelæ. Ducatur CF, quæ rectam GH bifariam in F ſecet. Quoniam $GF : G'F' = CF : CF' = FH : F'H'$; erit etiam $G'F' = F'H'$. Atqui $GE = D'H$, & $G'E' = DH'$; ergo $EF = FD'$, & $E'F' = F'D$. Proinde recta CF eſt diameter hyperbolæ, & ſpatia SEF, $SE'F'$ reſpective æqualia ſunt ſpatiis $SD'F$, SDF'. Sed triangula CEF, $CE'F'$ reſpective æqualia ſunt triangulis $CD'F$, CDF'; ergo & ſpatia CES, $CE'S$ reſpective æqualia ſunt ſpatiis $CD'S$, CDS; proinde & ſectores ECE', DCD' ſunt invicem æquales.

Rectæ AD, CD' ſibi invicem occurrant in I. Quoniam $CA \times AD = CA' \times A'D'$ ſemper eſt triangulum CAD æquale triangulo $CA'D'$: hinc, ſublato utrinque ſpatio communi CAI, & adjecto utrinque ſpatio communi DID', fit ſector DCD' æqualis trapezio mixtilineo $ADD'A'$; eodemque modo ſector ECE' æqualis eſt trapezio $BEE'B'$.

Corol.

Fig. 2[illegible].

Corollarium primum. Super aſymptoto CA ſumantur abſciſſæ quotcunque in progreſſione geometrica, & ducantur ordinatæ

$$CA,\ CA',\ CA'',\ CA''',\ CA^{\text{iv}} \ldots CA^{\text{N}}$$

$$AB,\ A'B',\ A''B'',\ A'''B''',\ A^{\text{iv}}B^{\text{iv}} \ldots A^{\text{N}}B^{\text{N}}$$

Omnia trapezia

$$AB',\ A'B'',\ A''B''',\ A'''B^{\text{iv}},\ A^{\text{iv}}B^{\text{v}} \ldots A^{\text{N}-1}B^{\text{N}}$$

ſunt invicem æqualia; & proinde ſpatia hyperbolica AB^{N} creſcunt ut logarithmi abſciſſarum CA^{N}, CA.

Corollarium ſecundum. Datis in aſymptoto duobus punctis quibuſcunque A, A^{N}; nullus eſt limes numeri abſciſſarum in progreſſione geometrica, cujus exponens $CA^{\text{N}} : CA$, creſcentium, quæ in aſymptoto producta CA ſumi poſſunt. Quare nullus etiam eſt limes numeri ſpatiorum trapezio AB^{N} æqualium, quæ in ſpatio aſymptotico ſumi poſſunt; ideoque nullus eſt limes magnitudinis hujus ſpatii.

Corollarium tertium. Detur ratio $CA^{\text{N}} : CA$. Ratio hæc dividatur in quotcunque n rationes $CA' : CA$ ſibi invicem æquales; proindeque trapezium hyperbolicum AB^{N} dividatur in eundem numerum partium æqualium AB', $A'B''$, $A''B'''$.... $A^{\text{N}-1}B^{\text{N}}$. Ducantur Bb', $B'b$ aſymptoto CA parallelæ, quæ ipſis $A'B'$, $A''B''$ in b' & b occurrant. Quoniam

$$CA : CA' = CA' : CA''$$

$$CA'-CA : CA''-CA' = CA : CA'$$

ſeu $AA' : A'A'' = A'B' : AB$, & $AA' \times AB = A'A'' \times A'B'$; & ſic deinceps: unde omnia parallelogramma ſpatio AB^{N} circumſcripta ſunt invicem æqualia. Quoniam ratio $CA^{\text{N}} : CA$ datur: eodem manente numero n, pariter datur ratio $CA' : CA$; adeoque & ratio $AA' : CA$, ſeu ratio parallelogrammi $ABb'A'$ ad potentiam hyperbolæ $CA \times AB \,\text{ſin.}A$. Et proinde etiam datur ratio ſummæ omnium parallelogrammorum ſpatio AB^{N} circumſcriptorum ad potentiam hyperbolæ; & proinde limes ſummæ horum parallelogrammorum ſeu ſpatium AB^{N} eſt etiam ad potentiam hyperbolæ in eadem ratione data. In diverſis itaque hyperbolis trapezia, duabus ordinatis, aut duabus abſciſſis, quarum ratio datur, reſpondentia, ſunt inter ſe uti potentiæ harum hyperbolarum; ſeu potentia hyperbolæ eſt modulus ſyſtematis logarithmici ad hanc hyperbolam pertinentis.

 Appli-

Applicatio 1. Hinc consequitur mensura spatii hyperbolici inter duas rectas asymptoto ordinatim applicatas, quarum ratio datur, comprehensi. Scilicet sumatur logarithmus naturalis rationis harum ordinatarum, & per hunc logarithmum multiplicetur potentia hyperbolæ.

Fig. 27. *Applicatio* 2. Hinc etiam fluit mensura spatii cujuslibet hyperbolici. Sit v. gr. $CS = a$ semi-axis transversus hyperbolæ, & quæratur spatium ESF. Sit $Sb = b$ semi-axis secundus; & sit Sb' asymptoto CB ordinatim applicata, seu ipsi EB parallela; atque potentia hyperbolæ, nempe $Cb' \times b'S \sin.C$, seu $\frac{1}{2}ab$, dicatur P.

$$ESF = ECF - ECS = ECF - EBb'S = ECF - P\log.\frac{Sb'}{EB} = ECF + P\log.\frac{EB}{Sb'}$$
$$= ECF + P\log.\frac{GE}{Sb} = ECF + P\log.\frac{GF - EF}{Sb} = ECF + P\log.\left(\frac{CF}{CS} - \frac{EF}{Sb}\right)$$
$$= \tfrac{1}{2}xy + P\log.\left(\frac{x}{a} - \frac{y}{b}\right) = \tfrac{1}{2}xy + P\log.\frac{bx-ay}{ab} = \tfrac{1}{2}\frac{b}{a}x\sqrt{(xx-aa)} + \tfrac{1}{2}ab\log.\frac{x-\sqrt{(xx-aa)}}{a}$$
$$= \tfrac{1}{2}\frac{b}{a}x\sqrt{(xx-aa)} + \tfrac{1}{2}ab\log.\frac{a}{x+\sqrt{(xx-aa)}}.$$

Caput Undecimum.

De rectificatione curvarum.

§. 108.

Ratio æqualitatis limes est rationis arcus cujusvis curvæ ad chordam suam (§. 47.). Hoc principio nititur rectificatio curvarum.

Etenim sint x & y abscissa axis & recta axi huic ordinatim applicata sub angulo recto. Sit S arcus curvæ: est $\lim.\Delta S : \sqrt{(\Delta x^2 + \Delta y^2)} = 1 : 1$

Fig. 11.

$$\text{seu} \quad \lim.\frac{\Delta S}{\Delta x} : \sqrt{\left(1 + \frac{\Delta y^2}{\Delta x^2}\right)} = 1 : 1$$
$$\text{seu} \quad \frac{dS}{dx} : \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)} = 1 : 1$$
$$\text{\& proinde} \quad \frac{dS}{dx} = \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)}.$$

Eadem

Eadem æquatio differentialis poteſt etiam ex §. 49. obtineri. Quoniam eſt $\frac{dx}{dS}$ = ſin.PMT, eſt $\frac{dS}{dx}$ = coſec.PMT: ſed cot.$PMT = \frac{dy}{dx}$ (§. 49.); proinde coſec.$PMT = \sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)}$; hinc $\frac{dS}{dx} = \sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)}$.

Quodſi autem angulus coordinatarum non eſt rectus, ponatur hic angulus $= \varphi$; eritque eodem modo $\frac{dS}{dx} = \sqrt{\left(1-2\frac{dy}{dx}\text{coſ.}\,\varphi+\left(\frac{dy}{dx}\right)^2\right)}$.

Data igitur æquatione curvæ rectificatio ejus ad calculum integralem reducitur.

Exemplum 1. Sit $pyy = x^3$, ſeu $p^{\frac{1}{2}}y = x^{\frac{3}{2}}$; erit $\frac{dS}{dx} = \frac{1}{2}\sqrt{\frac{4p+9x}{p}}$; unde $S = C+\frac{1}{27}\frac{(4p+9x)^{\frac{3}{2}}}{\sqrt{p}}$. Sit $S=0$, quando $x=0$; $S=\frac{1}{27}\left(\frac{(4p+9x)^{\frac{3}{2}}}{\sqrt{p}}-8p\right)$. Proinde parabolæ, æquatione $pyy = x^3$ determinatæ, rectificatio abſolute habetur.

Exemplum 2. Sit $yy = aa-xx$, ſeu $y = \sqrt{(aa-xx)}$: hinc $\frac{dS}{dx} = \frac{a}{\sqrt{(aa-xx)}}$, quæ eſt æquatio differentialis arcus circuli. Fit itaque

$$\frac{dS}{dx} = 1+\frac{1}{2}\cdot\frac{xx}{aa}+\frac{1}{2^2}\cdot\frac{1.3}{1.2}\cdot\frac{x^4}{a^4}+\frac{1}{2^3}\cdot\frac{1.3.5}{1.2.3}\cdot\frac{x^6}{a^6}+\frac{1}{2^4}\cdot\frac{1\ldots7}{1\ldots4}\cdot\frac{x^8}{a^8}+\ldots$$

$$S = x+\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{x^3}{aa}+\frac{1}{2^2}\cdot\frac{1.3}{1.2}\cdot\frac{1}{5}\cdot\frac{x^5}{a^4}+\frac{1}{2^3}\cdot\frac{1.3.5}{1.2.3}\cdot\frac{1}{7}\cdot\frac{x^7}{a^6}+\frac{1}{2^4}\cdot\frac{1\ldots7}{1\ldots4}\cdot\frac{1}{9}\cdot\frac{x^9}{a^8}+\ldots$$

Sit $x=a=1$; $p = 1+\frac{1}{2}\cdot\frac{1}{3}\cdot\ +\frac{1}{2^2}\cdot\frac{1.3}{1.2}\cdot\frac{1}{5}\ +\frac{1}{2^3}\cdot\frac{1.3.5}{1.2.3}\cdot\frac{1}{7}\ +\frac{1}{2^4}\cdot\frac{1\ldots7}{1\ldots4}\cdot\frac{1}{9}\ +\ldots$

Exemplum 3. Sit $yy = 4px$, $y = 2\sqrt{px}$, $\frac{dy}{dx} = \sqrt{\frac{p}{x}}$; $\frac{dS}{dx} = \sqrt{1+\frac{p}{x}}$, quæ eſt æquatio differentialis arcus parabolæ. Unde erit

$$S = \frac{y}{2p}\sqrt{(4pp+yy)}+2p\log.\frac{2p}{\sqrt{(4pp+yy)}-y}.$$

Exemplum 4. Sit $yy = \frac{bb}{aa}(aa-xx)$, $\frac{dS}{dx} = \frac{e}{a}\cdot\frac{\sqrt{\left(\frac{a^4}{ee}-xx\right)}}{\sqrt{(aa-xx)}}$, poſito $ee=a^2-b^2$, quæ eſt æquatio differentialis arcus ellipſeos. Hanc formulam terminis numero finitis integrare, aut ſaltem ad arcus circulares aut ad logarithmos terminis finitis reducere, fruſtra huc usque fuit tentatum.

Numerator $\sqrt{\left(\frac{a^4}{ee} - xx\right)} = a\sqrt{\left(\frac{aa}{ee} - \frac{xx}{aa}\right)}$ in feriem convertatur; fit

$$\sqrt{\left(\frac{a^4}{ee} - xx\right)} = a\left(\frac{a}{e} - \frac{1}{2}\cdot\frac{e}{a}\cdot\frac{xx}{aa} - \frac{1}{2^2}\cdot\frac{1}{1.2}\cdot\left(\frac{e}{a}\right)^3\cdot\left(\frac{x}{a}\right)^4 - \frac{1}{2^3}\cdot\frac{1.3}{1.2.3}\cdot\left(\frac{e}{a}\right)^5\cdot\left(\frac{x}{a}\right)^6 - \ldots\right.$$

unde $\frac{dS}{dx} = \frac{a}{\sqrt{aa-xx}}\left(1 - \frac{1}{2}\cdot\frac{ee}{aa}\cdot\frac{xx}{aa} - \frac{1}{2^2}\cdot\frac{1}{1.2}\cdot\left(\frac{e}{a}\right)^4\cdot\left(\frac{x}{a}\right)^4 - \frac{1}{2^4}\cdot\frac{1.3}{1.2.3}\cdot\left(\frac{e}{a}\right)^6\cdot\left(\frac{x}{a}\right)^6 \ldots\right.$

Integratio uniuscujusque termini hujus feriei reducitur ad arcus circulares; & nominatim, facto $x = a$, quadrans perimetri ellipfeos reducitur ad quadrantem circumferentiæ circuli ferie regulari, quæ fequitur.

$$S = ap\left(1 - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{ee}{aa} - \frac{1}{2^2}\cdot\frac{1}{1.2}\cdot\frac{3.1}{4.2}\cdot\frac{e^4}{a^4} - \frac{1}{2^3}\cdot\frac{1.3}{1.2.3}\cdot\frac{5.3.1}{6.4.2}\cdot\frac{e^6}{a^6} - \frac{1}{2^4}\cdot\frac{1.3.5}{1\ldots4}\cdot\frac{7\ldots1}{8\ldots2}\cdot\frac{e^8}{a^8} - \ldots\right.$$

Quæ feries promte convergit, fi exigua fuerit ellipfeos excentricitas e.

Æquatio differentialis rectificationis ellipfeos poteft etiam paulo fimplicius obtineri, ut fequitur. Sint CA, CB dimidii axes ellipfeos. Centro C radio CA (v. gr.) defcribatur quadrans circumferentiæ AND, cui CB occurrat in D. Tum angulo ACN dicto z, erit $CP = a\,\text{fin.}z$, $NP = a\,\text{cof.}z$, $MP = b\,\text{cof.}z$, (Fig. 29.)

$\frac{d.CP}{dz} = a\,\text{cof.}z$, $\frac{d.MP}{dz} = -b\,\text{fin.}z$; hinc

$$\frac{dS}{dz} = \sqrt{(aa\,\text{cof.}^2z + bb\,\text{fin.}^2z)} = a\sqrt{\left(1 - \frac{ee}{aa}\text{fin.}^2z\right)} = b\sqrt{\left(1 + \frac{ee}{bb}\text{cof.}^2z\right)}$$

$$= a\left(1 - \frac{1}{2}\frac{ee}{aa}\text{fin.}^2z - \frac{1}{2^2}\cdot\frac{1}{2}\frac{e^4}{a^4}\text{fin}^4z - \frac{1}{2^3}\cdot\frac{1.3}{1.2.3}\cdot\frac{e^6}{a^6}\text{fin.}^6z - \frac{1}{2^4}\cdot\frac{1.3.5}{1\ldots4}\cdot\frac{e^8}{a^8}\text{fin.}^8z - \ldots\right.$$

Facto $z = 90$ oritur eadem feries, quam prius obtinuimus.

Exemplum 5. Sit $yy = \frac{bb}{aa}(xx - aa)$, fit $\frac{dS}{dx} = \frac{e}{a}\cdot\frac{\sqrt{xx - \frac{a^4}{ee}}}{\sqrt{xx-aa}}$, pofito $ee = bb + aa$, quæ eft æquatio differentialis arcus hyperbolæ: numeratore in feriem converfo integratio ad logarithmos reducitur.

Exemplum 6. Sit $y = \sqrt{(2rx - xx)} + r\,\text{arc.fin.v.}\frac{x}{r}$, quæ eft æquatio cycloidis vulgaris. $\frac{dy}{dx} = \frac{r-x}{\sqrt{(2rx-xx)}} + \frac{r}{\sqrt{(2rx-xx)}} = \frac{2r-x}{\sqrt{(2rx-xx)}} = \sqrt{\frac{2r-x}{x}}$,

$\frac{dS}{dx} = \sqrt{\frac{2r}{x}}$, $S = 2\sqrt{rx}$. Facto $x = 2r$, $S = 4r$.

Fit

Exemplum 7. Sit $y = \sqrt{(2rx - xx)} + R$ arc.sin.v.$\frac{x}{r}$, quæ est æquatio cycloidum protractarum aut contractarum.

Fit $\frac{dS}{dx} = \frac{\sqrt{(RR+rr+2R(r-x)}}{\sqrt{(rr-(r-x)^2)}}$. Hæc formula reducitur ad rectificationem ellipseos methodo sequenti.

Fiat $r - x = \frac{AA-zz}{2B}$; erit $\frac{dS}{dz} = \frac{z}{B} \cdot \frac{\sqrt{(RR+rr+\frac{AA}{B}R - \frac{R}{B}zz)}}{\sqrt{\left(\left(r+\frac{AA-zz}{2B}\right)\left(r-\frac{AA-zz}{2B}\right)\right)}}$:

fiat $AA = 2Br$; erit $\frac{dS}{dz} = 2\frac{\sqrt{((R+r)^2 - \frac{R}{B}zz)}}{\sqrt{(4Br - zz)}}$:

fiat $B = R$; erit $\frac{dS}{dz} = 2\frac{\sqrt{((R+r)^2 - zz)}}{\sqrt{(4Rr - zz)}}$. Unde res ad rectificationem ellipseos reducitur. (Ex. 4.)

§. 109. Rectificatio curvarum ad focum aliquem relatarum eodem modo determinatur, aut etiam paulo aliter, ut sequitur.

Sin.$FMT = y\frac{dx}{dS}$ (§. 49.); hinc $\frac{dS}{dx} = y$ cosec.FMT. Atqui Fig. 15.

cot.$FMT = \frac{1}{y} \cdot \frac{dy}{dx}$; & proinde cosec.$FMT = \sqrt{\left(1 + \frac{1}{yy} \cdot \left(\frac{dy}{dx}\right)^2\right)}$; hinc $\frac{dS}{dx} = \sqrt{\left(yy + \left(\frac{dy}{dx}\right)^2\right)}$.

Exemplum 1. Sit $y = r \cdot \left(\frac{b}{a}\right)^x$, quæ est æquatio spiralis logarithmicæ.

$\frac{dy}{dx} = r \log.\frac{b}{a} \cdot \left(\frac{b}{a}\right)^x$, $\frac{dS}{dx} = r\left(\frac{b}{a}\right)^x \sqrt{(1 + \log.^2\frac{b}{a})}$; $S = y\sqrt{(1+\log.^2\frac{b}{a})}$: unde etiam arcus spiralis logarithmicæ crescunt in progressione geometrica, dum anguli crescunt in progressione arithmetica.

Exemplum 2. Sit $y = r\,\text{sec.}^2\tfrac{1}{2}x$, quæ est æquatio parabolæ conicæ. Erit $\frac{dS}{dx} = r\,\text{sec.}^3\tfrac{1}{2}x$. $S = r(\text{sec.}\tfrac{1}{2}x\,\text{tang.}\tfrac{1}{2}x + \log.(\text{sec.}\tfrac{1}{2}x + \text{tang.}\tfrac{1}{2}x)) =$
$r(\text{sec.}\tfrac{1}{2}x\,\text{tang.}\tfrac{1}{2}x + \log.\text{tang.}45° + \tfrac{1}{2}x)$.

Exemplum 3. Sit $y = \frac{aa-ee}{a+e\,\text{cos.}x}$, quæ est æquatio focalis ellipseos conicæ;

$\frac{dS}{dx} = \frac{aa-ee}{(a+e\,\text{cos.}x)^2}\sqrt{(aa+2ae\,\text{cos.}x+ee)} = \frac{aa-ee}{(a+e\,\text{cos.}x)^2}\sqrt{((a+e)^2 - 4ae\,\text{sin.}^2x)}$.

§. 110.

§. 110. Quoniam $\frac{dS}{dx} = \sqrt{(1+(\frac{dy}{dx})^2)}$; rectificatio cujusvis curvæ exprimi etiam poteſt per feriem Bernoullianam. Plerumque autem exponentes differentiales altiorum ordinum, quibus hæc feries conſtaret, adeo fiunt complexi, ut reductionis hujus utilitas valde dubia fiat.

Majoris momenti eſt obſervatio, quæ curvæ cujusvis rectificationem ad quadraturam alterius curvæ reducere docet.

Fig. 30. Sit nempe altera curva $M'm'$ ſuper eodem axe deſcripta, cujus ſuperficies ſit S', & ordinatæ y'. Erit (§. 100.) $\frac{dS'}{dx} = y'$, $\frac{dS}{dx} = \sqrt{(1+(\frac{dy}{dx})^2)}$. Ergo ſi fiat ſemper $y' = a\sqrt{(1+(\frac{dy}{dx})^2)}$; erit $\frac{dS'}{dx} = a.\frac{dS}{dx}$; & proinde $S' = aS$, ſi S & S' evaneſcant ſimul facto $x = 0$.

Natura autem alterius hujus curvæ per priorem facilius determinatur modo ſequente.

Definitio. Sit curva ad aliquem axem relata per rectas ipſi perpendiculariter ordinatim applicatas. Ad punctum aliquod hujus curvæ ducatur recta ipſam tangens. Tum per idem punctum ducatur recta tangenti perpendicularis; quæ axi (ſi fieri poſſit) occurrat. Pars hujus perpendicularis, punctum inter contactus & axem intercepta, dicitur *normalis;* & pars axis, normalem inter & rectam axi ex eodem puncto ordinatim applicatam comprehenſa, dicitur *ſubnormalis.*

Sit nempe MN tangenti MT perpendicularis, & quæ axi occurrat in N; linea MN dicitur *normalis*, & PN dicitur *ſubnormalis.*

In triangulo rectangulo TMN eſt $TP : PM = PM : PN$; proinde $PN = \frac{yy}{y\frac{dx}{dy}} = y\frac{dy}{dx}$. Hinc $TN = y(\frac{dx}{dy} + \frac{dy}{dx})$. Atqui $MN^2 = TN \times PN$; ergo $MN^2 = yy(1+(\frac{dy}{dx})^2)$, & $MN = y\sqrt{(1+(\frac{dy}{dx})^2)}$.

Quoniam autem $M'P = y' = a\sqrt{(1+(\frac{dy}{dx})^2)}$; fit $y' = a \times \frac{MN}{y}$, ſeu $y : MN = a : y'$. Proinde poſterioris curvæ ordinata eſt quarta proportionalis ordinatæ & normali prioris curvæ, & alicui rectæ magnitudine datæ.

Exem-

Exempla. 1°. Sit $yy = 2px$; $MN = y\sqrt{(1+\frac{pp}{yy})} = \sqrt{(pp+yy)}$; $y' = a.\frac{\sqrt{pp+yy}}{y}$. $a : y' = y : \sqrt{pp+yy}$; $aa : y'y' = yy : pp+yy$.

$$aa : y'y' - aa = yy : pp = 2px : pp = 2x : p; \quad y'y' = aa(1+\frac{p}{2x}).$$

2°. Sit $ayy = x^3$, $\frac{dy}{dx} = \frac{3xx}{2ay}$: $y'y' = \frac{1}{4}a(4a+9x) = \frac{9}{4}a(\frac{4}{9}a+x)$. Quoniam posterior curva, nempe parabola conica, est quadrabilis; prior curva est rectificabilis. (Vide §. 108. Ex. 1.)

§. 111. Data expressione arcus alicujus curvæ per coordinatas, ejus determinari potest ipsa curva (quantum permittit imperfecta calculi integralis conditio).

Exemplum 1. Quæratur linea, cujus longitudo est abscissæ proportionalis. Quoniam est $S = \frac{m}{n}x$, $\frac{dS}{dx} = \frac{m}{n}$; atqui $\frac{dS}{dx} = \sqrt{(1+(\frac{dy}{dx})^2)}$; ergo

$$1+(\frac{dy}{dx})^2 = \frac{mm}{nn}, \quad (\frac{dy}{dx})^2 = \frac{mm-nn}{nn}, \quad \frac{dy}{dx} = \frac{\sqrt{(mm-nn)}}{n}; \quad y = C + x\frac{\sqrt{(mm-nn)}}{n},$$

quæ est æquatio ad lineam rectam.

Exemplum 2. Quæritur curva; cujus arcus crescit in ratione subduplicata abscissæ. Est ideo $S = 2\sqrt{ax}$, $\frac{dS}{dx} = \sqrt{\frac{a}{x}} = \sqrt{(1+(\frac{dy}{dx})^2)}$; hinc

$$\frac{a}{x} = 1+(\frac{dy}{dx})^2, \quad \frac{dy}{dx} = \sqrt{\frac{a-x}{x}} = \frac{a-x}{\sqrt{ax-xx}} = \frac{\frac{1}{2}a}{\sqrt{ax-xx}} - \frac{\frac{1}{2}a-x}{\sqrt{ax-xx}},$$

$y = \frac{1}{2}a \times \text{arc. sin.v.} \frac{x}{\frac{1}{2}a} + \sqrt{(ax-xx)}$, quæ est æquatio cycloidis vulgaris.

Caput Duodecimum.

De capacitate solidorum rotundorum.

§. 112.

Sit curva quælibet ad axem aliquem relata per rectas axi huic perpendiculariter (v. gr. *a*) applicatas. Sit etiam rectangulum datam habens basim b, & cujus

a) Quæcunque dicentur de casu, quo rectæ axi ordinatim applicantur ad angulos rectos, facillime applicantur casui, quo eædem sub dato angulo obliquo axi occurrunt.

cujus altitudo mutabilis ſit abſciſſæ axis æqualis. Rectangulum & ſpatium curvilineum eidem aſciſſæ reſpondentia circa hujus axem ſimul revolvantur. Dico: limitem rationis, quæ inter mutationes ſimultaneas cylindri a rectangulo, & ſolidi a ſpatio curvilineo rotatione hac ſimul genitorum intercedit, æqualem eſſe rationi duplicatæ baſis rectanguli & ordinatæ huic abſciſſæ reſpondentis.

Fig. 24. Sit SMP curva ad axem SP per rectas MP huic perpendiculariter applicatas relata. Sit SA recta magnitudine data eidem axi perpendiculariter applicata. Sit SP abſciſſa axis, cui reſpondet ordinata MP; & compleatur rectangulum $SARP$, cujus altitudo creſcit ut abſciſſa SP. Spatium curvilineum SMP & rectangulum $SARP$ circa axem SP ſimul revolvantur. Dico, limitem rationis mutationum ſimultanearum cylindri a rectangulo $SARP$ & ſolidi a curva SMP ſimul genitorum æqualem eſſe rationi $SA^2 : MP^2$.

Sit PP' mutatio abſciſſæ axis; & proinde mutationes ſimultaneæ cylindri & prædicti ſolidi ſint ſolida rotatione rectanguli $SRR'P'$ & ſpatii $PMM'P'$ ſimul genita. Curvæ SMP inſcribantur & circumſcribantur rectangula $PMmP'$, $Pm'M'P'$. Solidum a ſpatio $PMM'P'$ genitum minus eſt cylindro a rectangulo $Pm'M'P'$ genito; majus autem cylindro ſimul genito a rectangulo inſcripto $PMmP'$. Atqui hi duo cylindri æque alti ſunt in ratione duplicata ordinatarum $M'P'$, MP: & proinde imminuta altitudine PP' ratio æqualitatis limes eſt rationis horum cylindrorum (§. 14.); tantoque magis ratio æqualitatis limes eſt rationis alterutrius horum cylindrorum ad ſolidum rotatione ſpatii $MPP'M'$ genitum. Quare ſolido a curva SMP genito dicto S, fit

$$
\begin{array}{llclcl}
 & \text{lim.}\,\Delta S : \text{cyl.}\,PMmP' & = & 1 & : & 1 \\
\text{Atqui} & \text{cyl.}\,PMmP' : \text{cyl.}\,PRR'P' & = & PM^2 & : & SA^2 \\
\text{Quare} & \text{lim.}\,\Delta S : \text{cyl.}\,PRR'P' & = & PM^2 & : & SA^2 \ (\S.\ 14.) \\
\text{ſeu} & \text{lim.}\,\Delta S : \pi.SA^2.PP' & = & PM^2 & : & SA^2 \\
 & & = & \multicolumn{3}{l}{\pi.PP'\times PM^2 : \pi.PP'.SA^2;} \\
\text{unde} & \text{lim.}\,\Delta S & = & \multicolumn{3}{l}{\pi.PP'\times PM^2} \\
\text{et} & \text{lim.}\,\frac{\Delta S}{\mathrm{d}x} & = & \multicolumn{3}{l}{\pi.PM^2} \\
\text{ſeu} & \frac{\mathrm{d}S}{\mathrm{d}x} & = & \multicolumn{3}{l}{\pi.PM^2 = \pi.yy.}
\end{array}
$$

Obſer-

Obſervatio. Eadem expreſſio differentialis eodem modo obtinetur, ſi ſpatium curvilineum *ABPM*, duabus ordinatis *AB*, *MP*, abſciſſa axis *BP*, & arcu *AM* terminatum, circa axem *SP* revolvatur. Fig. 25.

Exemplum 1. Sit $y = \frac{x^m}{a^{m-1}}$; erit $yy = \frac{x^{2m}}{a^{2m-2}} = \frac{dS}{dx}$, $S = C + \frac{1}{2m+1}\frac{x^{2m+1}}{a^{2m-2}}$.

1°. Sit m numerus poſitivus; & ſit $S = 0$, quando $x = 0$: erit Fig. 24.
$S = \frac{1}{2m+1}\frac{x^{2m+1}}{a^{2m-1}} = \frac{1}{2m+1}xyy$. Proinde data parabola, cujus æquatio eſt $y = \frac{x^m}{a^{m-1}}$; paraboloides dimidio parabolæ ſegmento circa axem revoluto genita eſt ad cylindrum æque altum ſuper eadem baſi, in ratione data $1 : 2m+1$.

Scholium. Exemplo hoc continetur caſus, quo ſpatium revolutum eſt triangulum, facto nempe $m = 1$.

2°. Sit m numerus negativus, ſeu ſit $y(a+x)^m = ba^m$. Ergo $\frac{dS}{dx} = bba^{2m}(a+x)^{-2m}$; Fig. 25.
$S = C + \frac{1}{1-2m}bba^{2m}(a+x)^{-2m+1} = C + \frac{1}{1-2m}\cdot\frac{bba^{2m}}{(a+x)^{2m-1}} = C + \frac{1}{1-2m}yy(a+x)$.
Sit $S = 0$, quando $x = 0$, $y = b$; erit $S = \frac{1}{1-2m}(yy(a+x) - abb) = \frac{1}{2m-1}(abb - yy(a+x))$.

1°. Sit $2m > 1$, ſeu $m > \frac{1}{2}$. Tum $S = \frac{1}{2m-1}(abb - yy(a+x))$, ſeu ſolidum rotatione ſpatii hyperbolici *ABPM* genitum proportionale eſt differentiæ cylindrorum a rectangulis *ABSD*, *MPSQ* genitorum. Quoniam autem $y = \frac{ba^m}{(a+x)^m}$, $yy = \frac{bba^{2m}}{(a+x)^{2m}}$, & $yy(a+x) = \frac{bba^{2m}}{(a+x)^{2m-1}}$; nec ullus eſt limes magnitudinis abſciſſæ $a+x$: nullus eſt limes parvitatis quantitatis $\frac{bba^{2m}}{(a+x)^{2m-1}}$, ſeu cylindri $yy(a+x)$. Hinc in formula $S = \frac{1}{2m-1}(abb - yy(a+x))$ ſolidum S habet limitem magnitudinis, nempe lim. $S = \frac{1}{2m-1}abb$.

2°. Sit $2m < 1$. Tunc $S = \frac{1}{1-2m}(yy(a+x) - abb) = \frac{1}{1-2m}(bba^{2m}(a+x)^{1-2m} - abb)$
$= \frac{1}{1-2m}abb\left(\frac{a+x}{a}\right)^{1-2m}$. Quoniam autem nullus eſt limes magnitudinis abſciſſæ

 x ſeu

x ſeu $a+x$, nullus etiam limes eſt magnitudinis quantitatis $abb\left(\frac{a+x}{a}\right)^{1-2m}$; & proinde, aucto x, nullus eſt limes magnitudinis ſolidi S.

3°. Sit $2m=1$. Ex æquatione $S=\frac{1}{1-1}(bba-abb)$ nihil deduci poteſt præ-æqualitatem cylindrorum $ABSD$, $MPSQ$. Æquatio differentialis $\frac{dS}{dx}=\frac{bba}{a+x}$ nos monet, hunc caſum ad logarithmos reduci. (§. 62.)

Scholium. Caſu, quo $2m>1$, & proinde $S=\frac{1}{2m-1}\left(abb-\frac{bba^{2m}}{(a+x)^{2m-1}}\right)$, ſumta x ex altera parte ordinatæ AB, fit $S=\frac{1}{2m-1}\left(abb-\frac{bba^{2m}}{(a-x)^{2m-1}}\right)$ $=\frac{1}{2m-1}abb\left(1-\left(\frac{a}{a-x}\right)^{2m-1}\right)=-\frac{1}{2m-1}abb\left(\left(\frac{a}{a-x}\right)^{2m-1}-1\right)$. Quoniam autem nullus eſt limes parvitatis differentiæ $a-x$, nullus etiam limes eſt magnitudinis quantitatis $\left(\frac{a}{a-x}\right)^{2m-1}$; & proinde nullus eſt limes magnitudinis ſolidi S. Quoniam autem $y=\frac{ba^m}{(a-x)^m}$; facto $x=a$, fieret $y=\frac{ba^m}{a^m o^m}=\frac{b}{o^m}$; unde (Cap. IX.) impoſſibile eſt, ut ſit $x=a$: & in æquatione $S=-\frac{1}{2m-1}abb\left(\left(\frac{a}{a-x}\right)^{2m-1}-1\right)$ poſito $x=a$, fit $S=-\frac{1}{2m-1}abb\left(\frac{1}{o^{m-1}}-1\right)$; quo rurſus monemur, impoſſibile eſſe, aſſignare ſolidum, etiam nunc fictum, ſpatio etiamnum ficto abſciſſæ BS reſpondens.

Cum omnia ratiocinia, expreſſioni $S=-\frac{1}{2m-1}abb\left(\frac{1}{o^{2m-1}}-1\right)$ ad varios infinitorum ordines illuſtrandos ſuperſtructa, contradictoria nitantur ſuppoſitione, poſſe fieri $x=a$; conſequentias inde ductas labi neceſſe eſt.

Idem dicatur de altero caſu, quo $2m<1$, & proinde $S=\frac{1}{1-2m}abb\left(\left(\frac{b}{y}\right)^{\frac{1-2m}{m}}-1\right)$; unde ſub contradictoria ſuppoſitione $y=o=b\cdot o$, fit $S=\frac{1}{1-2m}abb\left(\left(\frac{1}{o}\right)^{\frac{1-2m}{m}}-1\right)$.

§. 113. Determinatio capacitatis ſolidorum rotundorum nonnunquam etiam facilius peragitur modo ſequenti.

Baſi figuræ genitricis in partes quotcunque æquales diviſa, curvæ inſcribantur

bantur & circumſcribantur rectangula, quorum reliqua latera ſint axi revolutionis parallela. Rectangula hæc gyratione ſua circa axem gignent annulos cylindricos, ſolido inſcriptos & circumſcriptos. Solidum rotatione curvæ genitum majus eſt ſumma annulorum cylindricorum rectangulis inſcriptis genitorum; minus vero ſumma annulorum cylindricorum rectangulis circumſcriptis genitorum: ſed idem ſolidum limes eſt tam prioris ſummæ creſcentis, quam poſterioris ſummæ decreſcentis.

Sit $SMAB$ ſegmentum curvæ, quod rotatione circa axem SB baſi AB perpendicularem gignit aliquod ſolidum; & ſegmento huic circumſcribatur rectangulum $SBAD$, quod ſua circa eundem axem rotatione gignet cylindrum ſolido circumſcriptum. Dividatur AB in partes quotcunque æquales: ſit RR' una harum partium. Ducantur RM, $R'M'$ ipſi AB perpendiculares, quæ curvæ in M, M' occurrant, & ipſi SD in P, P'; ducantur Mm, $M'm'$ ipſi AB parallelæ, quæ rectis $M'R'$, MR in m & m' reſpective occurrant. Fig. 4.

Annuli cylindrici, gyratione rectangulorum RM', $R'M$, RP' geniti, ſunt inter ſe reſpective uti ipſa rectangula RM', $R'M$, RP'; ſeu uti lineæ $M'R'$, MR, RP. Solidum rotatione curvæ SMA genitum dicatur S, & cylindrus rotatione rectanguli $ABSD$ genitus dicatur C, ac mutationes ſimultaneæ horum ſolidorum dicantur ΔS, ΔC; erit $\Delta S : \Delta C \begin{matrix} > MR : SB \\ < M'R' : SB \end{matrix}$. Sed prima ratio poteſt fieri $\begin{matrix}\text{minor}\\ \text{major}\end{matrix}$ quacunque ratione propoſita, quæ $\begin{matrix}\text{major}\\ \text{minor}\end{matrix}$ ſit ratione $\begin{matrix} MR : SB \\ M'R' : SB \end{matrix}$. Proinde (§. 1.)

$$\lim. \Delta S : \Delta C = RM : SB.$$

Atqui
$$\Delta C : C = (BR + BR')RR' : BA^2; \text{ proinde}$$
$$\lim. \Delta C : C = 2BR . RR' : BA^2; \text{ hinc}$$
$$\lim. \Delta S : C = 2BR . RR' . RM : SB . BA^2$$
$$= 2\pi . BR . RR' . RM : SB . BA^2 . \pi$$

Facto igitur $BR = x$, $RR' = \Delta x$, $RM = y$,

eſt
$$\lim. \frac{\Delta S}{\Delta x} : 2\pi . xy = C : SB . BA^2 . \pi$$
$$= 1 : 1:$$

 unde

unde $\frac{dS}{dx} = 2\pi xy$; quæ igitur eſt æquatio differentialis ſolidi, ſpatio $SMAB$ circa axem SB rotato geniti.

Exemplum 1. Figura SAB ſit triangulum rectilineum. Sit $AB=b$: erit $SB = b\,\mathrm{tang}.A$, $y=(b-x)\,\mathrm{tang}.A$, $\frac{dS}{dx} = 2\pi x(b-x)\,\mathrm{tang}.A$, $S = C+\pi(bxx-\frac{2}{3}x^3)\,\mathrm{tang}.A$. Solidum autem evaneſcit, quando $x=0$; ergo $C=0$, $S=\pi x\,\mathrm{tang}.A(bx-\frac{2}{3}xx)=SB(bx-\frac{2}{3}xx)\pi$. Sit $x=b$; fit $S=\frac{1}{3}\pi.SB.BA^2$ (uti notum).

Exemplum 2. Figura $SMAB$ ſit quadrans circuli, cujus centrum B, & radii BS, $BA=r$; erit $y=\sqrt{(rr-xx)}$, $\frac{dS}{dx} = 2\pi x\sqrt{(rr-xx)}$, $S=\pi(C-\frac{2}{3}(rr-xx)^{\frac{3}{2}})$. Sit $x=0$; fit $S=\pi(C-\frac{2}{3}r^3)$: proinde $C=\frac{2}{3}r^3$, & $S=\pi(\frac{2}{3}r^3-\frac{2}{3}(rr-xx)^{\frac{3}{2}})$. Sit $x=r$; tum $S=\frac{2}{3}r^3.\pi$ (uti notum).

Exemplum 3. Figura $SMAB$ ſit parabola conica, circa axem SB rotata. Fit $y=\frac{bb-xx}{p}$, $\frac{dS}{dx}=2\pi\left(\frac{bbx-x^3}{p}\right)$, $S=2\pi\left(\frac{(\frac{1}{2}bbxx-\frac{1}{4}x^4)}{p}\right)=\frac{bbxx-\frac{1}{2}x^4}{p}\pi$. Sit $x=b$, erit $S=\pi.\frac{1}{2}\frac{x^4}{p}=\frac{1}{2}SB.AB^2.\pi$.

§. 114. Quoniam paraboloidum & hyperboloidum cubaturæ magni ſunt in determinanda ſolidorum rotundorum capacitate momenti; eaſdem paulo aliter explicare e re eſſe cenſeo.

Fig. 24. Sit $SMM'P'$ dimidium ſegmentum parabolæ, cujus æquatio eſt $y^{m+n}=a^m x^n$. Per verticem S ducatur tangens SA. Segmento SMP & ſpatio exteriori ASM inſcribantur v. gr. rectangula $MPP'm$, $MQQ'm'$. Spatio $SP'M'Q'$ circa axem SP revoluto, rectangulum $MMP'm$ gignit cylindrum ſegmento paraboloidico inſcriptum; & rectangulum $MQQ'm'$ gignit annulum cylindricum ſolido, quod rotatione ſpatii exterioris $SM'Q'$ gignitur, inſcriptum.

Atqui ratio cylindri $MPP'm$ ad annulum $MQQ'm'$ eadem eſt, quæ ſolidorum $Mm\times MP^2$, $Mm'\times QM(2MP\times Mm')$; & proinde limes rationis cylindri $MPP'm$ & annuli $MQQ'm'$ æqualis eſt limiti rationis ſolidorum $Mm\times MP^2$ & $Mm'\times MQ(2MP+Mm')$.

Sed

$$\begin{array}{llllll}
\text{Sed} & \lim. MP & : & 2MP+Mm' & = 1 : 2 & \\
& \lim. Mm & : & Mm' & = PT : MP & (\S. 40.) \\
& MP & : & QM & = MP : SP & \\
\text{Ergo} & \lim. MP^2 \times Mm & : & Mm'.QM(2MP+Mm') & = PT : 2SP & (\S. 14.) \\
& & & & = m+n : 2n & (\S. 42.)
\end{array}$$

Proinde lim. cyl. $MPP'm$: ann. cyl. $MQQ'm'$ $= m+n : 2n$. Proinde etiam (§. 15.) limes rationis ſummæ omnium cylindrorum paraboloidi inſcriptorum, ad ſummam omnium annulorum cylindricorum ſolido exteriori MSQ inſcriptorum, æqualis eſt eidem rationi conſtanti $m+n : 2n$. Unde etiam (§. 4.) ratio limitum harum ſummarum, nempe ratio paraboloidis SMP ad ſolidum exterius SQM, æqualis eſt eidem rationi datæ. Hinc paraboloides SMP eſt ad cylindrum $SPMQ$, uti $m+n$ ad $m+3n$.

2°. Spatium hyperbolicum $ABPM$, cujus æquatio eſt $y^m(a+x)^n = b^m a^n$, circa aſymptotum SP revolvatur. Circumſcribantur v. gr. rectangula $MPP'm$, $mQQ'M'$. Cylindrus rectangulo $MPP'm$, & annulus cylindricus rectangulo $mQQ'M'$ geniti, ſunt inter ſe uti ſolida $PP' \times MP^2$, $M'm.Q'M'(2MP-M'm)$. Proinde limes rationis cylindri hujus & annuli cylindrici æqualis eſt limiti rationis horum ſolidorum. Fig. 25.

$$\begin{array}{llllll}
\text{Atqui} & \lim. MP & : & 2MP-M'm & = 1 : 2 & \\
& \lim. PP' & : & M'm & = PT : MP & (\S. 40.) \\
& \lim. MP & : & M'Q' & = MP : MQ\ (=SP) & \\
\text{Ergo} & \lim. MP^2.PP' & : & M'm.M'Q'(2MP-Mm) & = PT : 2SP & (\S. 14.) \\
& & & & = m : 2n & (\S. 42.) \\
\text{Hinc} & \lim. \text{cyl.} MPP'm & : & \text{ann. cyl.} mQQ'M' & = m : 2n. &
\end{array}$$

Proinde etiam ratio ſummæ omnium cylindrorum, ſolido rotatione ſpatii $AMPB$ genito circumſcriptorum, ad ſummam omnium annulorum cylindricorum, ſolido rotatione ſpatii $AMQD$ genito circumſcriptorum, æqualis eſt eidem rationi conſtanti $m : 2n$: & proinde ſolida, quæ limites ſunt harum ſummarum, & quæ gignuntur rotatione ſpatiorum $AMPB$, $AMQD$, ſunt in eadem ratione data $m : 2n$ (§. 4.)

1°. Sit

1°. Sit $m > 2n$. Erit etiam sol.$AMPB >$ sol.$AMQD$, & sol.$AMPB$–sol.$AMQD$ $=$ cyl.$MPSQ$ — cyl.$ABSD$; proinde sol.$AMBP$: cyl.$MPSQ$ — cyl.$ABSD$ $=$ $m : m - 2n$.

2°. Sit $m < 2n$: erit sol.$AMQD >$ sol.$AMPB$, & sol.$AMQD$—sol.$AMPB$ $=$ cyl.$ABSD$—cyl.$MPSQ$; proinde sol.$AMPB$: cyl.$ABSD$—cyl.$MPSQ$ $= m : 2n - m$.

3°. Sit $m = 2n$. Nihil ulterius inde concludi poteſt, niſi quod ſolida $AMPB$, $AMQD$ ſint inter ſe æqualia; & proinde etiam cylindri $MPSQ$, $ABSD$ ſint inter ſe æquales.

§. 115. Curva generatrix ſolidi referatur ad focum aliquem per radios vectores ad hunc focum ductos, & per angulos, quos radii vectores cum recta poſitione data comprehendunt. Capacitas ſolidi determinabitur modo ſequenti.

Fig. 26. Sit $SM'M$ curva ad focum F relata per radios vectores FM, FM', & angulos SFM, SFM', quos radii vectores cum recta FS poſitione data comprehendunt. Centro F radio FS (pro unitate ſumto) deſcribatur circulus, qui radiis vectoribus FM, FM' in punctis X, X' occurrat. Tum centro F radiis FM, FM' deſcriptis arcubus Mm, $M'm'$, inſcribantur & circumſcribantur curvæ ſectores circulares MFm, $M'Fm'$. Solida ſectoribus his circa axem SF revolutis genita ſunt inter ſe in ratione triplicata radiorum FM, FM'. Sed ratio æqualitatis limes eſt rationis horum radiorum: proinde & ratio æqualitatis limes eſt rationis ſolidorum ab his ſectoribus genitorum (§. 4.); tantoque magis ratio æqualitatis limes eſt rationis alterutrius horum ſolidorum ad ſolidum rotatione ſectoris curvæ MFM' genitum.

Sit $Fm = y$, $FM' = y'$, $FS = r$, angulus $SFX = x$, $XFX' = \Delta x$. Notum eſt, ſolidi rotatione ſectoris circularis XFX' geniti expreſſionem eſſe $\frac{2}{3}\pi r^3(\text{coſ.}x' - \text{coſ.}x) = \frac{2}{3}\pi r^3 \Delta\text{coſ.}x$. Sit S ſolidum ſectore curvæ SFM genitum;

eſt	lim. ſol.MFM'	: ſol.MFm	$=$	1	:	1
	ſol.MFm	: ſol.XFX'	$=$	y^3	:	r^3
hinc	lim.ſol.MFM'	: ſol.XFX'	$=$	y^3	:	r^3 (§. 14.)
ſeu	lim.ΔS	: $\frac{2}{3}\pi r^3 \Delta\text{coſ.}x$	$=$	y^3	:	r^3

Atqui

$$
\begin{array}{llll}
\text{Atqui} & \lim.\Delta\cos.x : \Delta x & = \sin.x : 1 & (\S.\ 76.) \\
\text{Ergo} & \lim.\Delta S : \tfrac{2}{3}\pi r^3\Delta x & = y^3\sin.x : r^3 & \\
\text{et} & \lim.\dfrac{\Delta S}{\Delta x} : \tfrac{2}{3}\pi r^3 & = y^3\sin.x : r^3 & \\
& & = \tfrac{2}{3}\pi y^3\sin.x : \tfrac{2}{3}\pi r^3; & \\
\text{proinde} & \lim.\dfrac{\Delta S}{\Delta x} & = \tfrac{2}{3}\pi y^3\sin.x & \\
\text{seu} & \dfrac{dS}{dx} & = \tfrac{2}{3}\pi y^3\sin.x, &
\end{array}
$$

quæ est æquatio differentialis solidi propositi.

Exempla. 1°. Sit y quantitas data $= r$: $\frac{dS}{dx} = \frac{2}{3}\pi r^3\sin.x$, $S = C - \frac{2}{3}\pi r^3\cos.x$, Sit $S = 0$, quando $x = 0$: $C = \frac{2}{3}\pi r^3$, $S = \frac{2}{3}\pi r^3(1 - \cos.x) = \frac{4}{3}\pi r^3\sin.^2\frac{1}{2}x$. Sit $x = 180°$: $S = \frac{4}{3}\pi r^3$.

2°. Sit $y = a\sec.x$ (quæ est æquatio ad lineam rectam): $\frac{dS}{dx} = \frac{2}{3}\pi a^3(\sec.^3x\sin.x = \frac{2}{3}\pi a^3(\sec.^2x\tang.x)$, $S = \frac{2}{3}\pi a^3(C + \frac{1}{2}\sec.^2x)$. Sit $S = 0$, quando $x = 0$: $C = -\frac{1}{2}$, $S = \frac{1}{3}\pi a^3\tang.^2x$.

3°. Sit $y = a\sec.^2\frac{1}{2}x$ (quæ est æquatio parabolæ conicæ): $\frac{dS}{dx} = \frac{2}{3}\pi a^3\sec.^6\frac{1}{2}x\sin.x = \frac{4}{3}\pi a^3\sec.^4\frac{1}{2}x\tang.\frac{1}{2}x$, $S = \frac{4}{3}\pi a^3(C + \frac{1}{2}\sec.^4\frac{1}{2}x)$. Sit $S = 0$, quando $x = 0$: $C = -\frac{1}{2}$, $S = \frac{2}{3}\pi a^3(\sec.^4\frac{1}{2}x - 1) = \frac{2}{3}\pi a^3(\sec.^2\frac{1}{2}x + 1)(\sec.^2\frac{1}{2}x - 1)$ $= \frac{2}{3}\pi a^3\left(\left(\frac{y}{a} + 1\right)\left(\frac{y}{a} - 1\right)\right) = \frac{2}{3}\pi a(y + a)(y - a) = \frac{2}{3}\pi a(yy - aa)$.

4°. Sit $y = \dfrac{bb}{a + e\cos.x}$ (quæ est æquatio ellipseos conicæ): $\frac{dS}{dx} = \frac{2}{3}\pi\left(\frac{b^6}{(a + e\cos.x)^3}\sin.x\right)$, $S = \frac{1}{3}\pi\left(C + \frac{b^6}{e(a + e\cos.x)^2}\right)$. Sit $S = 0$, quando $x = 0$: $C = -\frac{b^6}{e(a+e)^2}$; $S = \frac{1}{3}\pi\left(\frac{b^6}{e(a + e\cos.x)^2} - \frac{b^6}{e(a+e)^2}\right) = \frac{1}{3}\pi\left(\frac{bbyy}{e} - \frac{bb(a-e)^2}{e}\right)$ $= \frac{1}{3}\pi\frac{bb(yy - (a-e)^2)}{e}$.

§. 116. Series Bernoulliana applicari etiam potest determinationi capacitatis solidorum rotundorum.

1°. Quoniam est (§. 112.) $\frac{dS}{dx} = \pi.yy$; fit per §. 36.

$$S = \pi\,(xyy$$
$$- 2.\frac{x^2}{1.2}.y\frac{dy}{dx}$$
$$+ 2.\frac{x^3}{1.2.3}\left(y\frac{ddy}{dx^2}+\left(\frac{dy}{dx}\right)^2\right)$$
$$- 2.\frac{x^4}{1.2...4}\left(y\frac{d^3y}{dx^3}+3\frac{dy}{dx}.\frac{ddy}{dx^2}\right)$$
$$+ 2.\frac{x^5}{1.2...5}\left(y\frac{d^4y}{dx^4}+4\frac{dy}{dx}.\frac{d^3y}{dx^3}+3\left(\frac{ddy}{dx^2}\right)^2\right)$$
$$- 2.\frac{x^6}{1.2...6}\left(y\frac{d^5y}{dx^5}+5\frac{dy}{dx}.\frac{d^4y}{dx^4}+10\frac{ddy}{dx^2}.\frac{d^3y}{dx^3}\right)$$

- - - - -
- - - - -

Quæ ſeries, utut regularis, non ſemper eſt calculo omnium promtiſſimo accommodata; imo & non abrumpitur quibusdam etiam caſibus, quibus alia via capacitas ſolidi terminis numero finitis exprimi poteſt.

2°. Quoniam eſt (§. 113.) $\frac{dS}{dx} = 2\pi xy$; erit (§. 36.)

$$S = 2\pi\left(\frac{1}{1.2}xxy - \frac{x^3}{1.2.3}.\frac{dy}{dx} + \frac{x^4}{1...4}\frac{ddy}{dx^2} - \frac{x^5}{1...5}\frac{d^3y}{dx^3} + \frac{x^6}{1...6}\frac{d^4y}{dx^4} - \ldots\right)$$

Cylindrorum ſolido rotundo tam inſcriptorum quam circumſcriptorum ſummæ eodem modo determinari poſſunt, quo rectangulorum ſpatio curvilineo inſcriptorum & circumſcriptorum ſummæ §. 103. determinatæ fuerunt.

§. 117. Solidorum rotundorum cubatura reduci poteſt ad rectificationem circuli & ad quadraturam alterius curvæ.

Etenim quoniam eſt (§. 112.) $\frac{dS}{dx} = \pi yy$; ſi deſcribatur ſuper eodem axe, & iisdem axis abſciſſis, altera curva, cujus ordinatæ y' ſint quadratis ordinatarum curvæ genitricis proportionales; ita ut $ay' = yy$: erit $\frac{dS}{dx} = \pi ay'$, & $\frac{dS'}{dx} = y'$; ideoque capacitas prioris ſolidi erit areæ poſterioris curvæ proportionalis.

Porro quoniam eſt (§. 113.) $\frac{dS}{dx} = 2\pi xy$: ſi fiat ſemper $ay' = 2xy$; capaci

tas solidi erit areae curvae, cujus aequatio differentialis est $\frac{dS'}{dx} = y'$ proportionalis.

§. 118. Data expressione capacitatis alicujus solidi rotundi per rectas axi ordinatim applicatas, determinari potest natura curvae genitricis.

Exempla. 1°. Sit $S = \pi aax$; ideoque $\frac{dS}{dx} = \pi aa$: hinc $yy = aa$, $y = a$; proinde basis est constans, & solidum propositum est cylindrus rectus.

2°. Sit $S = \pi \frac{x^m}{a^{m-3}}$: hinc $\frac{dS}{dx} (= \pi yy) = m\pi \frac{x^{m-1}}{a^{m-3}}$; $yy = m \frac{x^{m-1}}{a^{m-3}}$.

3°. Sit $S = \pi . \frac{m}{n} xyy$: $\frac{dS}{dx} (= \pi yy) = \pi . \frac{m}{n} yy + 2\pi xy \frac{dy}{dx}$; $yy = \frac{m}{n} yy + 2xy \frac{dx}{dy}$; $y = \frac{m}{n} y + 2x \frac{dy}{dx}$; $2x \frac{dy}{dx} = \frac{n-m}{n} . y$; $y^{2n} = C^{n+m} x^{n-m}$.

Caput decimum tertium.

De superficiebus solidorum rotundorum.

§. 119.

Linea curva quaecunque circa axem aliquem rotata generet superficiem curvam. Dico: rationem aequalitatis limitem esse superficierum a chorda & ab arcu curvae simul genitarum.

Sit SMM' curva, quae circa axem SP rotata superficiem curvam generat. Dico: rationem aequalitatis limitem esse rationis superficierum a chorda MM & ab arcu MZM simul genitarum. Fig. 31.

Per M & M' actae concipiantur rectae Mm & $M'm'$ axi SP parallelae & arcui MM' aequales, quae simul cum curva rotatae superficies curvas cylindricas gignent. Quoniam singulae partes rectae Mm axi SP parallelae minus ab hoc axe distant, quam pars quaelibet arcus MZM'; superficies a recta Mm genita minor est superficie ab arcu MZM' simul genita: contra quoniam singulae partes rectae $M'm'$ magis distant ab axe, quam pars quaelibet arcus MZM'; superficies a recta $M'm'$ genita major est superficie ab MZM' simul genita. Atqui superficies cylindricae rotatione rectarum Mm, $M'm'$ genitae sunt inter se uti or-

 dinatae

dinatæ MP, $M'P'$; & proinde ratio æqualitatis limes eſt rationis harum ſuperficierum; & a fortiori ratio æqualitatis limes eſt rationis alterutrius harum ſuperficierum ad ſuperficiem rotatione arcus genitam.

Scilicet arcu MZM' poſito $= \Delta z$, & ſuperficie ab eo genita $= \Delta S$; eſt $\lim. \Delta S : 2\pi . y\Delta z = 1 : 1$.

Superficies rotatione chordæ & rectæ Mm genitæ ſunt inter ſe uti rectangula $\sqrt{(\Delta x^2 + \Delta y^2)}\frac{2y+\Delta y}{2}$ & $y\Delta z$.

$$\begin{array}{llll}
\text{Atqui} & \lim.\sqrt{(\Delta x^2+\Delta y^2)} & : \Delta z = 1 : 1 & (\S.\ 47.) \\
 & \lim.\frac{2y+\Delta y}{2} & : y = 1 : 1 & \\
\text{ergo} & \lim.\sqrt{(\Delta x^2+\Delta y^2)}.\frac{2y+\Delta y}{2} & : y\Delta z = 1 : 1 & (\S.\ 14.) \\
\text{Sed} & \lim. y\Delta z & : \Delta S = 1 : 2\pi & \\
\text{ergo} & \lim.\sqrt{(\Delta x^2+\Delta y^2)}.\frac{2y+\Delta y}{2} & : \Delta S = 1 : 2\pi & (\S.\ 14.) \\
\text{ſeu} & \lim.\sqrt{\left(1+\frac{\Delta y^2}{\Delta x^2}\right)}.\frac{2y+\Delta y}{2} & : \frac{\Delta S}{\Delta x} = 1 : 2\pi & \\
\text{ſeu} & \lim.\frac{\Delta S}{dx} = 2\pi \lim.\sqrt{\left(1+\frac{\Delta\ ^2}{\Delta x^2}\right)}.\frac{2y+\Delta y}{2}; & &
\end{array}$$

et proinde $\frac{dS}{dx} = 2\pi y\sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)}$; quæ eſt æquatio differentialis ſuperficiei rotatione curvæ genitæ.

1°. Arcus MZM' totus ſit concavus verſus axem rotationis SP. Quoniam arcus major eſt chorda; & partes arcus inter rectas quaslibet axi ordinatim applicatas ab axe rotationis magis diſtant, quam ab eodem axe diſtant partes chordæ ordinatis iisdem interjacentes: duplici hoc reſpectu ſuperficies rotatione arcus genita major eſt ſuperficie, quæ rotatione chordæ gignitur.

2°. Quodſi autem arcus eſt verſus axem rotationis convexus; generatim de æqualitate aut inæqualitate ſuperficierum ab arcu & chorda ſimul genitarum ſtatui nihil poteſt. Nempe quatenus partes arcus duas inter quaslibet axi ordinatim applicatas contentæ huic viciniores ſunt, quam partes chordæ ordinatis iisdem interjacentes; ſuperficies arcu genita minor fit ſuperficie a chorda genita: dum ex altera parte ob arcum chorda majorem prior ſuperficies major

eſt

eſt poſteriore. Quare hoc caſu æquatio differentialis $\frac{dS}{dx} = 2\pi y \sqrt{(1 + (\frac{dy}{dx})^2)}$, limitum notione, prouti §. 13. fuit extenſa, nititur.

§. 120. *Exemplum* 1. Sit $yy = rr - xx$: $\frac{dy}{dx} = -\frac{x}{y}$, $\sqrt{(1 + (\frac{dy}{dx})^2)} = \frac{r}{y}$, $\frac{dS}{dx} = 2\pi r$, $S = 2\pi rx$. Nempe ſuperficies ſphæricæ inter duo plana ſibi invicem parallela comprehenſæ ſunt inter ſe uti diſtantiæ horum planorum.

Exemplum 2. Sit $y = x \,\text{tang.}\varphi$ (quæ eſt æquatio lineæ rectæ): $\frac{dS}{dx} = 2\pi x \,\text{tang.}\varphi \,\text{ſec.}\varphi$, $S = \pi xx \,\text{tang.}\varphi \,\text{ſec.}\varphi = \pi xy \,\text{ſec.}\varphi$.

Exemplum 3. Sit $yy = 2px$ (quæ eſt æquatio parabolæ): $\frac{dS}{dx} = 2\pi \sqrt{(pp+yy)}$ $= 2\pi \sqrt{(pp+2px)}$, $S = 2\pi(C + \frac{1}{3p}(pp+2px)^{\frac{3}{2}})$. Sit $S = 0$, quando $x = 0$: $C = -\frac{1}{3}pp$, $C = 2\pi(\frac{1}{3p}(pp+2px)^{\frac{3}{2}} - \frac{1}{3}pp) = \frac{2}{3}\pi p((p+2x)\sqrt{\frac{p+2x}{p}} - p)$.

Exemplum 4. Sit $yy = \frac{bb}{aa}(aa - xx)$.

1°. Sit $b = a$: $\frac{dS}{dx} = 2\pi a$ (quæ eſt æquatio differentialis ſuperficiei ſphæricæ).

2°. Sit $b < a$, ſeu ellipſis gyretur circa axem transverſum; ſit $aa - bb = ee$: $\frac{dS}{dx} = 2\pi \frac{be}{aa}\sqrt{(\frac{a^4}{ee} - xx)}$, $S = \pi(C + \frac{aab}{e}(\text{arc.ſin.}\frac{xe}{aa} + \frac{ex}{aa}\sqrt{(1 - \frac{ee}{aa}\cdot\frac{xx}{aa})}))$. Sit $S = 0$, quando $x = 0$: $C = 0$. Sit $x = a$: $S = \pi\frac{aab}{e}(\text{arc.ſin.}\frac{e}{a} + \frac{eb}{aa})$ $= \pi(bb + \frac{aab}{e}\text{arc.ſin.}\frac{e}{a})$.

Obſervatio. $\text{Arc.ſin.}\frac{e}{a} = \text{arc.tang.}\frac{e}{b} = \frac{e}{b} - \frac{1}{3}\cdot\frac{e^3}{b^3} + \frac{1}{5}\cdot\frac{e^5}{b^5} - \frac{1}{7}\cdot\frac{e^7}{b^7} + \dots$ (§. 79.)

Hinc $\frac{aab}{e}\text{arc.ſin.}\frac{e}{a} = aa(1 - \frac{1}{3}\frac{ee}{bb} + \frac{1}{5}\frac{e^4}{b^4} - \frac{1}{6}\frac{e^6}{b^6} + \dots)$

Proinde facto $e = 0$, $S = \pi(bb + aa) = 2bb\pi$, juxta Ex. 1.

3°. Ellipſis gyretur circa axem ſecundum; unde $yy = \frac{aa}{bb}(bb - xx)$: $\frac{dS}{dx} = 2\pi\cdot\frac{ae}{bb}\sqrt{(\frac{b^4}{ee} + xx)}$, cujus integratio reducitur ad aream hyperbolæ conicæ ſeu ad logarithmos. Scilicet

$S = \pi$

$S = \pi\frac{ae}{bb}\left(C + x\sqrt{\left(\frac{b^4}{ee} + xx\right)} + \frac{b^4}{ee}\log.\left(\sqrt{\frac{b^4}{ee} + xx}\right) + x\right)$. Sit $S = 0$, quando $x = 0$:

$C = -\log.\frac{bb}{e}$, $S = \pi.\frac{ae}{bb}\left(x\sqrt{\left(\frac{b^4}{ee} + xx\right)} + \frac{b^4}{ee}\log.\frac{\sqrt{\left(\frac{b^4}{ee} + xx\right)} + x}{\frac{bb}{e}}\right]$. Sit $x = b$:

$$S = \pi.\frac{ae}{bb}\left(b\times\frac{ab}{e} + \frac{b^4}{ee}\log.\frac{\frac{ab}{e} + b}{\frac{bb}{e}}\right]$$

$$= \pi\left(aa + \frac{abb}{e}\log.\frac{a+e}{b}\right) = \pi\left(aa + \frac{abb}{e}\log.\frac{b}{a-e}\right) = \pi\left(aa + \frac{abb}{e}\log.\sqrt{\frac{a+e}{a-e}}\right).$$

Observatio. $\log.\sqrt{\frac{a+e}{a-e}} = \frac{e}{a} + \frac{1}{3}.\frac{e^3}{a^3} + \frac{1}{5}.\frac{e^5}{a^5} + \ldots$ (§. 61.)

hinc $\frac{abb}{e}\log.\sqrt{\frac{a+e}{a-e}} = ab\left(1 + \frac{1}{3}.\frac{ee}{aa} + \frac{1}{5}.\frac{e^4}{a^4} + \ldots\right)$

proinde facto $e = 0$, $S = \pi(aa + ab) = 2\pi aa$, juxta Ex. 1.
Superficies hyperboloidum, rotatione hyperbolæ circa alterutrum axem genitæ, eodem modo determinantur.

Exemplum 5. Sit $y(a+x) = ab$, seu hyperbola conica gyretur circa asymptotum. $\frac{dS}{dx} = 2\pi\frac{ab}{a+x}\sqrt{\left(1 + \frac{aabb}{(a+x)^4}\right)}$: unde facto $S = 0$, quando $x = 0$;

$$S = \pi.ab\left(\sqrt{\left(1+\frac{bb}{aa}\right)} - \sqrt{\left(1 + \frac{aabb}{(a+x)^4}\right)} + \log.\frac{\sqrt{((aa+bb)} - a}{\sqrt{\left(\frac{(a+x)^4}{aa} + bb\right)} - \frac{(a+x)^2}{a}}\right].$$

Exemplum 6. Sit $y = \sqrt{(2rx - xx)} + \text{arc.sin.v.}\frac{x}{r}$, quæ est æquatio cycloidis vulgaris. Fit $\frac{dS}{dx} = 2\pi\left(\sqrt{(2r(2r-x)} + r\sqrt{\frac{2r}{x}}\,\text{arc.sin.v.}\frac{x}{r}\right)$:

unde $S = 2\pi\left(C - \frac{2}{3}(2r-x)\sqrt{2r(2r-x)} + 4r\sqrt{2r(2r-x)} + 2r\sqrt{2rx}\,\text{arc.sin.v.}\frac{x}{r}\right)$.
Sit $S = 0$, quando $x = 0$: $C = -\frac{16}{3}rr$. Sit $x = 2r$: $S = 8\pi rr\left(\pi - \frac{4}{3}\right)$.

§. 121. Quoniam $\frac{dS}{dx} = 2\pi y\sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)}$; & normalis expressio est $y\sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)}$ (§. 110.): quadratura superficiei solidorum rotundorum pendet,

ab

ab normali curvæ genitricis. Scilicet ſuper axe curvæ genitricis deſcribatur curva talis, ut rectæ axi ordinatim applicatæ æquales ſint (aut proportionales) normalibus curvæ genitricis iisdem abſciſſis reſpondentibus. Superficies ſolidi rotatione prioris curvæ geniti proportionalis erit areæ curvæ poſterioris. Ratio autem harum ſuperficierum ea eſt, quæ circumferentiæ circuli & radii.

Scholium. Seriei Bernoullianæ ad determinandas ſolidorum rotundorum ſuperficies applicandæ non immoror; quoniam exponentes differentiales ordinum ſucceſſivorum quantitatis $y\sqrt{(1+(\frac{dy}{dx})^2)}$ adeo fiunt compoſiti, ut nulla inde utilitas duci poſſit.

§. 122. Ratio differentialis ſuperficierum, a curvis ad punctum datum relatis genitarum, eodem modo poteſt determinari. Sit nempe angulus $SFM=z$, & radius vector $FM=r$; eſt $\frac{dS}{dx}=2\pi r\,\text{ſin.}z\sqrt{(rr+(\frac{dy}{dx})^2)}$. Exercitii cauſa oſtendere ſufficiat, quomodo ratio hæc differentialis ex priore deducatur. Eſt ideo Fig. 26.

$$y=r\,\text{ſin.}z;\quad \text{igitur}\quad \frac{dy}{dz}=\frac{dr}{dz}\text{ſin.}z+r\,\text{coſ.}z$$

$$x=r\,\text{coſ.}z\qquad \frac{dx}{dz}=\frac{dr}{dz}\text{coſ.}z-r\,\text{ſin.}z$$

$$\frac{dy}{dx}=\frac{\frac{dr}{dz}\text{ſin.}z+r\,\text{coſ.}z}{\frac{dr}{dz}\text{coſ.}z-r\,\text{ſin.}z}$$

$$\text{hinc}\quad \sqrt{(1+(\frac{dy}{dx})^2)}=\frac{\sqrt{(rr+(\frac{dr}{dz})^2)}}{\frac{dr}{dz}\text{coſ.}z-r\,\text{ſin.}z}=\frac{\sqrt{(rr+(\frac{dr}{dz})^2)}}{\frac{dx}{dz}}=\frac{dz}{dx}\sqrt{(rr+(\frac{dr}{dz})^2)}.$$

$$\text{Atqui}\quad \frac{dS}{dx}=2\pi y\sqrt{(1+(\frac{dy}{dx})^2)};$$

$$\text{ergo}\quad \frac{dS}{dx}=2\pi r\,\text{ſin.}z\frac{dz}{dx}\sqrt{(rr+(\frac{dr}{dz})^2)}$$

$$\text{et}\quad \frac{dS}{dz}=2\pi r\,\text{ſin.}z\sqrt{(rr+(\frac{dr}{dz})^2)}.$$

Exem-

Exemplum 1. Sit r datæ magnitudinis: $\frac{dS}{dz} = 2\pi rr\,\text{fin.}\,z$, $S = 2\pi rr\,(C - \text{cof.}\,z)$. Sit $S = 0$, quando $z = 0$: $C = 1$, $S = 2\pi rr(1 - \text{cof.}z) = 4\pi rr\,\text{fin.}^2\tfrac{1}{2}z$.

Exemplum 2. Sit $r = a\,\text{fec.}\,z$ (quæ eft æquatio lineæ rectæ): $\frac{dr}{dz} = a\,\text{fec.}z\,\text{tang.}z$, $\frac{dS}{dz} = 2\pi a\,\text{fec.}z\,\text{fin.}z\,\sqrt{(aa\,\text{fec.}^2z + aa\,\text{fec.}^2z\,\text{tang.}^2z)} = 2\pi aa\,\text{tang.}z\,\text{fec.}^2z$, $S = \pi aa(C + \text{fec.}^2z)$. Sit $S = 0$, quando $z = 0$: $C = -1$; $S = \pi aa\,\text{tang.}^2z$.

Exemplum 3. Sit $r = p\,\text{fec.}^2\tfrac{1}{2}z$ (quæ eft æquatio focalis parabolæ): $\frac{dr}{dz} = p\,\text{fec.}^2\tfrac{1}{2}z\,\text{tang.}\tfrac{1}{2}z$, $\frac{dS}{dz} = 4\pi.pp\,\text{fec.}^3\tfrac{1}{2}z\,\text{tang.}\tfrac{1}{2}z$, $S = C + \tfrac{8}{3}\pi pp.\text{fec.}^3\tfrac{1}{2}z = \tfrac{8}{3}\pi.pp(C + \text{fec.}^3\tfrac{1}{2}z)$. Sit $S = 0$, quando $z = 0$: $C = -1$, $S = \tfrac{8}{3}\pi.pp(\text{fec.}^3\tfrac{1}{2}z - 1) = \tfrac{8}{3}\pi.pp\left(\left(\frac{r}{p}\right)^{\frac{3}{2}} - \right) = \tfrac{8}{3}\pi.(r\sqrt{pr} - pp)$.

§. 123. Quemadmodum fuperficies folidorum rotundorum æquatione curvæ genitricis determinatur: ita viciffim, expreffione fuperficiei folidorum rotundorum data, erui poteft æquatio curvæ genitricis; uti paucis exemplis oftendam.

Exemplum 1. Superficies folidi crefcat uti abfciffa axis. Fit ideo $S = 2\pi rx$, & $\frac{dS}{dx} = 2\pi r = 2\pi y\sqrt{(1 + \left(\frac{dy}{dx}\right)^2}$; hinc $1 + \left(\frac{dy}{dx}\right)^2 = \frac{rr}{yy}$, $\left(\frac{dy}{dx}\right)^2 = \frac{rr - yy}{yy}$; $\frac{dx}{dy} = \frac{y}{\sqrt{(rr - yy)}}$; $x = C - \sqrt{(rr - yy)}$, quæ eft æquatio circuli.

Exemplum 2. Superficies crefcat in ratione duplicata abfciffæ axis. Eft ideo $S = \frac{2m}{n}\pi.xx$, $\frac{dS}{dx} = \frac{4m}{n}\pi x = 2\pi y\sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)}$; $\frac{2m}{n}x = y\sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)}$. Huic æquationi fatisfacit (in cafu particulari) linea recta, quando nempe $\frac{dy}{dx}$ eft ratio conftans, unde & ratio $x : y$ eft conftans.

Exemplum 3. Curva referatur ad aliquem focum, & fuperficies crefcat in ratione radii vectoris. Eft ideo $S = 2\pi ar$; $\frac{dS}{dz} = 2\pi a\frac{dr}{dz} = 2\pi r\,\text{fin.}z\sqrt{\left(rr + \left(\frac{dr}{dz}\right)^2\right)}$; unde $a\frac{dr}{dz} = r\text{fin.}z\sqrt{\left(rr + \left(\frac{dr}{dz}\right)^2\right)}$, $\left(\frac{dr}{dz}\right)^2(aa - rr\text{fin.}^2z) = r^4\text{fin.}^2z$, $\frac{dr}{dz} = \frac{rr\text{fin.}z}{\sqrt{(aa - rr\text{fin.}^2z)}}$, cujus æquationis integrationem me exhibere non poffe fateor.

Caput decimum quartum.
De regula Guldini *dicta.*

§. 124.

Recta magnitudine data circa axem quemcunque in eodem plano situm revolvatur. Superficies cylindrica vel conica, rotatione rectæ hujus genita, æqualis est rectangulo sub hac linea & sub circumferentia a puncto lineæ hujus medio descripta. Proinde superficies, a recta magnitudine data circa varios axes revoluta genitæ, proportionales sunt distantiis puncti ejus medii ab axe revolutionis.

Pariter cylindrus vel annulus cylindricus, a rectangulo circa unum ipsius latus, vel circa rectam uni ex lateribus ejus parallelam revoluto genitus, æqualis est prismati, cujus basis est ipsum rectangulum, & cujus altitudo æqualis est circumferentiæ, quam centrum figuræ rectanguli hac revolutione percurrit seu generat. Solida igitur ab eodem rectangulo, circa varios axes uni laterum ejus parallelos revoluto, genita proportionalia sunt distantiis centri hujus rectanguli ab axe revolutionis.

His exemplis (omnium simplicissimis) præmissis, facile intelligitur, quid sibi velit regula capite hoc explicanda. Scilicet lineæ vel superficies quæcunque tam specie quam magnitudine datæ circa axem aliquem revolvantur: superficies vel solida, rotatione hac genita, sunt inter se in ratione composita ex rationibus magnitudinum atque circumferentiarum a puncto quodam, cujus situm magnitudo genitrix determinat, descriptarum.

Cum regula hæc determinationem capacitatis & superficiei solidorum rotundorum (de qua duobus postremis capitibus actum fuit) admodum juvet; e re esse censeo, fundamenta ejus hoc loco exponere.

§. 125. *Lemma.* Sint quotlibet puncta in eodem plano positione data, & sint totidem rectæ magnitudine datæ. Ex omnibus punctis datis demittantur in rectam quamlibet in eodem plano sitam rectæ perpendiculares. Sumatur summa rectangulorum factorum ex his perpendiculis & ex lineis magnitudine datis respective. Dico: dari in eodem plano punctum, ex quo si in eandem

rectam demittatur perpendiculum, rectangulum, sub hoc perpendiculo & sub summa rectarum magnitudine datarum contentum, æquale sit summæ priorum rectangulorum.

Fig. 32. *Exemplum primum.* Sint duo puncta A, B, a quibus in rectam quamcunque $A'B'$ demittantur perpendicula AA', BB'. Sint etiam duæ rectæ a, b magnitudine datæ. Recta AB in puncto Z ita secetur, ut sit $AZ\times a = BZ\times b$; & a puncto Z in rectam $A'B'$ demittatur perpendiculum ZY. Dico: fore $AA'\times a + BB'\times b = ZY(a+b)$.

Constr. Per Z agatur recta ab ipsi $A'B'$ parallela, quæ rectis AA', BB' in a & b occurrat.

Dem. Propter similia triangula AZa, BZb est $Aa : Bb = AZ : BZ = b : a$; hinc $a(ZY - AA') = b(BB' - ZY)$; proinde $ZY(a+b) = a\times AA' + b\times BB'$.

Fig. 33. *Exemplum secundum.* Sint tria puncta A, B, C positione data, & tres rectæ a, b, c magnitudine datæ; & sint AA', BB', CC' perpendicula in rectam quamcunque demissa.

Puncto Z respectu punctorum A, B ita determinato, ut sit $ZY(a+b) = AA'\times a + BB'\times b$; determinetur punctum Z' respectu punctorum Z, C, ita ut $Z'Y'(a+b+c) = ZY(a+b) + CC'\times c$: erit $Z'Y'(a+b+c) = AA'\times a + BB'\times b + CC'\times c$.

Exemplum tertium. Sint quatuor puncta A, B, C, D positione data, & quatuor lineæ a, b, c, d magnitudine datæ.

Determinetur punctum Z' respectu punctorum A, B, C (juxta exemplum secundum); tum determinetur punctum Z'' respectu punctorum Z', D, sic, ut $Z''Y''(a+b+c+d) = Z'Y'(a+b+c) + DD'\times d$: erit $Z''Y''(a+b+c+d) = AA'\times a + BB'\times b + CC'\times c + DD'\times d$.

Ex his exemplis patet methodus generalis procedendi ac demonstrandi. Propositione nimirum evicta pro punctis quotlibet n positione datis, & pro totidem rectis magnitudine datis; eadem etiam vera esse demonstratur, si tam punctorum quam rectarum numerus unitate augeatur. Cum igitur vera sit propositio de paucis punctis & rectis, 2, 3, 4 . . . .; ea etiam vera est pro quinque punctis: inde pro sex, & sic deinceps. Ideo semper determinari potest punctum Z tale, ut sit $a\times AA' + b\times BB' + c\times CC' + \ldots\, l\times LL' = ZZ'(a+b+c+\ldots+l)$.

Scholium.

Scholium. Quoniam puncti Z in plano positione dato situs determinatur perpendiculis, quae in duas rectas positione datas, se mutuo secantes, ab eo ducuntur; demonstrandum est: quod, si praecedens propositio locum habeat pro ejusmodi duabus rectis, eadem quoque valeat pro alia quacunque recta.

Et primo quidem praedicta aequatione locum habente pro recta qualibet positione data, eadem valet pro quavis recta priori parallela; cum utrumque aequationis prioris membrum ita augeatur vel minuatur rectangulo sub distantia duarum parallelarum & sub summa $a+b+c+d+\ldots+l$.

2°. Sint duae rectae SN, SN' sibi invicem perpendiculares; & demonstratum fuerit duas aequationes sequentes locum habere: Fig. 34.

$$a\times AA'+b\times BB'+c\times CC'+\ldots l\times LL' = ZZ'(a+b+c+d+\ldots l)$$
$$a\times SA'+b\times SB'+c\times SC'+\ldots l\times SL' = SZ'(a+b+c+d+\ldots l).$$

Ducatur per S recta quaevis SN'', in quam demittantur perpendicula AA'', BB'', CC'', $DD''\ldots LL''$, ZZ''; dico, fore etiam $a\times AA''+b\times BB''+c\times CC''+\ldots l\times LL''=ZZ''(a+b+c+\ldots+l)$.

Etenim quam facillime demonstratur (sive immediate, sive ex primis trigonometriae planae principiis de sinu & cosinu summae ac differentiae duorum angulorum) esse

$$\begin{aligned}
AA'' &= AA'\text{cos}.NSN''-SA'\text{sin}.NSN''\\
BB'' &= BB'\text{cos}.NSN''-SB'\text{sin}.NSN''\\
CC'' &= CC'\text{cos}.NSN''-SC'\text{sin}.NSN''\\
&\vdots\\
LL'' &= LL'\text{cos}.NSN''-SL'\text{sin}.NSN''\\
ZZ'' &= ZZ'\text{cos}.NSN''-SZ'\text{sin}.NSN''.
\end{aligned}$$

Proinde

$$a\times AA''+b\times BB''c\times CC''+\ldots+l\times LL''=\begin{matrix}\text{cos}.NSN''(a\times AA'+b\times BB'+c\times CC'+\ldots+l\times LL')\\-\text{sin}.NSN''(a\times SA'+b\times SB'+c\times SC+\ldots+l\times SL')\end{matrix}$$
$$=\begin{matrix}ZZ'(a+b+c+\ldots l)\text{cos}.NSN''\\-SZ'(a+b+c+\ldots l)\text{sin}.NSN''\end{matrix}=(a+b+c+\ldots l)(ZZ'\text{cos}.NSN''-SZ'\text{sin}.NSN'')$$
$$=(a+b+c+\ldots l)ZZ''.$$

3°. Vicissim: si propositio locum habet pro duabus rectis SN, SN'', eadem locum habet pro tertia recta SN' uni priorum perpendiculari & per S ducta.

Propositione igitur pro duabus quibuscunque rectis sibi mutuo occurrenti-

 bus

bus ſtabilita, eadem vera eſt pro alia quacunque recta; ſive transeat per punctum occurſus priorum, ſive non.

Obſervatio. In enunciatis præcedentibus ſuppoſui, puncta data jacere ad easdem partes rectæ, in quam perpendicula aguntur. Si ſecus fuerit: mutatis ſignis perpendiculorum ex punctis ad diverſas ejusdem rectæ partes ſitis demiſſorum; quæcunque de ſumma dicta fuerunt, applicantur exceſſui, quo ſumma rectangulorum, punctis ab una hujus lineæ parte ſitis reſpondentium, ſuperat ſummam reliquorum. Quo monito præmiſſo, in ſequentibus pariter de ſumma tantum dicere ſufficiet; quaſi omnia puncta ad easdem rectæ, in quam perpendicula aguntur, partes jacerent.

§. 126. In Mechanica oſtenditur: punctum Z commune eſſe *gravitatis centrum* totidem maſſarum a, b, c, . . . l, quarum ſingularum centra gravitatis ſint A, B, C, . . . L. Qua puncti Z denominatione hic etiam (brevitatis cauſa) uti liceat.

Itaque punctum rectæ magnitudine datæ medium eſt centrum gravitatis hujus lineæ; & ſuperficies, quam recta magnitudine data circa axem quemlibet revoluta generat, æqualis eſt rectangulo ſub hac recta & ſub circumferentia, quam centrum gravitatis ejus rotatione hac deſcribit.

Item centrum figuræ cujusvis rectanguli ſimul eſt ejus centrum gravitatis; & ſolida rotatione rectangulorum circa axes quoscunque, uni ex lateribus eorum parallelos, genita ſunt in ratione compoſita ex rationibus ipſorum rectangulorum atque circumferentiarum, quas earum centra gravitatis deſcribunt.

Sint quotlibet rectæ in eodem plano poſitione ac magnitudine datæ, quæ ſimul circa axem quemlibet revolvantur. In punctis rectarum harum mediis fingantur maſſæ ſingulis rectis proportionales, & quæratur centrum gravitatis commune harum maſſarum. Summa ſuperficierum a rectis illis rotatione hac genitarum æqualis eſt rectangulo facto ex ſumma harum rectarum, & ex circumferentia ab communi maſſarum harum centro gravitatis eadem revolutione percurſa.

Sint quotlibet rectangula in eodem plano poſitione & magnitudine data; ſic ut ſingulorum horum rectangulorum unum latus ſit rectæ alicui poſitione datæ parallelum. Rectangula hæc ſimul revolvantur circa axem rectæ huic paral-

parallelum. Tum in centris figuræ horum rectangulorum concipiantur massæ iisdem proportionales, & quæratur centrum gravitatis commune omnium harum massarum; solidum ab rectangulis his revolutione illa genitum est in ratione composita ex summa rectangulorum, & ex circumferentia a communi gravitatis centro revolutione hac genita.

§. 187. *Lemma notum.* Trapezium $AA'B'B$, cujus anguli B, B' sunt recti, rotetur circa latus $A'B'$; solidum revolutione hac genitum est ad cylindrum ejusdem altitudinis $A'B'$, & cujus radius baseos est r, uti $\frac{AA'^2+AA'\times BB'+BB'^2}{3}$ ad r^2. Proinde, posita π dimidia circumferentia circuli, cujus radius est unitas, capacitatis solidi hujus expressio est $A'B'\times\frac{AA'^2+AA'\times BB'+BB'^2}{3}\pi$. Fig. 32.

Theorema. Rectangulum quodcunque circa axem quemcunque extra rectangulum in plano ejus situm revolvatur: dico, solidum rotatione hac genitum esse in ratione composita ex magnitudine rectanguli & ex circumferentia a centro ejus rotatione hac genita.

Sit $ABCD$ rectangulum, cujus centrum S, quod revolvatur circa axem $A'D'$ extra illud in plano ipsius situm; ex S demittatur in hunc axem perpendiculum SS': dico, solidum rotatione rectanguli $ABCD$ circa axem $A'D'$ genitum esse in ratione composita ex rectangulo $ABCD$, & ex circumferentia, cujus radius est SS'; seu expressionem capacitatis hujus solidi esse $ABCD\times SS'\times 2\pi$. Fig. 35.

Sint a, b, c, d puncta media laterum AB, BC, CD, DA.
Ex omnibus punctis A, B, C, D, a, b, c, d, demittantur in axem $A'D'$ perpendiculares AA', BB', CC', DD', aa', bb', cc', dd'. Sit φ angulus, sub quo SS' inclinatur ad latera opposita rectanguli AB, CD.

Solidum rotatione rectanguli genitum est excessus, quo solidum rotatione spatii $A'ABCC'$ genitum superat solidum genitum rotatione spatii $A'ADCC'$;

seu fol. $ABCD$ = fol. $A'ABB'$ − fol. $D'DCC'$ + fol. $B'BCC'$ − fol. $A'ADD'$

$$= \tfrac{1}{3}\pi . A'B'\left\{\begin{array}{l} A'A^2+A'A\times BB'+BB'^2 \\ -(D'D^2+D'D\times CC'+CC'^2)\end{array}\right\}$$

$$+ \tfrac{1}{3}\pi . A'D'\left\{\begin{array}{l} B'B^2+B'B\times CC'+CC'^2 \\ -(A'A^2+A'A\times DD'+DD'^2)\end{array}\right\}$$

$= \tfrac{1}{3}\pi.$

$$=\tfrac{1}{3}\pi.A'B'\left\{(BB'^2-DD'^2)-(CC'^2-AA'^2)+\frac{(BB'+AA')^2-(BB'-AA')^2}{4}-\frac{(CC'+DD')^2-(CC'-DD')^2}{4}\right\}.$$

$$=\tfrac{1}{3}\pi.A'D'\left\{(BB'^2-DD'^2)+(CC'^2-AA'^2)+\frac{(BB'+CC')^2-(BB'-CC')^2}{4}-\frac{(AA'+DD')^2-(AA'-DD')^2}{4}\right\}.$$

$$=\tfrac{1}{3}\pi.A'B'\{2SS'(BB'-DD')-2SS'(CC'-AA')+SS'(BB'+AA'-CC'-DD')\}$$

$$+\tfrac{1}{3}\pi.A'D'\{2SS'(BB'-DD')+2SS'(CC'-AA')+SS'(BB'-AA'+CC'-DD')\}.$$

$$\begin{matrix}=\pi.A'B'.SS'(\ AA'+BB'-CC'-DD')\\ +\pi.A'D'.SS'(-AA'+BB'+CC'-DD')\end{matrix}=\begin{matrix}2\pi.A'B'.SS'(aa'-cc')\\ +2\pi.A'D'.SS'(bb'-dd')\end{matrix}$$

Atqui $\begin{matrix}(aa'-cc') = BC\text{ſin.}\varphi\\ (bb'-cc') = AB\text{coſ.}\varphi\end{matrix}$ Item $\begin{matrix}A'B' = AB\text{ſin.}\varphi\\ A'D' = BC\text{coſ.}\varphi\end{matrix}$

Itaque $\begin{matrix}2\pi.A'B'.SS'(aa'-cc')\\ +2\pi.A'D'.SS'(bb'-dd')\end{matrix} = 2\pi.SS'\times\begin{matrix}AB.BC\text{ſin.}^2\varphi\\ +AB\times BC\text{coſ.}^2\varphi\end{matrix} = 2\pi.SS'.AB.BC=$

$= ABCD.SS'.2\pi.$

Corollarium primum. Rectangula quotcunque circa eundem axem revolvantur: ſumma ſolidorum, quæ rotatione rectangulorum horum generantur, æqualis eſt prismati, cujus baſis eſt ſumma horum rectangulorum, & cujus altitudo eſt circumferentia percurſa a centro gravitatis communi totidem maſſarum in centris rectangulorum coadunatarum ſingulisque rectangulis proportionalium; quod punctum dicitur centrum gravitatis commune omnium horum rectangulorum.

Corollarium ſecundum. Sint quotcunque rectangula poſitione & magnitudine data, quæ circa axem quempiam revolvuntur. Solidum rotatione hac genitum proportionale eſt diſtantiæ centri gravitatis communis omnium rectangulorum ab axe rotationis.

Scholium. Modo haud multum abſimili demonſtratur: ſolidum a parallelogrammo obliquangulo circa axem quemcunque revoluto genitum æquale eſſe prismati, cujus baſis eſt ipſum parallelogrammum, & cujus altitudo æqualis eſt circumferentiæ a centro figuræ parallelogrammi rotatione hac genitæ. Et ſimilia inde nectuntur corollaria.

§. 128.

§. 128. Solidum, factum ex aliqua figura atque ex distantia centri gravitatis figuræ ab aliqua recta, vocatur *momentum figuræ* respectu hujus rectæ. Proinde momentum alicujus rectanguli respectu axis, circa quem rotatur, est ad solidum ab eodem rectangulo hac rotatione genitum in ratione constanti; nempe in ratione radii ad circumferentiam, seu ut $1 : 2\pi$. Et momentum quotlibet rectangulorum respectu axis, circa quem rotantur, est ad summam spatiorum solidorum ab iisdem rectangulis hac rotatione genitorum, in eadem ratione constanti.

Pariter rectangulum, factum ex aliqua linea atque ex distantia centri ejus gravitatis a recta quapiam, dicitur *momentum* prioris *lineæ* respectu posterioris; & rectangulum, factum ex summa quotcunque linearum & ex distantia communis earum centri gravitatis ab aliqua recta, vocatur momentum harum linearum respectu hujus rectæ. Proinde si rectæ quotlibet circa axem quempiam revolvantur, momentum harum rectarum respectu hujus axis est ad summam superficierum, quas eædem lineæ hac rotatione generant, in prædicta ratione constanti, $1 : 2\pi$.

His præmissis pergo ad superficiem, a linea quacunque circa axem aliquem rotata genitam, & ad solidum rotatione cujuscunque figuræ circa axem aliquem genitum.

§. 129. Brevitatis causa denotent $S.R$, $S.F$ solida a rectangulis figuræ alicui inscriptis aut circumscriptis, & a figura ipsa rotatione circa eundem axem simul genita; porro denotent $M.R$, $M.F$ momenta horum extensorum respectu hujus axis; $G.R$, $G.F$ distantias centrorum gravitatis horum extensorum ab eodem axe; & $A.R$, $A.F$ areas horum extensorum.

$$
\begin{aligned}
1^\circ.\ \text{Est}\quad M.F(=A.F\times G.F) &= \lim.M.R\\
&= \lim.(A.R\times G.R)\\
&= \lim.A.R\times\lim.G.R\\
&= \quad A.F\times\lim.G.R\\
\text{proinde}\quad G.F &= \qquad\lim.G.R.
\end{aligned}
$$

2°. Est

2°. Est $S.F = \lim. S.R$

$$= 2\pi . \lim. M.R$$
$$= 2\pi . \lim.(A.R \times G.R)$$
$$= 2\pi . \lim. A.R \times \lim. G.R$$
$$= 2\pi . A.F \times G.F$$
$$= 2\pi \times M.F.$$

Proinde *ſolidum rotatione figuræ F genitum æquale eſt ſolido facto ex hac figura atque ex circumferentia a centro gravitatis ejus rotatione hac genita; ſeu ſolidum a figura F genitum eſt ad momentum ejus reſpectu axis rotationis in ratione conſtanti circumferentiæ circuli ad radium; & ſolida rotatione ejusdem figuræ circa varios axes genita ſunt inter ſe uti diſtantiæ centri gravitatis hujus figuræ ab iisdem axibus.*

Eodem modo *ſuperficies rotatione alicujus lineæ circa axem quempiam genita æqualis eſt rectangulo, facto ſub hac linea & ſub circumferentia a centro gravitatis ejus rotatione hac genita; ſeu ſuperficies hæc eſt ad momentum lineæ genitricis reſpectu axis, circa quem rotatur, in ratione conſtanti circumferentiæ circuli ad radium.* (a)

§. 130.

(a) Theorema hoc, ſuppreſſa demonſtratione, Pappus jam ſub finem *Præfationis Lib. VII. Collect. math.* expoſuit his verbis: Ο *των τελειων αμφοισικων λογος συνηπται εκ τε των αμφοισματων και των επι τας αξονας ομοιως κατηγμενων ευθειων απο των εν αυτοις κεντροβαρικων σημειων.* Ο *δε των ατελων εκ τε των αμφοισματων και των περιφερειων, οσας εποιησε τα εν αυτοις κεντροβαρικα σημεια.* Quæ Halleius (*Apollonii De ſectione rationis & ſpatii Libri.* Oxon. 1706.) ita vertit: „Figuræ perfecto gyro genitæ rationem habent compoſitam ex ratione gyrantium & ex illa rectarum ſimiliter ad axes ductarum ab ipſarum gyrantium gravitatis centris. Ratio vero incompleto gyro genitarum fit ex ratione gyrantium & arcuum, quos deſcripſere earundem centra gravitatis.„ Commandinus (Pappi *mathemat. Collect.* Piſauri 1588.) verterat: „Perfectorum utrorumque ordinum proportio compoſita eſt ex proportione amphismatum & rectarum linearum ſimiliter ad axes ductarum a punctis, quæ in ipſis gravitatis centra ſunt. Imperfectorum autem proportio compoſita eſt ex proportione amphismatum & circumferentiarum a punctis, quæ in ipſis ſunt centra gravitatis, factarum.„ Keplerus (*Nova Stereometria doliorum.* Lincii 1615. P. I. Theor. XVIII. ſq.) omnem annulum, genitum rotatione cujuslibet figuræ ſymmetricæ circa axem diametro figuræ parallelum, ſive extra figuram ſitum, ſive perimetrum ejus contingentem, æqualem eſſe oſtendit cylindro, cujus altitudo æquet longitudinem circumferentiæ, quam centrum figuræ circumductæ deſcripſit,

baſis

§. 130. Hinc deducuntur methodi centrum gravitatis cujusvis figuræ determinandi. Nempe determinetur ſolidum rotatione figuræ hujus circa axem quemlibet genitum; quod transformetur in prisma, cujus baſis ſit ipſa figura rotata: altitudo hujus prismatis imminuta in ratione circumferentiæ ad radium erit diſtantia centri gravitatis hujus figuræ ab axe rotationis. Et proinde centrum gravitatis ipſum determinabitur, ſi determinentur ſolida rotatione figuræ hujus circa duos axes (ſibi invicem non parallelos) genita.

Viciſſim, centro gravitatis figuræ alicujus poſitione dato, determinantur ſolida rotatione figuræ hujus circa axem quemlibet poſitione datum genita.

Eadem applicantur ad ſuperficies rotatione alicujus lineæ circa duos axes genitas.

Quodſi autem figura ſymmetrica eſt, ſeu axem aliquem figuræ habet: quoniam centrum gravitatis ejus in hoc axe ſitum eſt, ut poſitio centri hujus determinetur, ſufficit momentum figuræ quærere, reſpectu unius rectæ, v. gr. reſpectu lineæ, quæ ſit axi huic ordinatim applicata.

Caput decimum quintum.

De ſolidis earumque ſuperficiebus in genere, et de curvis duplicis curvaturæ.

Solidorum rotundorum proprietates tum a figura genitrice, tum a poſitione axis, circa quem figura hæc rotatur, pendere, in tribus capitibus præcedentibus abunde fuit declaratum. Solida hæc præcipue mathematicos occuparunt. Dantur autem innumera alia a ſolidis rotundis diverſa; quorum potiores duas ſpecies

baſis vero eadem ſit cum ſectione annuli. Guldinus (*De centro gravitatis.* Lib. II. Viennæ 1640. Cap. VIII. Prop. II.) regulam, quam vocat, generalem compoſitionis poteſtatum rotundarum hanc tradidit, ſed per inductionem tantum comprobavit: „Quantitas rotunda in viam rotationis (lineam circularem, quam in rotatione deſcribit centrum gravitatis magnitudinis rotatæ) ducta producit poteſtatem rotundam uno gradu altiorem poteſtate ſive quantitate rotata.„ Demonſtrationes regulæ, quæ ad Guldinum tanquam inventorem referri conſuevit, varii deinde varias propoſuerunt. Conf. Montucla *Hiſt. des Math.* T. II. p. 19. ſqq.

species primum breviter pertractare, tum generatim de illis disquirere non alienum ab re erit.

§. 131. Sit figura quæcunque plana positione & magnitudine data. Recta quæpiam ita moveatur, ut sibi semper parallela maneat, & ut aliquod ejus punctum circa perimetrum figuræ hujus progrediatur. Singula rectæ hujus puncta describunt lineas perimetro figuræ datæ parallelas, eidemque similes & æquales: proinde solidi hoc modo geniti sectio, plano quocunque figuræ datæ parallelo facta, æqualis & similis est huic figuræ. Superficies a recta mobili genita dicitur *superficies cylindrica*; & recta mobilis dicitur ejus *latus*. Figura data, & sectio solidi, plano figuræ huic parallelo & per alterum lateris extremum transeunte facta, dicuntur *bases* solidi. Solidum ipsum, basibus & superficie cylindrica terminatum, dicitur *cylindrus* seu solidum cylindricum. Cylindrus est *rectus* aut *obliquus*, prouti latus est basi perpendiculare aut obliquum. Distantia basium cylindri dicitur ejus *altitudo*.

Solidum cylindricum limes est prismatum ipsi circumscriptorum aut inscriptorum. Sed capacitates horum prismatum sunt in ratione composita ex rationibus basium & altitudinum eorum; & bases solidi cylindrici sunt etiam limites basium horum prismatum: proinde etiam solida cylindrica sunt in ratione composita ex rationibus basium & altitudinum ipsorum. Ideo denotante S capacitatem solidi cylindrici, B basin ejus, & H altitudinem; erit $S = BH$.

Proinde quotiescunque area basis solidi cylindrici terminis finitis exprimi potest; capacitas quoque solidi cylindrici accurate determinatur.

Pariter superficies cylindricæ rectæ sunt in ratione composita ex rationibus perimetrorum basium, & altitudinum. Superficies autem cylindricæ obliquæ pendent a perimetro sectionis cylindri, plano ipsius lateribus perpendiculari factæ.

§. 132. Solida inter, quæ a cylindricis originem ducunt, ea contemplari sufficiat, quæ cylindrorum rectorum (inprimis) sectione basi obliqua generantur.

Fig. 36. Sit nempe AB recta quæcunque in plano basis cylindri recti acta; & sit $ANN'B$ segmentum basis recta hac AB abscissum. Cylindrus secetur plano per

AB

AB transeunte; ſitque *AMM'B* ſectio ſuperficiei cylindricæ plano hoc facta. Solidum, ſuperficie cylindrica *ANN'BM'M*, & ſegmentis *ANN'B*, *AMM'B* comprehenſum, vocatur *cono-cuneus* vel *ungula cylindrica*.

In plano baſis ducatur recta quævis *NP* ipſi *AB* perpendicularis; per *NP* agatur planum baſi perpendiculare; & ſit *MNP* ſectio ungulæ plano hoc facta. Trianguli rectanguli *MNP* angulus *MPN* æqualis eſt angulo inclinationis duorum planorum *ANN'B*, *AMM'B*, qui ſit φ; & proinde omnes ſectiones pari modo factæ dantur ſpecie; nempe eſt $NM = NP$ tang. φ, $N'M' = N'P'$ tang. φ.

Capacitas ungulæ cylindricæ limes eſt capacitatis ungularum prismaticarum, quæ oriuntur ex ſectionibus ſimul factis prismatum ſolido cylindrico inſcriptorum aut circumſcriptorum. Capacitate igitur ungulæ poſita $=S$, axis *AB* abſciſſa $AP = x$, & $NP = y$; fit $\frac{dS}{dx} = \frac{1}{2}yy$ tang. φ: proinde ungulæ capacitas ab integratione hujus formulæ pendet.

Exempla. Solidum cylindricum ſit cylindrus circularis rectus; & recta *AB* ſit diameter $2r$ baſis cylindri. Erit $yy = 2rx - xx$; $\frac{dS}{dx} = \frac{1}{2}(2rx - xx)$ tang. φ; $S = C + \frac{1}{2}(rxx - \frac{1}{3}x^3)$ tang. φ. Atqui $S = 0$, quando $x = 0$; proinde $S = \frac{1}{2}xx(r - \frac{1}{3}x)$ tang. φ. Sit $x = 2r$: erit $S = 2rr(\frac{1}{3}r)$ tang. $\varphi = \frac{2}{3}rr.r$ tang. φ. Cubatura ſolidi hujus accurate obtinetur, neque a quadratura circuli pendet. Par ratio eſt ungularum cylindricarum, quarum baſes ſunt ellipſes conicæ, & *AB* alteruter earum axis, v. gr. transverſus $2a$. Tum ſcilicet $yy = \frac{bb}{aa}(2ax - xx)$; unde $S = \frac{1}{2}\frac{bb}{aa}$ tang. $\varphi . xx(a - \frac{1}{3}x)$. Sit $x = 2a$; erit $S = 2bb$ tang. $\varphi . \frac{1}{3}a = \frac{2}{3}ab.b$ tang. φ.

Idem dicatur de ungulis ex ſectionibus cylindrorum parabolicorum & hyperbolicorum genitis.

Obſervatio. Ex formula $\frac{dS}{dx} = \frac{1}{2}yy$ tang. φ ſequitur: ungulas, ſectionibus ejusdem ſolidi per eandem in baſi rectam transeuntibus factas, inter ſe eſſe uti tangentes angulorum φ. Verum hæc ratio tantum ſubſiſtit, quamdiu tang. φ eſt poſſibilis, ſeu quamdiu planum *AMM'B* cylindri lateribus occurrit. Quando autem anguli φ tangens fit impoſſibilis, ſeu quando angulus φ fit 90°; ſigni $\frac{1}{0}$

seu ∞ introductione monemur: non amplius de ungulis agi posse; nec formulam pro ipsis traditam posse ad capacitatem solidi, plano basi normali a cylindro abscissi, determinandam applicari.

§. 133. Æquatio differentialis superficiei curvæ ungularum ex iisdem principiis deducitur. Sit nempe z arcus AN segmenti basis $NN'B$, & sit S superficies curva ungulæ; fit $\frac{dS}{dz} = MN = y\,\text{tang.}\,\varphi$: unde res ad calculum integralem reducitur.

Exemplum primum. Solidum cylindricum sit cylindrus circularis rectus, & sit $AB = 2r$ diameter basis; erit $\frac{dz}{dx} = \frac{r}{y}$: hinc $\frac{dS}{dx} = r\,\text{tang.}\,\varphi$; $S = rx\,\text{tang.}\,\varphi$. Sit $x = 2r$; erit $S = 2rr\,\text{tang.}\,\varphi = 2r \times r\,\text{tang.}\,\varphi$: proinde hoc casu etiam superficies curva ungulæ absolute habetur.

Scholium. Eadem, quæ §. præcedente, de impossibilitate applicationis harum formularum ad casum, quo $\varphi = 90°$, observentur.

Exemplum secundum. Solidum cylindricum sit cylindrus ellipticus, & sit AB axis alteruter ellipsis.

1°. Sit $AB = 2a$ axis transversus, $2b$ axis secundus, sitque $aa - bb = ee$; & abscissæ x axis AB sumantur a centro: erit $\frac{dS}{dx} = \frac{be}{aa}\text{tang.}\,\varphi \sqrt{\left(\frac{a^4}{ee} - xx\right)}$; cujus formulæ integratio non ab rectificatione ellipsis, sed a quadratura tantum circuli pendet. Nempe est $S = \frac{1}{2}\left(\frac{a^3}{e}\text{arc.\,sin.}\frac{ex}{aa} + \frac{ex}{a}\sqrt{\left(\frac{a^4}{ee} - xx\right)}\right)\text{tang.}\,\varphi$.

Sit $x = a$; fit $S = \frac{1}{2}\left(\frac{a^3}{e}\text{arc.\,sin.}\frac{e}{a} + ab\right)\text{tang.}\,\varphi$, & superficies integra $AMM'BN'N$ est $\left(\frac{a^3}{e}\text{arc.\,sin.}\frac{e}{a} + ab\right)\text{tang.}\,\varphi$.

Scholium. Arc. sin. $\frac{e}{a} = \frac{e}{b} - \frac{1}{3}\frac{e^3}{b^3} + \frac{1}{5}\frac{e^5}{b^5} - \ldots\ldots$ (§. 79.) hinc $\frac{a^3}{e}\text{arc.\,sin.}\frac{e}{a} = \frac{a^3}{b}\left(1 - \frac{1}{3}\frac{ee}{bb} + \frac{1}{5}\frac{e^4}{b^4} - \ldots\right)$: proinde casu, quo $e = 0$, seu quo basis est circulus, fit $S = (aa + ab)\,\text{tang.}\,\varphi = 2aa\,\text{tang.}\,\varphi$ (ut prius).

2°. Sit AB axis secundus $= 2b$: erit $\frac{dS}{dx} = \frac{ae}{bb}\text{tang.}\,\varphi\sqrt{\left(\frac{b^4}{ee} + xx\right)}$; cujus formulæ integratio non a rectificatione ellipsis, sed ab quadratura hyperbolæ seu a logarithmis pendet. Nempe est

$S =$

$$S = \tfrac{1}{2}\frac{ae}{bb}\text{tang.}\varphi\left(x\sqrt{\left(\frac{b^4}{ee}+xx\right)} + \frac{b^4}{ee}\log.\frac{\sqrt{\left(\frac{b^4}{ee}+xx\right)}+x}{\frac{bb}{e}}\right].$$

Sit $x = b$; superficies integra est $\frac{ae}{bb}\text{tang.}\varphi\left(\frac{bba}{e}+\frac{b^4}{ee}\log.\frac{\frac{bba}{e}+b}{\frac{bb}{e}}\right] =$

$$= \left(aa+\frac{bba}{e}\log.\frac{a+e}{b}\right)\text{tang.}\varphi = \left(aa+\frac{bba}{e}\log.\sqrt{\frac{a+e}{a-e}}\right)\text{tang.}\varphi =$$

$$= \left(aa+\frac{bba}{e}\log.\sqrt{\frac{1+\frac{e}{a}}{1-\frac{e}{a}}}\right]\text{tang.}\varphi.$$

Scholium. $\text{Log.}\sqrt{\frac{1+\frac{e}{a}}{1-\frac{e}{a}}} = \frac{e}{a} + \frac{1}{3}\frac{e^3}{a^3} + \frac{1}{5}\frac{e^5}{a^5} + \ldots$ (§. 61.);

hinc $\frac{bba}{e}\log.\sqrt{\frac{a+e}{a-e}} = bb\left(1+\frac{1}{3}\frac{ee}{aa}+\frac{1}{5}\frac{e^4}{a^4}+\ldots\right)$

Unde, facto $e = 0$, superficies proposita est $2aa\,\text{tang.}\varphi$ (ut prius).

Eodem modo ostenditur: superficiem ungularum parabolicarum aut absolute haberi, aut ad logarithmos reduci; & superficies ungularum hyperbolicarum (sectione per alterutrum axem genitarum) a rectificatione hyperbolæ non pendere.

§. 134. Solida conica aliud constituunt solidorum genus, quorum consideratio frequenter occurrit.

Sit figura quæcunque plana positione & magnitudine data. Sit etiam punctum quodvis extra planum figuræ positione datum; & recta per hoc punctum ducta circa perimetrum figuræ rotetur. Recta hæc ita revoluta gignit *superficiem conicam*, cujus *vertex* est punctum datum. Recta ex vertice ad punctum aliquod perimetri basis ducta coincidit cum linea genitrice ea positione, qua per punctum hoc perimetri basis transit; ideoque recta hæc tota in superficie conica jacet, & dicitur *latus* superficiei conicæ. Solidum, figura data & super-

ficie conica terminatum, dicitur *conus*, seu solidum conicum; & figura ipsa ejus *basis*. Recta ex vertice in planum basis perpendiculariter demissa vocatur *altitudo* coni. Si basis habet centrum figuræ, & altitudo coni plano basis in hoc centro occurrit; conus vocatur *rectus*: si secus; conus est *obliquus*.

Si conus plano secetur basi parallelo: figura sectionis basi similis est; & dimensiones homologæ sectionis ac basis sunt inter se uti distantiæ plani secantis & basis a vertice coni.

Sint duo coni æquealti; bases eorum in eodem plano jaceant, ita ut ipsi sint versus easdem hujus plani partes siti. Altitudo communis duorum conorum dividatur in partes quotcunque æquales: & utrique inscribantur ac circumscribantur prismata æquealta ad normam § primi exempli tertii; quorum bases sint figuræ similes sectionibus simul inscriptæ aut circumscriptæ. Duo prismata in utroque cono sibi invicem respondentia (seu a vertice coni æqualiter distantia) sunt in ratione constanti basium suarum, seu in ratione figurarum basibus conorum simul inscriptarum aut circumscriptarum; proinde & summæ horum prismatum sunt in eadem ratione constanti. Quare & limites harum summarum, nempe duo coni, sunt uti limites figurarum basibus inscriptarum & circumscriptarum, seu ut ipsæ bases. Ratio igitur duorum conorum æquealtorum æqualis est rationi basium eorum.

Atqui ratio basium æqualis est rationi cylindrorum æquealtorum basibus illis insistentium. Proinde coni æquealti inter se sunt uti cylindri æquealti iisdem basibus insistentes.

Sed conus circularis est pars tertia cylindri circularis æquealti super eadem basi. Proinde conus quilibet pars est tertia cylindri æquealti eidemque basi insistentis.

Determinatio igitur capacitatis cujusvis solidi conici reducitur ad determinationem basis.

§. 135. Secus autem res habet quod ad superficiem solidorum conicorum. Notum est, superficiem coni recti circularis a circumferentia circuli pendere. Determinatio autem superficiei coni circularis obliqui (cujus investigationi

tioni tot tantique mathematici incubuerunt) nequidem ad rectificationem sectionum conicarum terminis finitis potest reduci.

Omissis casibus particularibus, quibus compendia quædam possunt indagationi superficierum conicarum adhiberi, methodum omnium universalissimam disquisitionem hanc instituendi strictim exponam.

Sit *AMB* basis coni; *S* vertex ejus; *SP* ipsius altitudo, quæ plano basis in *P* occurrit. Plano basis circumscribatur figura quæcunque rectilinea; per singula figuræ basi circumscriptæ latera & per verticem coni agantur plana; quæ constituent superficiem pyramidis cono circumscriptæ. Sit *M* punctum contactus unius ex lateribus; in quod ex vertice *S* demittatur perpendiculum *ST*. Sit *c* latus figuræ basin in *M* contingens. Ducantur *SM*, *PM* rectæ. Superficies trianguli, cujus hoc latus est basis, & cujus vertex *S*, est $\frac{1}{2}c \times ST = \frac{1}{2}c \times SM \sin. SMT$. In angulo solido *M*, cujus acies sunt *MP*, *MS*, *MT*, duæ facies *SMP*, *PMT* sibi invicem sunt perpendiculares: proinde $\cos. SMT = \cos. PMT \cos. SMP$; & $\sin. SMT = \sqrt{(1 - \cos.^2 PMT \cos.^2 SMP)}$. Superficies conica ponatur $= S$, & arcus $AM = z$: erit
Fig. 37.

$$\frac{dS}{dz} = \tfrac{1}{2} SM \sqrt{(1 - \cos.^2 PMT \cos.^2 SMP)}$$
$$= \tfrac{1}{2}\sqrt{(SM^2 - PM^2 \cos.^2 PMT)}$$
$$= \tfrac{1}{2}\sqrt{(SP^2 + PM^2 \sin.^2 PMT)}.$$

Atqui ex æquatione data basis datur angulus *PMT* per *PM* & per angulum *APM* v. gr.; daturque etiam exponens differentialis $\frac{dz}{d.PM}$: proinde res ad calculum integralem semper reducitur.

Exemplum. Basis sit circulus, cujus centrum *C*, & radius *CM*; & punctum *P* jaceat intra circulum. Erit $\sin. PMT = \cos. CMP$; $PM \sin. PMT = PM \cos. CMP = CM - CP \cos. MCP$. Sit $CM = r$, $SP = h$, $CP = a$, $MCP = z$: erit

$$\frac{dS}{dz} = \tfrac{1}{2} r \sqrt{(hh + rr - 2ar \cos. z + aa \cos.^2 z)}.$$ Sit $hh + rr = bb$; $2r = c$; fit

$$\frac{dS}{dz} = \tfrac{1}{2} r \sqrt{(bb - a \cos. z (c - a \cos. z))} = \tfrac{1}{2} br \sqrt{\left(1 - \frac{a}{b}\cos. z \left(\frac{c - a \cos. z}{b}\right)\right)}$$

$= \frac{1}{2} br$

$$= \tfrac{1}{2}br \Big\{ 1 - \frac{\frac{1}{2}}{1} \frac{a\cos.z(c - a\cos.z)}{bb}$$

$$- \frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2} \frac{aa\cos.^2z(c - a\cos.z)^2}{b^4}$$

$$- \frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3} \frac{a^3\cos.^3z(c - a\cos.z)^3}{b^6}$$

$$- \frac{\frac{1}{2}}{1}\cdots\frac{\frac{5}{2}}{4} \frac{a^4\cos.^4z(c - a\cos.z)^4}{b^8}$$

$$- \frac{\frac{1}{2}}{1}\cdots\frac{\frac{7}{2}}{5} \frac{a^5\cos.^5z(c - a\cos.z)^5}{b^{10}}$$

$$- \;\text{-}\;\text{-}\;\text{-}\;\text{-}\;\text{-}\;\Big\}$$

Unde, ſumtis integralibus arcui $z = 180^\circ$ reſpondentibus, ſuperficies integra coni obliqui, fit

$$S = br\pi\Big(1 + \frac{\frac{1}{2}}{1}\cdot\frac{1}{2} \frac{aa}{bb}$$

$$- \frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2} \frac{\frac{1}{2}aacc + \frac{\frac{4}{1}\cdot\frac{3}{2}}{2^4}a^4}{b^4}$$

$$+ \frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3} \frac{3\cdot\frac{\frac{4}{1}\cdot\frac{3}{2}}{2^4}a^4cc + \frac{\frac{6}{1}\cdot\frac{5}{2}\cdot\frac{4}{3}}{2^6}a^6}{b^6}$$

$$- \frac{\frac{1}{2}}{1}\cdots\frac{\frac{5}{2}}{4} \frac{\frac{\frac{4}{1}\cdot\frac{3}{2}}{2^4}a^4c^4 + \frac{4}{1}\cdot\frac{3}{2}\,\frac{\frac{6}{1}\cdot\frac{5}{2}\cdot\frac{4}{3}}{2^6}a^6c^2 + \frac{\frac{8}{1}\cdots\frac{5}{4}}{2^8}a^8}{b^8}$$

$$+ \frac{\frac{1}{2}}{1}\cdots\frac{\frac{7}{2}}{5} \frac{5\cdot\frac{\frac{6}{1}\cdots\frac{4}{3}}{2^6}a^6c^4 + \frac{5}{1}\cdot\frac{4}{2}\cdot\frac{\frac{8}{1}\cdots\frac{5}{4}}{2^8}a^8c^2 + \frac{\frac{10}{1}\cdots\frac{6}{5}}{2^{10}}a^{10}}{b^{10}}$$

$$- \;\text{-}\;\text{-}\;\text{-}\;\text{-}\;\text{-}\;\text{-}\;\text{-}\;\text{-}$$

$$+ \;\text{-}\;\text{-}\;\text{-}\;\text{-}\;\text{-}\;\text{-}\;\text{-}\;\text{-}\Big)$$

Series hæc eo promtius convergit, quo b ſeu $\sqrt{(hh + rr)}$ major eſt reſpectu a & c, ſeu a & $2r$. Ea nititur hoc principio: quod, ſi fuerit $\frac{dZ}{dz} = \cos.^{2m}z$, eſt

$$Z = \frac{\frac{2m}{1}\cdot\frac{2m-1}{2}\cdots\frac{m+1}{m}}{2^{2m}}\pi,$$ integrali ſumto pro $z = 180^\circ$.

Eodem

Eodem modo ſeries obtinetur admodum regularis pro $z = 90^\circ$, ex integratione formularum $\frac{dZ}{dz} = \text{coſ.}^{2m+1} z$, facto $z = 90^\circ$.

His (introductionis loco) de ſolidis cylindricis & conicis breviter præmiſſis, pergo ad conſiderationem ſolidorum magis univerſalem.

§. 136. Quemadmodum punctorum in eodem plano jacentium ſitus mutui omnium frequentiſſime determinantur, puncta hæc ad duas rectas ſibi invicem normales referendo per rectas ad eas perpendiculariter actas; & quemadmodum curvarum in eodem plano jacentium ſymptomata felici ſucceſſu determinantur per relationem mutuam perpendiculorum, ex ſingulis curvarum harum punctis in duas rectas ſibi invicem normales demiſſorum: ita etiam ſitus mutui punctorum in ſpatio ſitorum ſæpiſſime determinantur, puncta hæc ad tria plana ſibi invicem normalia referendo per rectas ad hæc plana perpendiculariter actas; pariterque tam curvarum in eodem plano non jacentium, quam ſuperficierum curvarum, & ſolidorum ſuperficiebus his terminatorum ſymptomata frequentiſſime determinantur per relationem mutuam rectarum, ex ſingulis curvarum harum aut ſuperficierum punctis in tria plana poſitione data & ſibi invicem normalia perpendiculariter demiſſarum.

Sint $\begin{matrix} SX \\ SY \\ SZ \end{matrix}$ ſectiones communes planorum $\begin{matrix} YSX & XSZ \\ XSY, & YSZ \\ XSZ & ZSY \end{matrix}$ ſibi invicem normalium & in puncto S ſibi mutuo occurrentium. Sit M punctum aliquod in ſpatio, ex quo agatur recta SP plano XSZ perpendicularis. Tum ex puncto P agatur recta PQ rectæ SX perpendicularis. Rectæ PQ, SQ reſpective æquales ſunt rectis ex eodem puncto M in plana YSX, YSZ perpendiculariter actis. Proinde puncti M ſitus in ſpatio determinatur tribus rectis Fig. 38.

MP, PQ, SQ

in plana ZSX, YSX, YSZ perpendiculariter actis, quæ vocentur

y, z, x.

Hinc v. gr. determinantur tam diſtantia SM puncti M a vertice S, quam inclinationes rectæ hujus SM ad tria plana propoſita: fit enim

$$SM^2 = MP^2 + PS^2 = MP^2 + PQ^2 + SQ^2 = yy + zz + xx.$$

$$\text{tang.}\,MSP = \frac{MP}{SP} = \frac{y}{\sqrt{(zz+xx)}}$$

$$\text{fin.}\,MSP = \frac{MP}{SM} = \frac{y}{\sqrt{(yy+zz+xx)}}$$

$$\text{cof.}\,MSP = \frac{SP}{SM} = \sqrt{\left(\frac{zz+xx}{yy+zz+xx}\right)}.$$

§. 137. Tria plana, ZSX, YSX, YSZ nonnifi majoris facilitatis gratia fumuntur fibi invicem perpendicularia; quæcunque enim de hoc fitu illorum planorum dicuntur, facile poffunt ad alium quemcunque fitum transferri.

Fig. 38. 2°. 1°. Per S agatur quodvis aliud planum R, quod plano ZSX in recta SV occurrat fub angulo dato α; & angulus XSV vocetur β. Ex P puncto agatur PR ipfi SV perpendicularis; & planum per rectas MP, PR ductum occurrat plano propofito in Rr recta: in quam ex puncto M agatur MT perpendicularis, quæ proinde erit plano propofito TRV perpendicularis; & angulus PRr eft angulus α, fub quo rRV planum inclinatur ad planum ZSX.

Erit (§. 125.)
$$\begin{aligned} PR &= PQ\,\text{cof.}\beta + SQ\,\text{fin.}\beta \\ SR &= SQ\,\text{cof.}\beta - PQ\,\text{fin.}\beta \\ MT(&= PR\,\text{fin.}\alpha - MP\,\text{cof.}\alpha) \\ &= PQ\,\text{cof.}\beta\,\text{fin.}\alpha + SQ\,\text{fin.}\beta\,\text{fin.}\alpha - MP\,\text{cof.}\alpha \\ RT(&= PR\,\text{cof.}\alpha + MP\,\text{fin.}\alpha) \\ &= PQ\,\text{cof.}\beta\,\text{cof.}\alpha + SQ\,\text{fin.}\beta\,\text{cof.}\alpha + MP\,\text{fin.}\alpha. \end{aligned}$$

Proinde tres rectæ SR, RT, MT, quibus puncti M fitus refpectu plani R determinatur, habentur per angulos α, β, & per rectas MP, PQ, SQ expreffæ.

2°. Planum propofitum R' non tranfeat per S; fed rectæ SX occurrat in S', & plano ZSX in $S'V'$. Per S agatur recta SV ipfi $S'V'$ parallela, & per SV agatur planum R plano R' parallelum. Tum iisdem, quæ in cafu præcedente factis, occurrat PR ipfi $S'V'$ in R', & MT occurrat in T' plano R'; ac proinde fit $R'T'$ ipfi RT parallela. Erit

$RR'=SS'\text{fin.}\beta$, unde $PR'=PQ\,\text{cof.}\beta+SQ\,\text{fin.}\beta+SS'\text{fin.}\beta$

$S'R'=SR+SS'\text{cof.}\beta$, feu $S'R'=SQ\,\text{cof.}\beta-PQ\,\text{fin.}\beta+SS'\text{cof.}\beta$

$R'T'=RT+R'R\,\text{cof.}\alpha$, feu $R'T'=PQ\text{cof.}\beta\text{cof.}\alpha+SQ\text{fin.}\beta\text{cof.}\alpha+MP\text{fin.}\alpha+SS'\text{fin.}\beta\text{cof.}\alpha$

$MT'=MT+R'R\text{fin.}\alpha$, feu $MT'=PQ\text{cof.}\beta\text{fin.}\alpha+SQ\text{fin.}\beta\text{fin.}\alpha-MP\text{cof.}\alpha+SS'\text{fin.}\beta\text{fin.}\alpha.$

Vicif-

Viciſſim rectæ MP, PQ, SQ exprimi poſſunt per rectas $S'R'$, $R'T'$, MT'; ſimul cum recta SS' & angulis α & β. Fit enim

$$\begin{aligned} SQ &= S'R'\cos.\beta + R'T'\sin.\beta\cos.\alpha + MT'\sin.\beta\sin.\alpha - SS' \\ PQ &= -S'R'\sin.\beta + R'T'\cos.\beta\cos.\alpha + MT'\cos.\beta\sin.\alpha \\ MP &= R'T'\sin.\alpha - MT'\cos.\alpha. \end{aligned}$$

Corollarium primum. Sit $MT' = 0$; ſeu planum R' per M transeat. Erit

$$\begin{aligned} MP &= MR'\sin.\alpha \\ PQ &= -S'R'\sin.\beta + R'M\cos.\beta\cos.\alpha \\ SQ &= S'R'\cos.\beta + R'M\sin.\beta\cos.\alpha - SS'. \end{aligned}$$

Corollarium ſecundum. Obſervandum eſt: rectas $S'R'$, $R'T'$, MT' per rectas MP, PQ, SQ ita determinari, ut in expreſſionibus ipſarum rectæ MP, PQ, SQ nulla alia operatione invicem connectantur præter additionem aut ſubſtractionem, ſeu priorum rectarum expreſſiones eſſe primi tantum gradus functiones poſteriorum.

§. 138. Sit M' aliud punctum, ex quo agatur $M'P'$ recta plano ZSX perpendicularis; & ex puncto P' agatur $P'Q'$ recta ipſi SX perpendicularis. Tum agatur etiam PP' recta: ac ſint Pq' ipſi $P'Q'$, Mm' ipſi $M'P'$ perpendiculares. Sint $M'P'$, $P'Q'$, SQ', Fig. 38. 1°.

y', z', x' reſpective.

$$\begin{aligned} MM'^2 = Mm'^2 + M'm'^2 = PP'^2 + M'm'^2 &= Pq'^2 + P'q'^2 + M'm'^2 \\ &= (x'-x)^2 + (z'-z)^2 + (y'-y)^2 \end{aligned}$$

unde $MM' = \sqrt{((x'-x)^2 + (z'-z)^2 + (y'-y)^2)}$

Angulus $M'Mm'$ ille eſt, quo MM' recta ad planum ZSX inclinatur; atque

$$\text{tang.}M'Mm = \frac{M'm'}{PP'} = \frac{y'-y}{\sqrt{((x'-x)^2 + (z'-z)^2)}}$$

$$\text{ſin.}M'Mm = \frac{y'-y}{\sqrt{((x'-x)^2 + (z'-z)^2 + (y'-y)^2)}}$$

$$\text{coſ.}M'Mm = \sqrt{\left(\frac{(x'-x)^2 + (z'-z)^2}{(y'-y)^2 + (x'-x)^2 + (z'-z)^2}\right)}.$$

Obſervatio. Recta PP' projectio eſt orthographica rectæ MM' in planum

 ZSX;

ZSX; & ratio rectæ MM' ad projectionem ſuam PP' æqualis eſt rationi radii ad coſinum anguli, quo recta MM' ad planum ZSX inclinatur.

Univerſim notum eſt: figuræ planæ cujuslibet in planum aliquod orthographice projectæ rationem ad projectionem ſuam æqualem eſſe rationi radii ad coſinum anguli, quo figuræ hujus planum ad planum projectionis inclinatur.

§. 139. Porro ſit & tertium punctum M''. Sit pariter $M''P''$ plano ZSX, & $P''Q''$ rectæ SX perpendicularis. Agantur MM'', $M'M''$ rectæ; & ſint Pq'', Mm'' rectis $P''Q''$, $M''P''$ perpendiculares. Sint $\frac{M''P''}{y''}$, $\frac{P''Q''}{z''}$, $\frac{SQ''}{x''}$ reſpective.

Erit $MM'' = \sqrt{((x''-x)^2+(y''-y)^2+(z''-z)^2)}$

$M'M'' = \sqrt{((x''-x')^2+(y''-y')^2+(z''-z')^2)}$.

Hinc determinatur area trianguli $MM'M''$ per formulam notam

$$MM'M''=\sqrt{\left(\frac{MM'+MM''+M'M''}{2}\cdot\frac{MM'+M'M''-M'M''}{2}\cdot\frac{MM'-MM''+M'M''}{2}\cdot\frac{-MM'+MM''+M'M''}{2}\right)}$$

$$=\tfrac{1}{4}\sqrt{\left\{\begin{array}{l} 2MM'^2.MM''^2-MM'^4\\ +2MM'^2.M'M''^2-MM^4\\ +2MM''^2.M'M''^2-M'M''^4\end{array}\right\}}.$$

Item $PP' = \sqrt{((x'-x)^2 + (z'-z)^2)}$

$PP'' = \sqrt{((x''-x)^2 + (z''-z)^2)}$

$P'P''= \sqrt{((x''-x')^2 + (z''-z')^2)}$: unde pariter determinatur area trianguli $PP'P''$.

Hinc etiam determinatur capacitas trunci priſmatici triangularis $MM'M''P''P'P$, cujus expreſſio eſt $PP'P''\times\frac{MP+M'P'+M''P''}{3}$.

Hinc quoque infertur angulus, ſub quo planum trianguli $MM'M''$ ad planum ZSX inclinatur; coſinus enim hujus anguli eſt $\frac{PP'P''}{MM'M''}$. Ad ſequentia autem præſtat, angulum hunc paulo aliter inveſtigare, ut ſequitur.

§. 140. Rectæ $M'M$, $M''M$ productæ, rectis $P'P$, $P''P$ in punctis T', T'' occurrant. Rectæ PT', PT'' dantur magnitudine; ſit nempe

$$PT' = MP \times \frac{PP'}{M'm'} = y \times \frac{\sqrt{((x'-x)^2+(z'-z)^2)}}{y'-y}$$

$$PT'' = MP \times \frac{PP''}{M''m''} = y \times \frac{\sqrt{((x''-x)^2+(z''-z)^2)}}{y''-y}.$$

In

In triangulo $T'PT''$ dantur angulus $T'PT''(=P'PP'')$, & crura hujus anguli PT', PT''; proinde datur recta $T'T'' = \sqrt{(T'P^2 - 2T'P.PT''\text{cos}.T'PT'' + PT''^2)}$. Sit PT ipsi $T'T''$ perpendicularis; erit $PT = \frac{T'P.PT''\text{sin}.T'PT''}{T'T''}$
$= \frac{T'P.PT''\text{sin}.T'PT''}{\sqrt{(T'P^2 - 2T'P.PT''\text{cos}.T'PT'' + PT''^2)}}$.

Hinc tang.$MTP(=\frac{MP}{TP}) = \frac{MP\sqrt{(T'P^2 - 2T'P.PT''\text{cos}.T'PT'' + PT''^2)}}{T'P.PT''\text{sin}.T'PT''}$.

Exemplum. Angulus $P'PP''$ sit rectus: fit tang.$MTP =$ Fig. 38. 3°.
$\sqrt{\left(\frac{(y''-y)^2}{(x''-x)^2 + (z''-z)^2} + \frac{(y'-y)^2}{(x'-x)^2 + (z'-z)^2}\right)}$. Sint autem simul plana $MPP'M'$, $MPP''M''$, planis YSX, YSZ respective parallela; & proinde $x'' = x$, $z' = z$: erit tang.$MTP = \sqrt{\left(\left(\frac{y''-y}{z''-z}\right)^2 + \left(\frac{y'-y}{x'-x}\right)^2\right)}$. Porro angulus TPQ ille est, sub quo SX, $T'T''$ rectæ ad se invicem inclinantur; angulus hic sit æqualis angulo T', cujus tangens est $\frac{PT''}{PT'} = \frac{z''-z}{y''-y} : \frac{x'-x}{y'-y}$.

§. 141. Hisce præmissis de punctis in spatio, quorum situs mutui nulla relatione determinata secum invicem connectuntur: pergo ad præcipuum disquisitionis hujus caput, ad casum nempe, quo puncta relatione aliqua data inter se connectuntur; ita ut locus punctorum M sit superficies aut curva aliqua, relatione quadam inter perpendicula MP, PQ, SQ determinata. Quem ut, quantum potero, luculenter explicem, ab exemplis omnium simplicissimis ordiar.

Exemplum primum. Detur summa quadratorum perpendiculorum MP, PQ, Fig. 38. 1°. 3°.
SQ; summa hæc est MS^2 (§. 136.): proinde recta SM datur magnitudine, & puncta M jacent in superficie sphærica, cujus centrum est S, & cujus radius magnitudine datur.

Quoniam summa $MP^2 + PQ^2 + SQ^2$ datur magnitudine: si recta MP ejusdem manet magnitudinis, & proinde puncta M in plano positione dato & plano ZSX parallelo jacent; summa $PQ^2 + SQ^2$ seu SP^2 pariter datur magnitudine. Agatur MN rectæ SP parallela, quæ ipsi SY in N occurrat. Recta $NM = SP$

 etiam

etiam datur magnitudine; & proinde puncta M sita sunt in circumferentia circuli, cujus centrum N & radius $NM = SP$.

Hinc discimus: omnes solidi propositi sectiones planis ZSX, YSX, YSZ parallelis factas esse circulares, & centra harum sectionum jacere in recta perpendiculari plano dato, cui plana acta sunt parallela; ideoque solidum propositum est solidum rotationis circa quamlibet rectarum SX, SY, SZ.

Fig. 38. 2°. Secetur idem solidum alio quocunque plano per rectam $S'V'$ transeunte. Sit angulus, sub quo hoc planum ad planum ZSX inclinatur $= \alpha$, & angulus $V'S'X = \beta$; sitque M punctum aliquod perimetri sectionis.

Quoniam (§. 137.)

$$MP = MR' \text{fin.}\alpha$$
$$PQ = MR' \text{cof.}\alpha \,\text{cof.}\beta - S'R' \text{fin.}\beta$$
$$SQ = MR' \text{cof.}\alpha \,\text{fin.}\beta + S'R' \text{cof.}\beta - S'S$$

$$PQ^2 + SQ^2 = MR'^2 \text{cof.}^2\alpha - 2SS' \times MR' \text{cof.}\alpha \,\text{fin.}\beta + S'R'^2 - 2SS' \times S'R' \text{cof.}\beta + S'S^2$$

$$\& \ MP^2 + PQ^2 + SQ^2 = MR'^2 - 2SS' \times MR' \text{cof.}\alpha \,\text{fin.}\beta + S'R'^2 - 2SS' \times S'R' \text{cof.}\beta + S'S^2$$
$$= (MR' - S'S \,\text{cof.}\alpha \,\text{fin.}\beta)^2 + (S'R' - SS' \text{cof.}\beta)^2 + SS'^2 \text{fin.}^2\alpha \,\text{fin.}^2\beta,$$

quæ est æquatio circumferentiæ circuli; proinde sectio solidi plano quolibet facta est circulus (uti notum).

Scholium. Ex formula $MP^2 + PQ^2 + SQ^2 - SS'^2 \text{fin.}^2\alpha \,\text{fin.}^2\beta = \begin{matrix} (MR' - S'S \,\text{cof.}\alpha \,\text{fin.}\beta)^2 \\ + (S'R' - S'S \,\text{cof.}\beta)^2 \end{matrix}$, datis duabus trium quantitatum SS', α, β, tertia sic determinatur, ut planum ductum tangat (si fieri possit) solidum propositum; facto nempe $SS' \text{fin.}\alpha \,\text{fin.}\beta = \sqrt{(MP^2 + PQ^2 + SQ^2)}$: sed de hoc situ planorum actorum seorsim agere præstat.

Exemplum secundum. Summa quadratorum rectarum PQ, SQ habeat rationem datam ad quadratum reliquæ MP.

Fig. 38. 1°. & 3°. I. Magnitudine rectæ MP manente eadem; summa quadratorum rectarum PQ, SQ, seu quadratum rectæ SP, pariter est datæ magnitudinis. Proinde sectiones solidi propositi, planis plano ZSX parallelis factæ, sunt circulares. Quoniam ratio $MP : SP$ seu $SN : NM$ datur: in triangulo MSN rectangulo ad N angulus MSN datur magnitudine; ideoque SM est latus coni recti, cujus axis est SY. Sit φ angulus datus YSM.

II. Recta

II. Recta SQ eadem manente, quadratum rectæ MP eſt ad quadratum rectæ PQ ſpatio data auctum, in ratione data; proinde omnes ſectiones planis plano ZSY parallelis factæ ſunt hyperbolæ ſimiles. Ejuſdemque indolis ſunt ſectiones planis plano YSX parallelis factæ.

III. Solidum propoſitum plano ſecetur quocunque, quod plano ZSX in $S'V'$ recta occurrat ſub angulo α, & ſit angulus $XS'V'=\beta$. Fig. 38. 2°.

$$\text{Eſt } PQ^2+SQ^2=SR^2 = MR'^2\text{coſ.}^2\alpha+S'R'^2 \begin{matrix}-2SS'\times MR'\text{coſ.}\alpha\,\text{ſin.}\beta \\ -2SS'\times S'R'\text{coſ.}\beta\end{matrix} +SS'^2$$

$$\text{et } MP^2 = MR'^2\text{ſin.}^2\alpha;$$

$$\text{proinde } MR'^2\text{ſin.}^2\alpha : MR'^2\text{coſ.}^2\alpha+S'R'^2-2SS'\times\begin{matrix}MR'\text{coſ.}\alpha\,\text{ſin.}\beta \\ +S'R'\text{coſ.}\beta\end{matrix}+SS'^2= 1:\text{tang.}^2\varphi:$$

$$\text{unde } MR'^2\text{coſ.}^2\alpha:(SR'-SS'\text{coſ.}\beta)^2+(MR'\text{coſ.}\alpha-SS'\text{ſin.}\beta)^2 =\text{cot.}^2\alpha:\text{tang.}^2\varphi.$$

1°. Sit $\text{cot.}\alpha=\text{tang.}\varphi$: erit $(SR'-SS'\text{coſ.}\beta)^2=MR'^2\text{coſ.}^2\alpha-(MR'\text{coſ.}\alpha-SS'\text{ſin.}\beta)^2$ $=(2MR'\text{coſ.}\alpha-SS'\text{ſin.}\beta)\times SS'\text{ſin.}\beta$; proinde ſectio ſolidi eſt parabolica.

2°. Sit $\text{cot.}\alpha \gtrless \text{tang.}\varphi$;

$$(SR'-SS'\text{coſ.}\beta)^2=MR'^2\text{ſin.}^2\alpha\,\text{tang.}^2\varphi-(MR'\text{coſ.}\alpha-SS'\text{ſin.}\beta)^2$$

$$=(MR'(\text{ſin.}\alpha\,\text{tang.}\varphi+\text{coſ.}\alpha)-SS'\text{ſin.}\beta)(SS'\text{ſin.}\beta-MR'(\text{coſ.}\alpha-\text{ſin.}\alpha\,\text{tang.}\varphi))$$

$$=\left(MR'\frac{\text{coſ.}\varphi-\alpha}{\text{coſ.}\varphi}-SS'\text{ſin.}\beta\right)\left(SS'\text{ſin.}\beta-MR'\frac{\text{coſ.}\varphi+\alpha}{\text{coſ.}\varphi}\right)$$

$$=\frac{\text{coſ.}\varphi-\alpha\,\text{coſ.}\varphi+\alpha}{\text{coſ.}^2\varphi}\left(SS'^2\frac{\text{ſin.}^2\alpha\,\text{ſin.}^2\beta\,\text{coſ.}^2\varphi}{\text{coſ.}\varphi-\alpha\,\text{coſ.}\varphi+\alpha}-\left(MR'-SS'\frac{\text{ſin.}\beta\,\text{coſ.}\alpha\,\text{coſ.}^2\varphi}{\text{coſ.}\varphi-\alpha\,\text{coſ.}\varphi+\alpha}\right)^2\right).$$

Quare ſectio eſt ellipſis, ſi fuerit $\text{coſ.}\varphi-\alpha\,\text{coſ.}\varphi+\alpha$ quantitas poſitiva, ſeu $\varphi+\alpha<180°$; hyperbola autem, ſi fuerit $\varphi+\alpha>180°$.

Et quoniam ſpecies ſectionis pendet ab coëfficiente $\text{coſ.}\varphi-\alpha\,\text{coſ.}\varphi+\alpha$, omnes ſectiones planis ſibi invicem parallelis factæ ſunt inter ſe ſimiles.

Exemplum tertium. Summa MP^2+PQ^2 ſemper ſit rectangulo ſub recta NS & ſub recta aliqua data $2p$ æqualis. Fig. 38. 1°. & 3°.

Igitur $SP^2=MN^2=2p\times SN$; proinde ſolidum propoſitum gignitur rotatione parabolæ, cujus vertex S, parameter $2p$, & axis SY.

Secetur ſolidum plano ipſi YSZ parallelo: recta SQ manente eadem, erit $PQ^2=2p\times MP-SQ^2$; proinde ſectio eſt parabolica: idemque valet de ſectionibus plano YSX parallelis.

Sece-

Secetur autem solidum plano per $S'V'$ transeunte. Erit

Fig. 38. 2°. $(S'R' - SS'\text{cos.}\beta)^2 + (MR'\text{cos.}\alpha - SS'\text{sin.}\beta)^2 = 2p \times MR'\text{sin.}\alpha$.

1°. Sit $\alpha = 90°$; ideoque $\text{cos.}\alpha = 0$: erit $(SR' - SS'\text{cos.}\beta)^2 - (SS'\text{sin.}\beta)^2 = 2p \times MR'$. Proinde omnes sectiones planis plano ZSX perpendicularibus factæ sunt parabolæ, quarum parameter $2p$ est eadem quæ parabolæ genitricis.

2°. Sit $\alpha \gtrless 90°$. Fit $(S'R' - SS'\text{cos.}\beta)^2 =$
$\text{cos.}^2\alpha\,(p\,\text{tang.}\alpha\,\text{sec.}^2\alpha\,(p\,\text{tang.}\alpha + 2SS'\text{sin.}\beta) - (MR' - \text{sec.}\alpha\,(p\,\text{tang.}\alpha + SS'\text{sin.}\beta))^2$.

Sectio igitur est elliptica, & species ellipseos pendet ab angulo α. Proinde omnes sectiones planis sibi invicem parallelis factæ sunt inter se similes.

Ut sectio possibilis sit, debet esse
$MR' - \text{sec.}\alpha(p\,\text{tang.}\alpha + SS'\text{sin.}\beta) \overline{\gtrless} \text{sec.}\alpha\sqrt{(p\,\text{tang.}\alpha(p\,\text{tang.}\alpha + 2SS'\text{sin.}\beta))}$; & si fuerit
$MR' = \text{sec.}\alpha\,(p\,\text{tang.}\alpha + SS'\text{sin.}\beta) + \text{sec.}\alpha\sqrt{(p\,\text{tang.}\alpha(p\,\text{tang.}\alpha + 2SS'\text{sin.}\beta))}$, planum ductum tangit superficiem propositam.

Pauca hæc exempla sufficiunt ad dijudicandum, quomodo ex relatione data perpendiculorum MP, PQ, SQ symptomata solidi possunt deduci; & nominatim, quomodo ex æquatione hac proprietates sectionum ejus, planis positione datis factarum, inferuntur. Et quoniam æquatio curvæ, quæ est sectio solidi propositi per aliquod planum, oritur ex æquatione data inter tres perpendiculares MP, PQ, SQ, substituendo valores harum linearum per rectas MR', SR' expressos; neque hæ in valoribus istis alia operatione præter additionem & substractionem afficiuntur (§. 137.): gradus æquationis pro sectione ortæ superare non potest gradum æquationis, quæ relationem trium perpendiculorum MP, PQ, SQ determinat.

§. 142. Transeo ad plana contactus solidorum generaliter consideratorum.

Situs plani cujuscunque determinatur per duas rectas sibi mutuo vel parallelas vel occurrentes. Proinde ut determinetur planum, quod propositam superficiem curvam in puncto dato contingat; determinandæ sunt duæ rectæ ex hoc puncto ductæ, per quas planum hoc agi debeat. Secetur itaque solidum propositum duobus planis per punctum datum transeuntibus: tum determinatis sectionum æquationibus (juxta §. 141.) ducantur (Cap. V.) rectæ, quæ sectiones

ctiones genitas in puncto dato contingant. Planum per duas has tangentes transiens erit planum contactus.

Facillime autem propositum peragitur, si solidum ad tria plana *ZSX*, *YSX*, *YSZ* sibi invicem perpendicularia referatur per rectas planis his perpendiculariter ordinatim applicatas; pariterque duæ sectiones fiant planis duobus ex illis parallelis; proinde tertio perpendicularibus.

Sit igitur *M* punctum datum in superficie solidi, ex quo in planum *ZSX* perpendicularis demittatur *MP*. Tum per rectam *MP* agantur duo plana *M'MP*, *M''MP*, planis *YSX*, *YSZ* respective parallela; & rectæ, quæ in puncto *M* sectiones planis *M'MP*, *M''MP* factas tangunt, plano *ZSX* in *T'* & *T''* puncτis occurrant. Erit (§. 40.) Fig. 38. 3°.

$$PT' = y\frac{dx}{dy}, \quad PT'' = y\frac{dz}{dy}; \text{ hinc } T'T'' = y\sqrt{\left(\left(\frac{dx}{dy}\right)^2 + \left(\frac{dz}{dy}\right)^2\right)}.$$

Sit *PT* ipsi *T'T''* perpendicularis; erit

$$PT = y \times \frac{\frac{dx}{dy} \cdot \frac{dz}{dy}}{\sqrt{\left(\left(\frac{dx}{dy}\right)^2 + \left(\frac{dz}{dy}\right)^2\right)}} = y \times \frac{1}{\sqrt{\left(\left(\frac{dy}{dz}\right)^2 + \left(\frac{dy}{dx}\right)^2\right)}}$$

$$\text{tang.} MTP = \frac{MP}{PT} = \sqrt{\left(\left(\frac{dy}{dz}\right)^2 + \left(\frac{dy}{dx}\right)^2\right)} = \sqrt{(\text{tang.}^2 MT'P + \text{tang.}^2 MT''P)}.$$

$$\text{sec.} MTP = \sqrt{\left(1 + \left(\frac{dy}{dz}\right)^2 + \left(\frac{dy}{dx}\right)^2\right)}$$

$$\text{cos.} MTP = \frac{1}{\sqrt{\left(1 + \left(\frac{dy}{dz}\right)^2 + \left(\frac{dy}{dx}\right)^2\right)}}$$

$$\text{sin.} MTP = \sqrt{\left[\frac{\left(\frac{dy}{dz}\right)^2 + \left(\frac{dy}{dx}\right)^2}{1 + \left(\frac{dy}{dz}\right)^2 + \left(\frac{dy}{dx}\right)^2}\right]}.$$

Exemplum primum. Curva in plano *M'MP* descripta sit circulus, cujus radius *r*; & curva in plano *M''MP* sit parabola, cujus parameter *2p*.

Itaque $yy = rr - zz; \frac{dy}{dz} = -\frac{z}{y}; \left(\frac{dy}{dz}\right)^2 = \frac{zz}{rr - zz}$

$yy = 2px; \frac{dy}{dx} = \frac{p}{y}; \left(\frac{dy}{dx}\right)^2 = \frac{p}{2x}$

$\text{tang.} MTP = \sqrt{\left(\frac{zz}{rr - zz} + \frac{p}{2x}\right)}.$

 Exem-

Exemplum secundum. Ambæ curvæ in planis $M'MP$, $M''MP$ sint circuli, quorum radii r & r':

erit $\begin{matrix} \left(\frac{dy}{dz}\right)^2 = \frac{zz}{rr-zz} \\ \left(\frac{dy}{dx}\right)^2 = \frac{xx}{rr-xx} \end{matrix}$; tang. $MPT = \sqrt{\left(\frac{zz}{rr-zz} + \frac{xx}{rr-xx}\right)}$.

§. 143. Ex præcedentibus deduci etiam possunt capacitates solidorum (sive non sint rotunda, sive ut talia non considerentur), dummodo sectiones eorum planis sibi invicem parallelis factæ juxta datam legem crescant aut decrescant.

Etenim solidum propositum secetur planis sibi invicem parallelis, & quorum distantiæ sint v. gr. æquales. Tum solido inscribantur & circumscribantur solida cylindrica, quorum bases sint sectiones solidi, & altitudo communis sit distantia duarum sectionum sibi proximarum. Frustum solidi inter duas sectiones vicinas comprehensum, majus est solido cylindrico ipsi inscripto; minus autem circumscripto. Atqui ratio æqualitatis limes est rationis horum solidorum cylindricorum; ergo a fortiori ratio æqualitatis limes est rationis frusti solidi ad alterutrum horum cylindrorum. Quare posita distantia communi duarum sectionum vicinarum $= \Delta x$, area sectionis solidi $= s$, & capacitate solidi $= S$: erit $\lim. \frac{\Delta S}{dx} = s$; ideoque $\frac{dS}{dx} = s$. Data igitur lege, juxta quam sectiones s crescunt aut decrescunt; datur exponens differentialis $\frac{dS}{dx}$; & proinde res ad calculum integralem reducitur.

Fig. 39. *Exemplum primum.* Sit $ASA'DD'$ paraboloides, gyratione dimidii segmenti parabolici SAC circa axem SC genita, quæ secetur plano MZM' basi ADA' perpendiculari, & plano SDD' per axem transeunti parallelo. Sectio MZM' pariter erit segmentum parabolicum: nempe $ZP = SC \times \frac{AP.PA'}{AC^2} = SC \times \frac{MP^2}{AC^2}$. Igitur $ZMP (= \frac{2}{3} MP.ZP) = \frac{2}{3} SC. \frac{MP^3}{AC^2}$, & $ZMM' = \frac{4}{3} SC \times \frac{MP^3}{AC^2}$. Quare positis $CP = x$, $SC = b$, $AC = r$: erit $\frac{dS}{dx} = \frac{4}{3} b \times \frac{(rr-xx)^{\frac{3}{2}}}{rr}$. Unde

$S=$

$S = b \times \frac{\frac{1}{3}x(5rr-2xx)\sqrt{(rr-xx)}+\frac{1}{2}r^4 \text{arc.fin.} \frac{x}{r}}{rr}$. Sit $x = r$: erit $S = \frac{1}{2}brrp$. Unde fegmentum $MAZM' = b\left(\frac{1}{2}rr \text{arc.fin.} \frac{\sqrt{(rr-xx)}}{r} - \frac{1}{3}x\sqrt{(rr-xx)}\frac{5rr-2xx}{2rr}\right)$.

Exemplum fecundum. Solidum $ASA'DD'$ fit dimidia ellipfois, quadrante ellipfis SAC circa axem SC rotato genita. Sint ACS, DSD', ACD tria plana fibi invicem perpendicularia; & fit ZMM' planum plano DSD' parallelum.

Sint $SC = a$, $AC = b$, $CP = x$: erit $ZP = \frac{a}{b}.MP$; $MZP = \frac{1}{2}p \times MP \times ZP = \frac{1}{2}p \times \frac{a}{b}.MP^2$; $MZM' = p.\frac{a}{b}.MP^2 = p.\frac{a}{b}(bb-xx) = \frac{dS}{dx}$; unde $S = p.\frac{a}{b}(bbx - \frac{1}{3}x^3) = p.\frac{a}{b}x(\frac{2}{3}bb + \frac{1}{3}MP^2)$. Sit $x = b$; fit $S = \frac{2}{3}abb \times p$. Hinc fegmenti $MAZM'$ capacitas eft $(\frac{2}{3}ab(b-x) - \frac{1}{3}(\frac{a}{b}x.MP^2)p$.

Exemplum tertium. Solidum propofitum fit conus rectus ASA', qui fecetur plano parabolico MZM' lateri SA' parallelo. Fig. 40.

Hoc cafu eft $\frac{dS}{dx} = MZM' \text{fin.} ZPA$.

$MZM' = \frac{4}{3}.MP.ZP$; $\frac{dS}{dx} = \frac{4}{3}MP \times ZP \text{fin.} ZPA = \frac{4}{3}(r-x)\sqrt{(rr-xx)} \text{fin.} SAA'$;

$S = \left(\frac{2}{3}rx\sqrt{(rr-xx)} + \frac{2}{3}r^3 \text{ arc. fin.} \frac{x}{r} - \frac{4}{9}(rr-xx)^{\frac{3}{2}}\right) \text{fin.} SAA' + C$.

Sit $S = 0$, quando $x = 0$; $C = \frac{4}{9}r^3 \text{fin.} SAA'$.

Hinc $S = \frac{4}{9}(AC^3 - MP^3)\text{fin.} SAA' + \frac{2}{3}AC \times CP \times MP \text{fin.} SAA' + \frac{2}{3}AC^2 \times MD \text{ fin.} SAA'$.

Scholium. Principiorum capite hoc explicatorum ad fuperficies folidorum non rotundorum determinandas applicatio non æque facilis eft; ob perpetuam mutationem angulorum, fub quibus plana fuperficiem tangentia ad plana fecantia contigua inclinantur: neque magnam ad hunc fcopum utilitatem formulæ inclinationum planorum tangentium ad plana, ad quæ fuperficies refertur, §. 142. exhibita, afferre poffe mihi videntur.

§. 144. Progredior ad curvas duplicis curvaturæ dictas, feu ad curvas in eodem plano non jacentes, quarum fymptomata principiis capite hoc ftabilitis determinantur. (a)

 Sit

(a) De his curvis primus egregium opufculum fcripfit fagaciffimus CLAIRAUT, vix fexdecim annos natus, fub titulo: *Traité des Courbes à double Courbure*, Paris 1729.

Sit M punctum quodlibet alicujus curvæ duplicis curvaturæ, quæ referatur ad tria plana ZSX, YSX, YSZ, sibi invicem normalia, per rectas MP, PQ, SQ, perpendiculariter sibi mutuo ordinatim applicatas. Curvæ hujus in planum ZSX projectio orthographica determinatur per relationem mutuam abscissarum SQ & ordinatim applicatarum PQ. Pariter ejusdem curvæ in planum YSZ projectio orthographica determinatur per relationem rectarum MP, PQ; denique curvæ hujus in planum YSX projectio orthographica determinatur per relationem rectarum MP, SQ. Hinc symptomatum curvæ duplicis curvaturæ determinatio revocatur ad considerationem curvarum in eodem plano jacentium, quæ in planis ZSX, YSZ, YSX sunt projectiones orthographicæ curvæ propositæ.

Fig. 38.

Duabus autem harum projectionum cognitis innotescit tertia. Etenim data relatione perpendiculi MP ad unamquamque rectarum PQ, SQ (seu datis curvæ propositæ in plana YSX, YSZ projectionibus orthographicis); datur etiam (quantum permittit imperfecta functionum theoria) relatio rectarum PQ, SQ (seu projectio in planum ZSX). Pariterque data æquatione superficiei curvæ, in qua curva duplicis curvaturæ jacet, & una trium curvæ in plana ZSX, YSX, YSZ projectionum; dantur duæ reliquæ. Data enim relatione mutua trium perpendiculorum MP, PQ, SQ per æquationem superficiei propositæ determinata; dataque præterea relatione duarum v. gr. rectarum PQ, SQ per projectionem in planum ZSX: datur etiam relatio utriusque horum perpendiculorum ad tertium MP; seu dantur curvæ propositæ in plana YSZ, YSX projectiones.

Exemplum primum. Projectio in planum ZSX sit circulus, cujus centrum S, radius r; erit ideo $zz+xx=rr$. Projectio autem in planum YSX sit ellipsis, cujus centrum S, & cujus axes a, b in rectis SX, SY jaceant: erit itaque $yy=\frac{bb}{aa}(aa-xx)$, seu $xx=\frac{aa}{bb}(bb-yy)$;

$$\text{hinc } zz+xx=\frac{aa}{bb}(bb-yy)+zz=rr:$$

$$\text{unde } zz=\frac{aa}{bb}\left(yy+\frac{bb}{aa}(rr-aa)\right).$$ Curvæ igitur propositæ in planum YSZ projectio orthographica est hyperbolica.

Exem-

Exemplum secundum. Utraque projectio in plana ΥSX, ΥSZ sint parabolæ conicæ, quarum axes jaceant in axe $S\Upsilon$, quarum parametri sint $2p$, $2p'$; & quarum vextex communis sit S.

Fit ideo $\begin{matrix} 2py = xx \\ 2p'y = zz \end{matrix}$; hinc $xx : zz = p : p'$, & $x : z = \sqrt{p} : \sqrt{p'}$: quare projectio in planum ZSX est linea recta; unde curva proposita non est duplicis curvaturæ, sed tota in eodem plano jacet per S transeunte, & plano ZSX perpendiculari.

Secus erit, si punctum S non sit vertex communis utriusque parabolæ. Sit v. gr. $\begin{matrix} 2p(a+y) = xx \\ 2p'(b+y) = zz \end{matrix}$; hinc $\begin{matrix} 2py = xx-2ap \\ 2p'y = zz-2bp' \end{matrix}$:

$$\begin{aligned} \text{unde } p'(xx-2ap) &= p(zz-2bp') \\ \text{seu } \quad xx &= \frac{p}{p'}zz-2bp+2ap \\ &= \frac{p}{p'}(zz-2bp'+2ap'); \end{aligned}$$

& proinde projectio in planum ZSX est hyperbola, nisi fuerit $b=a$.

Exemplum tertium. Curva proposita jaceat in superficie sphærica, cujus centrum S, & radius datus $SM = R$. Curvæ autem in planum ZSX projectio orthographica sit circumferentia circuli, cujus radius r, & centri C in plano ZSX positio determinetur per rectas $CA = a$, $Cb = b$, magnitudine datas. Fig. 38. 3°. & 4°.

Relatio trium perpendiculorum x, y, z determinatur duabus æquationibus $\begin{matrix} xx+yy+zz = RR \\ (b-x)^2 + (a-z)^2 = rr \end{matrix}$; hinc $a-z = \pm\sqrt{(rr-(b-x)^2)}$

$$zz = aa \pm 2a\sqrt{(rr-(b-x)^2)} + rr - (b-x)^2$$

$$xx+zz = aa \pm 2a\sqrt{(rr-(b-x)^2)} + rr - bb + 2bx = RR - yy:$$

unde $yy = RR - aa - rr + bb - 2bx \mp 2a\sqrt{(rr-(b-x)^2)}$.

Sit v. gr. $a = 0$, $b = 0$; ideoque centrum projectionis sit in S, $yy = RR = rr$, quo docemur, projectiones in utroque plano ΥSX, ΥSZ esse rectas plano ZSX parallelas; & proinde hoc casu lineam propositam esse simplicis curvaturæ.

Scholium. Exemplo hoc continetur investigatio curvæ, quæ est sectio communis superficiei sphæricæ, cujus centrum S & radius R; ac superficiei cylindricæ plano ZSX perpendicularis, cujus centrum basis est C & radius r. Pa-

 tetque

tetque hoc exemplo: quomodo investigatio symptomatum sectionis communis duarum superficierum ad determinationem symptomatum curvarum duplicis curvaturæ reducatur, quod alio adhuc exemplo familiari illustrabo.

Exemplum quartum. Sit hemisphærium, cujus centrum S, planum basis ZSX, & radius R: sit porro conus rectus; cujus vertex C jaceat in plano basis hemisphærii, ita ut basis coni sit basi hemisphærii parallela, seu ut axis coni sit basi hemisphærii perpendicularis. Distantiæ verticis coni a planis YSX, YSZ dicantur a, b; & sit φ dimidius angulus sectionis coni plano per axem transeunte factæ. Erit

$$xx + zz + yy = RR$$

$$(b-x)^2 + (a-z)^2 = yy\,\text{tang.}^2\varphi$$

hinc $xx\,\text{sec.}^2\varphi - 2bx + bb + zz\,\text{sec.}^2\varphi - 2az + aa = RR\,\text{tang.}^2\varphi$

seu $xx - 2bx\,\text{cos.}^2\varphi + bb\,\text{cos.}^2\varphi + zz - 2az\,\text{cos.}^2\varphi + aa\,\text{cos.}^2\varphi = RR\,\text{sin.}^2\varphi$

seu $(x - b\,\text{cos.}^2\varphi)^2 + (z - a\,\text{cos.}^2\varphi)^2 = \text{cos.}^2(RR - (aa+bb)\text{cos.}^2\varphi)$.

Projectio igitur sectionis communis superficierum coni & sphæræ in planum ZSX est circumferentia circuli.

Item sit $RR = aa \mp 2a\sqrt{(yy\,\text{tang.}^2\varphi - (b-x)^2)} + yy\,\text{sec.}^2\varphi - bb + 2bx$

$RR = bb \mp 2b\sqrt{(yy\,\text{tang.}^2\varphi - (a-z)^2)} + yy\,\text{sec.}^2\varphi - aa + 2az$, quæ sunt æquationes projectionum ejusdem sectionis in plana YSX, YSZ respective. (a)

Fig. 38. §. 145. Sint M, M' duo puncta curvæ alicujus curvaturæ, quorum projectiones in plano ZSX sint P, P' respective; unde est PP' projectio chordæ MM', puncta M, M' jungentis, & $P'T'$ projectio secantis.

Quare $MM' = \sqrt{(Mm'^2 + M'm'^2)} = \sqrt{(PP'^2 + M'm'^2)} = \sqrt{(\Delta x^2 + \Delta z^2 + \Delta y^2)}$, et $\frac{MM'}{\Delta x} = \sqrt{(1 + (\frac{\Delta z}{\Delta x})^2 + (\frac{\Delta y}{\Delta x})^2)}$. Extendendo igitur ad curvas duplicis curvaturæ, quæ vera sunt de curvis simplicis curvaturæ, quod scilicet ratio æqualitatis limes sit rationis arcus ad chordam: $\frac{dS}{dx} = \sqrt{(1 + (\frac{dy}{dx})^2 + (\frac{dz}{dx})^2)}$; & proinde exponens differentialis arcus curvæ duplicis curvaturæ, & cujusvis perpen-

(a) Qui plura de hoc capite voluerit, consulat EULERI *Introductionem ad Analysin infinitorum: Appendix de Superficiebus.*

perpendiculorum ex puncto aliquo curvæ hujus in plana ZSX, ZSY, YSX demissorum, per projectiones curvæ hujus in hæc tria plana determinatur.

Porro anguli $MT'P$ limes est angulus, sub quo tangens curvæ in M ad planum ZSX inclinatur, & tangens hujus anguli est $\frac{\frac{dy}{dx}}{\sqrt{\left(1+\left(\frac{dz}{dx}\right)^2\right)}}$. Item anguli $T'PQ$ seu $PP'q'$ limes est angulus, sub quo recta PQ inclinatur ad planum per tangentem in M ductum ac plano ZSX perpendiculare; & cotangens hujus anguli est $\frac{dz}{dx}$.

§. 146. Quamvis methodus hic delineata proprietates curvarum duplicis curvaturæ investigandi sit omnium universalissima; occurrunt tamen casus, quibus affectiones harum curvarum brevius & luculentius aliis modis eruuntur; perpendendo v. gr., quæ ex proposita palmaria quadam proprietate harum curvarum consequuntur.

Proponatur v. gr. curva in superficie curva cylindri, quæ ad singula ejus latera sub dato angulo inclinatur. Expandatur superficies curva cylindri in planum; in plano hoc ducatur recta, quæ sub dato angulo ad unum latus cylindri inclinatur: recta hæc erit expansio curvæ propositæ.

Pariter describenda sit in superficie curva coni circularis recti curva, quæ ad singula ejus latera sub dato angulo inclinatur: superficies hæc in planum expandatur; tum in plano hoc describatur spiralis logarithmica, quæ ad radios sectoris superficiei cono æqualis sub hoc angulo dato inclinatur. Spiralis hæc erit expansio curvæ propositæ.

Idem potissimum illustratur exemplo *curvarum loxodromicarum* in superficie sphæræ descriptarum, quarum palmaria proprietas est, quod meridianos sub eodem angulo dato intersecent. Sint P polus, PM, PM' meridiani ad duo puncta M, M' curvæ loxodromicæ ducti, MT tangens curvæ loxodromicæ in M, & $M'm$ arcus circuli paralleli centro P descripti. Sit $PMT = \varphi$. Sit PA meridianus positione datus, ad quem anguli APM referantur. Sit $MPM' = \Delta x$, $PM = y$, $Mm = -\Delta y$. Fig. 41.

lim.

$\lim.\frac{Mm}{M'm} = \cot.\varphi$, seu $\lim.\frac{-\Delta y}{\Delta x \sin. y} = \cot.\varphi$; $\lim.\frac{-\Delta y}{\Delta x} = \sin. y \cot.\varphi$.

$\lim. -\frac{\Delta x}{\Delta y} = \tang.\varphi \operatorname{cosec}. y$, seu $-\frac{dx}{dy} = \tang.\varphi \operatorname{cosec}. y$: unde $x = C + \tang.\varphi \log.\cot.\frac{1}{2}y$. Sit $x = 0$, quando $y = 90° = p$: erit $C = 0$, & $x = \tang.\varphi \log.\cot.\frac{1}{2}y$. Proinde cotangentibus dimidiorum arcuum *PM* in geometrica progressione crescentibus, anguli *APM* crescunt in progressione arithmetica.

Porro arcu *AM* posito $= Z$, est $-\frac{dZ}{dy} = \sec.\varphi$; hinc $Z = C - y \sec.\varphi$. Sit $Z = 0$, quando $y = p$; $C = p \sec.\varphi$, $Z = MX \sec.\varphi$, cujus limes est $p \sec.\varphi$.

Hinc etiam determinatur area sphærica *APM* inter radios vectores *PA*, *PM*, & arcum loxodromicum *AM* comprehensa.

Etenim superficie quadrantis hemisphærii dicta *H*, & area *APM* dicta *S*.

Fit $\frac{dS}{dx} = \frac{2H}{p} \sin.^2 \frac{1}{2}y$;

atqui $\frac{dx}{dy} = -\tang.\varphi \operatorname{cosec}. y$;

proinde $\frac{dS}{dy} = -\frac{H}{p} \tang.\varphi \tang.\frac{1}{2}y$: unde $S = C - 2\frac{H}{p} \tang.\varphi \log.\sec.\frac{1}{2}y$. Sit $S = 0$, quando $y = p$; $C = 2\frac{H}{p} \tang.\varphi \log.\sec. 45°$. $S = 2\frac{H}{p} \tang.\varphi \left(\log.\frac{\sec. 45°}{\sec.\frac{1}{2}y}\right)$, cujus limes est $2\frac{H}{p} \tang.\varphi \log.\sec. 45° = \frac{H}{p} \tang.\varphi \log. 2 = rr \tang.\varphi \times \log. 2 = \tang.\varphi \log. 2$ (radio sphæræ pro unitate sumto).

Scholium. Formulæ hæ iis tantum casibus applicantur, quibus de loxodromica agitur, seu φ non est $= 90°$.

Caput decimum sextum.

De significatione expressionis $\frac{0}{0}$ et æquipollentium.

§. 147.

Sit P quantitatis mutabilis x functio, quæ habeat factorem simplicem $x - a$, aut factorem compositum $(x - a)^m$, in quo m est numerus positivus: functio P evanescit facto $x = a$.

Vicis-

Viciſſim ſi functio P evaneſcit, facto $x=a$; functio hæc admittit diviſores formarum præcedentium, in quibus m eſt numerus poſitivus. Æquatione $P=0$ liberata a functionibus ſurdis & transcendentibus, quas poteſt continere: inverſa hæc unum eſt ex præcipuis fundamentis theoriæ æquationum, & a mathematicis admittitur, caſu ſaltem quo m eſt numerus integer poſitivus. Plerumque vero eadem per ſe evidens fit, functiones ſurdas aut transcendentes functionis P in ſeries convertendo, aut alias adhibendo transformationes. Quod in gratiam tironum pluribus exemplis illuſtrare e re eſſe cenſeo.

Exemplum 1. Sit $P=a-\sqrt{(aa-xx)}$, quæ evaneſcit facto $x=0$; dico, functionem hanc admittere diviſorem $x-0=x$.

Etenim $P=(a-\sqrt{(aa-xx)})=\dfrac{(a-\sqrt{(aa-xx)})(a+\sqrt{(aa-xx)})}{a+\sqrt{(aa-xx)}}=\dfrac{aa-(aa-xx)}{a+\sqrt{(aa-xx)}}=\dfrac{xx}{a+\sqrt{(aa-xx)}}$.

$$\text{Vel } \sqrt{(aa-xx)}=a-\frac{\frac{1}{2}}{1}\cdot\frac{xx}{a}-\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{x^4}{a^3}-\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3}\cdot\frac{x^6}{a^5}-\ldots\ldots$$

$$\text{unde } P = \frac{\frac{1}{2}}{1}\cdot\frac{xx}{a}+\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{x^4}{a^3}+\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3}\cdot\frac{x^6}{a^5}+\ldots\ldots$$

Exemplum 2. Sit $P=\sqrt{(a+x)}-\sqrt{(a-x)}$, quæ evaneſcit facto $x=0$.

$$P=\frac{(\sqrt{(a+x)}-\sqrt{(a-x)})(\sqrt{(a+x)}+\sqrt{(a-x)})}{\sqrt{(a+x)}+\sqrt{(a-x)}}=\frac{a+x-(a-x)}{\sqrt{(a+x)}+\sqrt{(a-x)}}=\frac{2x}{\sqrt{(a+x)}+\sqrt{(a-x)}}.$$

$$\text{Vel } \sqrt{(a+x)}=\sqrt{a}+\frac{\frac{1}{2}}{1}\cdot\frac{x}{\sqrt{a}}-\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{xx}{a\sqrt{a}}+\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3}\cdot\frac{x^3}{aa\sqrt{a}}-\ldots\ldots$$

$$\sqrt{(a-x)}=\sqrt{a}-\frac{\frac{1}{2}}{1}\cdot\frac{x}{\sqrt{a}}-\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{xx}{a\sqrt{a}}-\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3}\cdot\frac{x^3}{aa\sqrt{a}}-\ldots\ldots$$

$$\text{unde } P = 2\cdot\frac{\frac{1}{2}}{1}\cdot\frac{x}{\sqrt{a}} \qquad +2\cdot\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3}\cdot\frac{x^3}{aa\sqrt{a}}-\ldots\ldots$$

Exemplum 3. Sit $P=a^m-x^m$, quæ evaneſcit facto $x=a$.

Per theorema Coteſianum functio P admittit factorem $x-a$, ſi m eſt numerus integer poſitivus.

Sit autem m quivis alius numerus.

$$\text{Eſt } x^m=(x-(a-x))^m=a^m-\frac{m}{1}a^{m-1}(a-x)+\frac{m}{1}\cdot\frac{m-1}{2}a^{m-2}(a-x)^2-\frac{m}{1}\ldots\frac{m-2}{3}a^{m-3}(a-x)^3+\ldots\ldots$$

$$\text{unde } P(=a^m-x^m) = \frac{m}{1}a^{m-1}(a-x)-\frac{m}{1}\cdot\frac{m-1}{2}a^{m-2}(a-x)^2+\frac{m}{1}\ldots\frac{m-2}{3}a^{m-3}(a-x)^3-\ldots\ldots$$

Exemplum 4. Sit $P = (a+x)^m - (a-x)^m = 0$ facto $x = 0$.

$$(a+x)^m = a^m + \frac{m}{1}a^{m-1}x + \frac{m}{1}.\frac{m-1}{2}a^{m-2}x^2 + \frac{m}{1}\ldots\frac{m-2}{3}a^{m-3}x^3 + \ldots\ldots$$

$$(a-x)^m = a^m - \frac{m}{1}a^{m-1}x + \frac{m}{1}.\frac{m-1}{2}a^{m-2}x^2 - \frac{m}{1}\ldots\frac{m-2}{3}a^{m-3}x^3 + \ldots\ldots$$

$$P = 2.\frac{m}{1}a^{m-1}x \qquad\qquad + 2.\frac{m}{1}\ldots\frac{m-2}{3}a^{m-3}x^3 + \ldots\ldots$$

Exemplum 5. Sit $P = \sqrt{}(aa+ax+xx) - \sqrt{}(aa-ax+xx)$, evanescens facto $x=0$.

$$P = \frac{(aa+ax+xx)-(aa-ax+xx)}{\sqrt{}(aa+ax+xx)+\sqrt{}(aa-ax+xx)} = \frac{2ax}{\sqrt{}(aa+ax+xx)+\sqrt{}(aa-ax+xx)},$$ quæ admittit factorem x.

Idem patet convertendo in series quantitates $\sqrt{}((a+x)^2-ax)$, $\sqrt{}((a-x)^2+ax)$.

Exemplum 6. Sit $P = a - \sqrt[4]{}a^3x$, quæ evanescit facto $x = a$.

$\sqrt[4]{}a^3x = \sqrt[4]{}(a^4 - a^3(a-x))$, quæ in seriem conversa, & ab a subtracta, residuum relinquit, cujus factor est $a-x$.

Pariter $P = a - \sqrt[4]{}aa-xx = a - \sqrt[4]{}(a^4 - aa(aa-xx))$, in seriem conversa, offert factorem $aa-xx = (a-x)(a+x)$.

$P = a - \sqrt[4]{}ax^3 = a - \sqrt[4]{}(a^4 - a(a^3-x^3))$, in seriem conversa, offert factorem $a^3 - x^3 = (a-x)(aa+ax+xx)$.

Generatim $P = a - \sqrt[m]{}a^{m-n}x^n = a - \sqrt[m]{}(a^m - a^{m-n}(a^n - x^n))$, in seriem conversa, offert factorem $a^n - x^n$, qui admittit factorem $a-x$.

Exemplum 7. Sit $P = \sqrt{}(2aaxx - x^4) - a\sqrt[3]{}aax$, quæ evanescit posito $x = a$.

$$\sqrt{}(2aaxx-x^4)) = \sqrt{}(a^4-(aa-xx)^2) = aa - \frac{\frac{1}{2}}{1}\frac{(aa-xx)^2}{aa} - \frac{\frac{1}{2}}{1}.\frac{\frac{1}{2}}{2}.\frac{(aa-xx)^4}{a^6} - \frac{\frac{1}{2}}{1}.\frac{\frac{1}{2}}{2}.\frac{\frac{3}{2}}{3}\frac{(aa-xx)^6}{a^{10}} - \ldots$$

$$a\sqrt[3]{}aax = a\sqrt[3]{}a^3 - aa(a-x)) = aa - \frac{\frac{1}{3}}{1}.a(a-x) - \frac{\frac{1}{3}}{1}.\frac{\frac{2}{3}}{2}.(a-x)^2 - \frac{\frac{1}{3}}{1}.\frac{\frac{2}{3}}{2}.\frac{\frac{5}{3}}{3}.\frac{(a-x)^3}{a} - \ldots$$

Subtractione facta remanet factor $a-x$.

Exemplum 8. Sit $P = x \pm \text{fin}.x = 0$, quando $x = 0$.

$$\text{Quoniam fin}.x = x - \frac{1}{1.2.3}x^3 + \frac{1}{1\ldots5}x^5 - \frac{1}{1\ldots7}x^7 + \ldots\ldots$$

$$x + \text{fin}.x = 2x - \frac{1}{1.2.3}x^3 + \frac{1}{1\ldots5}x^5 - \frac{1}{1\ldots7}x^7 - \ldots\ldots$$

$$x - \text{fin}.x = \frac{1}{1.2.3}x^3 - \frac{1}{1\ldots5}x^5 + \frac{1}{1\ldots7}x^7 - \ldots\ldots$$

Exem-

Exemplum 9. Sit $P = e^x - e^{-x} = 0$, quando $x = 0$.

$$P = 2x\left(\frac{1}{1.2} + \frac{1}{1.2..4}xx + \frac{1}{1...6}x^4 + \ldots\right)$$

§. 148. 1°. Sit P functio variabilis x evanescens casu $x = a$, quæ igitur reduci potest ad formam $(x-a)P'$; non autem evanescat P' casu $x = a$, nec proinde P' reduci possit ad formam $(x-a)P''$; sed casu $x = a$ determinatum obtineat valorem A. Quæritur hic valor.

Ob $(x-a)P' = P$; fit differentiando $P' + (x-a)\frac{dP'}{dx} = \frac{dP}{dx}$:

unde $A = \frac{dP}{dx}$, quando in exponente differentiali $\frac{dP}{dx}$ quantitas constans a loco mutabilis x substituitur. Brevitatis causa designet ${}^a\!\left(\frac{dP}{dx}\right)$ valorem exponentis differentialis $\frac{dP}{dx}$, quando in eo loco x quantitas constans a substituitur. Fit ideo $A = {}^a\!\left(\frac{dP}{dx}\right)$.

2°. Pariter sit $Q = (x-a)Q'$, nec Q' evanescat casu $x = a$; valor determinatus B, quem Q' recipit casu $x = a$, est $B = {}^a\!\left(\frac{dQ}{dx}\right)$.

3°. Tum vero $\frac{P}{Q}\left(= \frac{(x-a)P'}{(x-a)Q'}\right) = \frac{P'}{Q'}$; casu $x = a$ fit $\frac{A}{B} = \frac{{}^a\!\left(\frac{dP}{dx}\right)}{{}^a\!\left(\frac{dQ}{dx}\right)} = {}^a\!\left(\frac{dP}{dQ}\right)$.

Exemplum. Sit $P = x^m - a^m$, quæ evanescit facto $x = a$: $\frac{dP}{dx} = mx^{m-1}$, & ${}^a\!\left(\frac{dP}{dx}\right) = ma^{m-1} = A$. Pariter sit $Q = x^n - a^n$, quæ evanescit facto $x = a$: $\frac{dQ}{dx} = nx^{n-1}$, & ${}^a\!\left(\frac{dQ}{dx}\right) = na^{n-1} = B$. Hinc $\frac{A}{B} = \frac{m}{n}a^{m-n}$.

§. 149. 1°. Quodsi (§. 148. 1°.) etiam P' evanescit casu $x = a$; proindeque est $P' = (x-a)P''$; sed P'' non evanescit casu $x = a$: eodem modo consequitur, valorem determinatum A, quem P'' casu $x = a$ obtinet, esse $A = {}^a\!\left(\frac{dP'}{dx}\right)$.

 Atqui

Atqui ob $P' + (x-a)\frac{dP'}{dx} = \frac{dP}{dx}$

est $\frac{2dP'}{dx} + (x-a)\frac{ddP'}{dx^2} = \frac{ddP}{dx^2}$: proinde casu $x=a$, $2A = {}^{a}\!\left(\frac{ddP}{dx^2}\right)$, & $A = \frac{1}{1.2}\,{}^{a}\!\left(\frac{ddP}{dx^2}\right)$.

2°. Pariter si (§. 148. 2°.) etiam $Q' = (x-a)Q''$, nec Q'' evanescit facto $x=a$; valor determinatus B, quem Q'' casu $x=a$ obtinet, est $B = \frac{1}{1.2}\,{}^{a}\!\left(\frac{ddQ}{dx^2}\right)$.

3°. Tum vero $\frac{P}{Q}\left(=\frac{(x-a)^2.\,P''}{(x-a)^2.\,Q''}\right) = \frac{P''}{Q''}$, fit $\frac{B}{A} = \frac{{}^{a}\left(\frac{ddP}{dx^2}\right)}{{}^{a}\left(\frac{ddQ}{dx^2}\right)}$ casu $x=a$.

Exemplum. Sit $P = a^{n+1} - (n+1)ax^n + nx^{n+1}$

$$\frac{dP}{dx} = -n.\overline{n+1}\,ax^{n-1} + n.\overline{n+1}\,x^n = 0, \text{ facto } x=a,$$

$$\frac{ddP}{dx^2} = -n.\overline{n-1}.\overline{n+1}\,ax^{n-2} + n.n.\overline{n+1}\,x^{n-1}$$

$${}^{a}\!\left(\frac{ddP}{dx^2}\right) = n.\overline{n+1}\,a^{n-1} = 2.A.$$

Pariter fit $Q = a^{m+1} - (m+1)ax^m + mx^{m+1}$

$${}^{a}\!\left(\frac{ddQ}{dx^2}\right) = m.\overline{m+1}\,a^{m-1} = 2B.$$

unde $$\frac{A}{B} = \frac{{}^{a}\left(\frac{ddP}{dx^2}\right)}{{}^{a}\left(\frac{ddQ}{dx^2}\right)} = \frac{n.\overline{n+1}}{m.\overline{m+1}}\,a^{n-m}.$$

§. 150. 1°. Quodsi (§. 149. 1°.) & P'' evanescit casu $x=a$, & proinde est $P'' = (x-a)P'''$; sed P''' non evanescit casu $x=a$: eodem modo consequitur, valorem determinatum A, quem P''' casu $x=a$ obtinet, esse $A = {}^{a}\!\left(\frac{dP''}{dx}\right)$.

Sed propter $P'' + (x-a)\frac{dP''}{dx} = \frac{dP'}{dx}$

est $2\frac{dP''}{dx^2} + (x-a)\frac{ddP''}{dx^2} = \frac{ddP'}{dx^2}$, quare facto $x=a$, est $A = \frac{1}{1.2}\,{}^{a}\!\left(\frac{ddP'}{dx^2}\right)$.

et

et propter $2\frac{dP'}{dx}+(x-a)\frac{ddP'}{dx^2}=\frac{ddP}{dx^2}$

est $3\frac{ddP}{dx^2}+(x-a)\frac{d^3P'}{dx^3}=\frac{d^3P}{dx^3}$, & $3\,{}^a\!\left(\frac{ddP'}{dx^2}\right)={}^a\!\left(\frac{d^3P}{dx^3}\right)$, quare

$A=\frac{1}{1.2.3}\,{}^a\!\left(\frac{d^3P'}{dx^3}\right)$.

2°. Pariter si (§. 149. 2°.) & Q'' evanescit casu $x=a$; ac proinde est $Q''=(x-a)Q'''$; sed Q''' non evanescit casu $x=a$: valor determinatus B, quem Q'' casu $x=a$ obtinet, erit $B=\frac{1}{1.2.3}\,{}^a\!\left(\frac{d^3Q}{dx^3}\right)$.

3°. Tum vero $\frac{P}{Q}\left(=\frac{(x-a)^3P'''}{(x-a)^3Q'''}\right)=\frac{P'''}{Q'''}$; fit $\frac{A}{B}=\frac{{}^a\!\left(\frac{d^3P}{dx^3}\right)}{{}^a\!\left(\frac{d^3Q}{dx^3}\right)}$.

Exemplum. Sit $\begin{aligned}P&=a(x-a)^3(xx+aa)\\ Q&=(x-a)^3(x^3+a^3)\end{aligned}$

$${}^a\!\left(\frac{d^3P}{dx^3}\right)=1.2.3.\,2a^3 \quad A=2a^3 \qquad \frac{A}{B}=\frac{{}^a\!\left(\frac{d^3P}{dx^3}\right)}{{}^a\!\left(\frac{d^3Q}{dx^3}\right)}=\frac{2a^3}{2a^3}=1$$
$${}^a\!\left(\frac{d^3Q}{dx^3}\right)=1.2.3.\,2a^3 \quad B=2a^3$$

§. 151. Si rursus (§. 150.) P''' & Q''' evanescunt, posito $x=a$; & proinde sunt $P'''=(x-a)P^{\text{iv}}$, $Q'''=(x-a)Q^{\text{iv}}$; ita tamen ut P^{iv} & Q^{iv} non evanescant casu $x=a$; sed P^{iv} & Q^{iv} hoc casu valores habent determinatos A, B: erit

$$A={}^a\!\left(\frac{dP'''}{dx}\right),\quad B={}^a\!\left(\frac{dQ'''}{dx}\right),\ \&$$

$$A=\frac{1}{1\ldots4}\,{}^a\!\left(\frac{d^4P}{dx^4}\right),\quad B=\frac{1}{1\ldots4}\,{}^a\!\left(\frac{d^4Q}{dx^4}\right):\ \text{unde}\ \frac{A}{B}=\frac{{}^a\!\left(\frac{d^4P}{dx^4}\right)}{{}^a\!\left(\frac{d^4Q}{dx^4}\right)}.$$

§. 152. *Generatim.* Sit $P=(x-a)^mP'$, exponente m denotante numerum integrum positivum; & P' non sit divisibilis per $x-a$.

Erit $\frac{dP}{dx}=m(x-a)^{m-1}.P'$

$$+\ (x-a)^m.\frac{dP'}{dx}.$$

 $\frac{ddP}{dx^2}$

$$\frac{ddP}{dx^2} = m.m-1(x-a)^{m-2}.P'$$
$$+2m \quad (x-a)^{m-1}.\frac{dP'}{dx}$$
$$+ \quad (x-a)^{m}.\frac{ddP'}{dx^2}$$

$$\frac{d^3P}{dx^3} = m\ldots m-2(x-a)^{m-3}.P'$$
$$+3m.m-1(x-a)^{m-2}.\frac{dP'}{dx}$$
$$+3m \quad (x-a)^{m-1}.\frac{ddP'}{dx^2}$$
$$+ \quad (x-a)^{m}.\frac{d^3P'}{dx^3}$$

$$\frac{d^4P}{dx^4} = m\ldots m-3(x-a)^{m-4}.P'$$
$$+4m\ldots m-2(x-a)^{m-3}.\frac{dP'}{dx}$$
$$+6m\ldots m-2(x-a)^{m-2}.\frac{ddP'}{dx^2}$$
$$+4m \quad (x-a)^{m-1}.\frac{d^3P'}{dx^3}$$
$$+ \quad (x-a)^{m}.\frac{d^4P'}{dx^4}$$

$$\vdots \qquad \vdots$$

$$\frac{d^nP}{dx^n} = m\ldots m-(n-1).(x-a)^{m-n}.P'$$
$$+\frac{n}{1}m\ldots m-(n-2).(x-a)^{m-(n-1)}.\frac{dP'}{dx}$$
$$+\frac{n}{1}.\frac{n-1}{2}m\ldots m-(n-3).(x-a)^{m-(n-2)}.\frac{ddP'}{dx^2}$$
$$+\frac{n}{1}\ldots\frac{n-2}{3}m\ldots m-(n-4).(x-a)^{m-(n-3)}.\frac{d^3P'}{dx^3}$$
$$\vdots \qquad \vdots$$
$$+\frac{n}{1}.m \quad (x-a)^{m-1}.\frac{d^{n-1}P'}{dx^{n-1}}$$
$$+ \quad (x-a)^{m}.\frac{d^nP}{dx^n}.$$

$$\begin{aligned}\frac{d^m P}{dx^m} = & \quad m \ldots 1 \,.\, P' \\ & + \frac{m}{1} \cdot m \ldots 2\,(x-a) \,.\, \frac{dP'}{dx} \\ & + \frac{m}{1} \cdot \frac{m-1}{2} \cdot m \ldots 3\,(x-a)^2 \,.\, \frac{ddP'}{dx^3} \\ & + \frac{m}{1} \cdot \frac{m-2}{3} \cdot m \ldots 4\,(x-a)^3 \,.\, \frac{d^3P'}{dx^3} \\ & \quad \vdots \qquad\qquad\qquad\qquad \vdots \\ & + \frac{m}{1} \cdot m \qquad (x-a)^{m-1} \,.\, \frac{d^{m-1}P'}{dx^{m-1}} \\ & + \qquad\qquad (x-a)^m \,.\, \frac{d^m P'}{dx^m}.\end{aligned}$$

Proinde $A = \frac{1}{1.2\ldots m}\,{}^{a}\!\left(\frac{d^m P}{dx^m}\right)$.

Pariter si $Q = (x-a)^m Q'$,

$$B = \frac{1}{1.2\ldots m}\,{}^{a}\!\left(\frac{d^m Q}{dx^m}\right): \text{ unde } \frac{A}{B} = \frac{{}^{a}\!\left(\frac{d^m P}{dx^m}\right)}{{}^{a}\!\left(\frac{d^m Q}{dx^m}\right)}.$$

§. 153. Sit nunc $P = (x-a)^{\frac{1}{n}} p$, denotante n numerum integrum positivum. Fit ideo

$$\begin{aligned} P^n &= (x-a)p^n, \\ \frac{dP^n}{dx} &= p^n + n(x-a)p^{n-1}\frac{dp}{dx}: \end{aligned}$$

unde $A = \sqrt[n]{\left({}^{a}\!\left(\frac{dP^n}{dx}\right)\right)}$.

Sit pariter $Q = (x-a)^{\frac{1}{n}} q$: erit $Q^n = (x-a)q^n$, & $B = \sqrt[n]{\left({}^{a}\!\left(\frac{dQ^n}{dx}\right)\right)}$:

unde $\frac{A}{B} = \sqrt[n]{\left[\frac{{}^{a}\left(\frac{dP^n}{dx}\right)}{{}^{a}\left(\frac{dQ^n}{dx}\right)}\right]}$.

§. 154. Sit tandem $P = (x-a)^{\frac{m}{n}} p$, ideoque $P^n = (x-a)^m p^n$: erit ${}^{a}\!\left(\frac{d^m P^n}{dx^m}\right) = 1 \ldots m A^n$.

Sit

Sit quoque $Q = (x-a)^{\frac{m}{n}} q$; ideoque $Q^n = (x-a)^m q^n$: erit ${}^a\left(\frac{d^m Q^n}{dx^m}\right) = 1 \ldots m B^n$;

unde $\frac{A^n}{B^n} = \frac{{}^a\left(\frac{d^m P^n}{dx^m}\right)}{{}^a\left(\frac{d^m Q^n}{dx^m}\right)}$, & $\frac{A}{B} = \sqrt[n]{\left[\frac{{}^a\left(\frac{d^m P^n}{dx^m}\right)}{{}^a\left(\frac{d^m Q^n}{dx^n}\right)}\right]}$.

Exempla. Sit $P = \sqrt{(xx-aa)} - (x-a) = \sqrt{(x-a)}(\sqrt{(x+a)} - \sqrt{(x-a)})$;
proinde $p = \sqrt{(x+a)} - \sqrt{(x-a)}$.

Erit $PP = 2(x-a)(x - \sqrt{(xx-aa)})$

$$\frac{dPP}{dx} = (x - \sqrt{xx-aa}) + 2(x-a)\left(1 - \frac{x}{\sqrt{xx-aa}}\right) = 2(x - \sqrt{(xx-aa)} + 2(x-a) - 2\sqrt{\frac{x-a}{x+a}}):$$

unde $AA = 2a$, $A = \sqrt{2a}$.

Sit $Q = \sqrt{(x^3-a^3)} - (x-a)\sqrt{x} = \sqrt{(x-a)}(\sqrt{(xx+ax+aa)} - \sqrt{x(x-a)})$

$$QQ = (x-a)(2xx+aa-2\sqrt{(x(x-a)(xx+ax+aa)})$$

$$\frac{dQQ}{dx} = (xx+aa-2\sqrt{(x(x-a)(xx+ax+aa)})+(x-a)\left(4x - \frac{(x-a)(xx+ax+aa)+x(xx+ax+aa)+x(x-a)(2x+a)}{\sqrt{x(x-a)(xx+ax+aa)})}\right)$$

$$= (xx+aa-2\sqrt{(x(x-a)(xx+ax+aa)})+(x-a)4x - \sqrt{(x-a)}\frac{(x-a)(xx+ax+aa)+x(xx+ax+aa)+x(x-a)(2x+a)}{\sqrt{(x(xx+ax+aa))}}$$

proinde $BB = 2aa$, $B = a\sqrt{2}$.

hinc $\frac{A}{B} = \frac{\sqrt{2a}}{a\sqrt{2}} = \frac{1}{\sqrt{a}}$.

§. 155. *Observatio.* Ex formula (§. 152.)

$$\begin{aligned}
\frac{d^m P}{dx^m} = {} & \quad m \ldots 1 P' \\
& + \frac{m}{1} \cdot m \ldots 2(x-a)\frac{dP'}{dx} \\
& + \frac{m}{1} \cdot \frac{m-1}{2} \cdot m \ldots 3(x-a)^2 \frac{ddP'}{dx^2} \\
& + \frac{m}{1} \cdots \frac{m-2}{3} \cdot m \ldots 4(x-a)^3 \frac{d^3 P'}{dx^3} \\
& \quad \vdots \\
& + \frac{m}{1} \cdot m \quad (x-a)^{m-1} \frac{d^{m-1} P'}{dx^{m-1}} \\
& + \quad (x-a)^m \frac{d^m P'}{dx^m}
\end{aligned}$$

sequitur, quantitatem $m \ldots 1 A$,

quae

quæ eſt limes poſterioris membri hujus æquationis, limitem eſſe etiam exponentis differentialis $\frac{d^mP}{dx^m}$, ſeu $A = \frac{1}{1.2\ldots m}\lim.\frac{d^mP}{dx^m}$.

Pariter $B = \frac{1}{1.2\ldots m}\lim.\frac{d^mQ}{dx^m}$; unde $\frac{A}{B} = \frac{\lim.\frac{d^mP}{dx^m}}{\lim.\frac{d^mQ}{dx^m}} = \lim.\frac{\frac{d^mP}{dx^m}}{\frac{d^mQ}{dx^m}}$.

Hanc obſervationem paucis exemplis illuſtrare haud alienum abs re erit.

Sit $\frac{P}{Q} = \frac{\text{ſin.}x}{x}$; $\frac{dP}{dx} = \text{coſ.}x$ $\frac{dQ}{dx} = 1.$ $\lim.\frac{\frac{dP}{dx}}{\frac{dQ}{dx}} = \lim.\frac{\text{coſ.}x}{1} = 1.$

Sit $\frac{P}{Q} = \frac{\text{tang.}x}{x}$, $\frac{dP}{dx} = \text{ſec.}^2x$ $\frac{dQ}{dx} = 1$ $\lim.\frac{\frac{dP}{dx}}{\frac{dQ}{dx}} = \lim.\text{ſec.}^2x = 1.$

Sit S ſumma progreſſionis geometricæ $S = 1 + x + x^2 + x^3 + \ldots + x^{n-1}$: eſt $S = \frac{1-x^n}{1-x} = \frac{x^n-1}{x-1}$, prouti progreſſio decreſcit, aut creſcit inde ab unitate. Quodſi autem ſeries propoſita neque creſcit neque decreſcit; ſeu ſi ſeries propoſita terminis conſtet inter ſe æqualibus: fit $S = \frac{1-1^n}{1-1} = \frac{1-1}{1-1}$.

Dico autem: methodos, quibus progreſſionum geometricarum ſummæ inveſtigantur, ad hunc caſum non poſſe applicari.

Prima methodus eo redit, ut inferatur (ex *Prop.* 12. *Lib. V. Elem.*): ſumma omnium terminorum excepto ultimo eſt ad ſummam omnium terminorum excepto primo, uti primus terminus ad ſecundum. Unde, caſu æqualitatis omnium terminorum, fit $S-1 : S-1 = 1 : 1$, $S-1 = S-1$; quæ eſt æquatio identica, ex qua nihil concludi poteſt: & ſi computus ulterius continuetur, fit $S(1-1) = 1-1$, $S = \frac{1-1}{1-1}$, quæ eſt expreſſio indeterminata.

Transeo ad alteram methodum.

Sit $S = 1 + x + x^2 + x^3 + x^4 + \ldots x^{n-1}$

erit $Sx = x + x^2 + x^4 + x^4 + \ldots x^{n-1} + x^n$

unde $S(x-1) = -1 \quad + x^n$ (ſi progreſſio creſcit)

$S(1-x) = 1 \quad - x^n$ (ſi progreſſio decreſcit)

 Sit

Sit autem feries neque crefcens neque decrefcens: erit

$$S = 1 + 1 + 1 + 1 + \dots + 1$$
$$S.1 = \quad 1 + 1 + 1 + \dots + 1 + 1$$
$$S(1-1) = 1 \qquad\qquad -1 \qquad S = \frac{1-1}{1-1},$$ quæ iterum eft expreffio indeterminata.

Cum autem ratio æqualitatis limes fit tam rationis decrefcentis $x:1$ (pofito $x > 1$), quam rationis crefcentis $x:1$, pofito $x < 1$; pariter fumma terminorum fibi invicem æqualium tam limes eft parvitatis fummæ decrefcentis totidem terminorum in progreffione geometrica a primo inde prioribus æquali crefcentium, quam limes magnitudinis fummæ crefcentis totidem terminorum in progreffione geometrica a primo inde prioribus æquali decrefcentium. Unde fumma terminorum fibi invicem æqualium (quamvis feriem geometricam non conftituant) elici poteft ex fumma totidem terminorum ab eodem primo termino in ferie geometrica progredientium; quærendo fcilicet limitem fummæ hujus progreffionis, quatenus exponens progreffionis ad unitatem accedit.

Eft nempe

$S = \frac{x^n - 1}{x - 1}$. Sit $x = 1 + z$; $\frac{x^n - 1}{x - 1} = \frac{n}{1} + \frac{n}{1} \cdot \frac{n-1}{2} z + \frac{n}{1} \dots \frac{n-2}{3} z^2 + \frac{n}{1} \dots \frac{n-3}{4} z^3 + \dots$ cujus feriei limes parvitatis eft n. Sed hic limes eft feries totidem terminorum priori æqualium: ergo fumma n terminorum unitati æqualium eft n.

Pariter fit $S = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots + n \quad x^{n-1}$

$$Sx = \quad x + 2x^2 + 3x^3 + 4x^4 + \dots + \overline{n-1}.x^{n-1} + nx^n$$
$$S - Sx = 1 + x + x^2 + x^3 + x^4 + \dots + \quad x^{n-1} - nx^n$$
$$Sx - Sx^2 = \quad x + x^2 + x^3 + x^4 + \dots + \quad x^{n-1} + \quad x^n - nx^{n+1}$$
$$S - 2Sx + Sx^2 = 1 \qquad -(n+1)x^n + nx^{n+1} = n(x^{n+1} - x^n) - (x^n - 1).$$

Facto $x = 1$, erit $S((1-1)-(1-1)) = (n-n)-(1-1))$; ex qua expreffione nihil deducere licet.

Fiat autem $x = 1 + z$, & quæratur limes expreffionis $\frac{n(x^{n+1} - x^n) - (x^n - 1)}{(x-1)^2}$: erit

$S =$

$$S = {}^{0}\frac{\left\{\begin{array}{l} 1+\frac{n+1}{1}z+\frac{n+1}{1}\cdot\frac{n}{2}z^2+\frac{n+1}{1}\cdots\frac{n-1}{3}z^3+\ldots \\ -(1+\frac{n}{1}z+\frac{n}{1}\cdot\frac{n-1}{2}z^2+\frac{n}{1}\cdots\frac{n-2}{3}z^3+\ldots \end{array}\right. -\left(\frac{n}{1}z+\frac{n}{1}\cdot\frac{n-1}{2}z^2+\frac{n}{1}\cdots\frac{n-2}{3}z^3+\ldots\right)}{zz}$$

$$=\frac{n.\,n+1}{1.2}+2.\frac{n-1.\,n.\,n+1}{1.2.3}z+3.\frac{n-2.\,n-1\ldots n+1}{1\ldots 4}z^2+\ldots\text{ cujus limes eſt }\frac{n.\,n+1}{1.2}.$$

Eademque computandi methodus ad plurimas ſeries applicatur, numerorum nempe figuratorum, & generatim terminorum ad differentias conſtantes perducentium, eorundemque per terminos ſeriei alicujus geometricæ multiplicatorum.

§. 156. Hinc intelligitur, quomodo theorema Taylorianum poſſit ad hanc inveſtigationem adhiberi. Scilicet ſit $P' = \frac{P}{x-a}$; ſitque P' functio integra ipſius x. Sit Δx valor ipſius $x-a$, præcedens valorem $x-a=0$: valor ipſius P huic valori Δx reſpondens erit $\frac{\Delta x}{1}\frac{\mathrm{d}P}{\mathrm{d}x}+\frac{\Delta x^2}{1.2}\frac{\mathrm{dd}P}{\mathrm{d}x^2}+\frac{\Delta x^3}{1.2.3}\frac{\mathrm{d}^3P}{\mathrm{d}x^3}+\ldots$;

proinde $P' = \frac{\mathrm{d}P}{\mathrm{d}x}+\frac{\Delta x}{1.2}\frac{\mathrm{dd}P}{\mathrm{d}x^2}+\frac{\Delta x^2}{1.2.3}\frac{\mathrm{d}^3P}{\mathrm{d}x^3}+\ldots$, cujus limes eſt ${}^{a}\!\left(\frac{\mathrm{d}P}{\mathrm{d}x}\right)$.

Eodemque modo $Q' = \frac{\mathrm{d}Q}{\mathrm{d}x}+\frac{\Delta x}{1.2}\frac{\mathrm{dd}Q}{\mathrm{d}x^2}+\frac{\Delta x^2}{1.2.3}\frac{\mathrm{d}^3Q}{\mathrm{d}x^3}+\ldots$, cujus limes eſt ${}^{a}\!\left(\frac{\mathrm{d}Q}{\mathrm{d}x}\right)$:

unde $\frac{A}{B} = {}^{a}\!\left(\frac{\mathrm{d}P}{\mathrm{d}Q}\right)$.

Eodemque modo ſi ${}^{a}\!\left(\frac{\mathrm{d}P}{\mathrm{d}x}\right)=0$, & ${}^{a}\!\left(\frac{\mathrm{d}Q}{\mathrm{d}x}\right)=0$: erit $\frac{A}{B} = \frac{{}^{a}\left(\frac{\mathrm{dd}P}{\mathrm{d}x^2}\right)}{{}^{a}\left(\frac{\mathrm{dd}Q}{\mathrm{d}x^2}\right)}$, & ſic deinceps; quousque ad exponentes differentiales ſimul non evaneſcentes perveniatur.

§. 157. Huc pertinet determinatio (ſi poſſibilis ſit) differentiæ expreſſionum infiniti ſeu impoſſibilis, quæ prodeunt, dum operationes quædam ultra caſus, ad quos quadrant, extenduntur; ſeu expreſſiones $\frac{1}{0}-\frac{1}{0}$, $\infty-\infty$, $(1-1)\infty$, $0\times\infty$, $0\times\frac{1}{0}$; quippe quæ ita redeunt ad $\frac{0}{0}$.

Exempli loco ſit applicatio reſolutionis fractionum, quarum denominatores facto-

factores habent a se invicem diversos, ad casum, quo factores hi fiunt inter se æquales.

Nempe $\frac{1}{x-a}\cdot\frac{1}{x-a'} = \begin{array}{l} \frac{1}{a-a'}\cdot\frac{1}{x-a} \\ +\frac{1}{a'-a}\cdot\frac{1}{x-a'} \end{array} = \frac{1}{a-a'}\left(\frac{1}{x-a}-\frac{1}{x-a'}\right)$; facto autem $a=a'$, signum impossibilis $\frac{1}{a-a'}$ nos monet, resolutionem propositam esse impossibilem; & quærenda est differentia duarum expressionum impossibilium $\frac{1}{a-a'}\cdot\frac{1}{x-a}$, $\frac{1}{a-a'}\cdot\frac{1}{x-a'}$.

Est autem $\frac{1}{x-a'} = \frac{1}{x-a+(a-a')} = \frac{1}{x-a} - \frac{a-a'}{(x-a)^2} + \frac{(a-a')^2}{(x-a)^3} - \frac{(a-a')^3}{(x-a)^4} + \dots$

$$\frac{1}{a-a'}\left(\frac{1}{x-a}-\frac{1}{x-a'}\right) = \frac{1}{a-a'}\left(\frac{a-a'}{(x-a)^2} - \frac{(a-a')^2}{(x-a)^3} + \frac{(a-a')^3}{(x-a)^4} - \dots\right)$$

$$= \frac{1}{(x-a)^2} - \frac{a-a'}{(x-a)^3} + \frac{(a-a')^2}{(x-a)^4} - \dots = \frac{1}{(x-a)^2}$$ casu, quo $a=a'$.

Pariter
$$\frac{1}{\overline{x-a}\,.\,\overline{x-a'}\,.\,\overline{x-a''}} = \frac{1}{\overline{a-a'}\,.\,\overline{a-a''}}\cdot\frac{1}{x-a}$$
$$+\frac{1}{\overline{a'-a}\,.\,\overline{a'-a''}}\cdot\frac{1}{x-a'}$$
$$+\frac{1}{\overline{a''-a}\,.\,\overline{a''-a'}}\cdot\frac{1}{x-a''} = \frac{1}{\overline{a-a'}\,.\,\overline{a-a''}\,.\,\overline{a'-a''}}\left(\frac{a'-a''}{x-a} - \frac{a-a''}{x-a'} + \frac{a-a'}{x-a''}\right)$$

$$= \frac{1}{\overline{a-a'}\,.\,\overline{a-a''}\,.\,\overline{a'-a''}}\left(\frac{a'-a''}{x-a} - \frac{a'-a''}{x-a'} - \left(\frac{a-a'}{x-a'} - \frac{a-a'}{x-a''}\right)\right)$$

$$= \frac{1}{\overline{a-a'}\,.\,\overline{a-a''}}\left(\frac{1}{x-a}-\frac{1}{x-a'}\right) - \frac{1}{\overline{a-a''}\,.\,\overline{a'-a''}}\left(\frac{1}{x-a'}-\frac{1}{x-a''}\right)$$
$$= \frac{1}{\overline{a-a'}\,.\,\overline{a-a''}}\left(\frac{1}{(x-a')-(a-a')} - \frac{1}{x-a'}\right) - \frac{1}{\overline{a-a''}\,.\,\overline{a'-a''}}\left(\frac{1}{x-a'} - \frac{1}{(x-a')+(a'-a'')}\right)$$

$$= \frac{1}{\overline{a-a'}\,.\,\overline{a-a''}}\left(\frac{a-a'}{(x-a')^2} + \frac{(a-a')^2}{(x-a')^3} + \frac{(a-a')^3}{(x-a')^4} + \frac{(a-a')^4}{(x-a')^5} + \dots\right)$$
$$- \frac{1}{\overline{a-a''}\,.\,\overline{a'-a''}}\left(\frac{a'-a''}{(x-a')^2} - \frac{(a'-a'')^2}{(x-a')^3} + \frac{(a'-a'')^3}{(x-a')^4} - \frac{(a'-a'')^4}{(x-a')^5} + \dots\right)$$

$=$

$$= \frac{1}{a-a''}\left(\frac{1}{(x-a')^2} + \frac{a-a'}{(x-a')^3} + \frac{(a-a')^2}{(x-a')^4} + \frac{(a-a')^3}{(x-a')^5} + \dots\right)$$

$$- \frac{1}{a-a''}\left(\frac{1}{(x-a')^2} - \frac{a'-a''}{(x-a')^3} + \frac{(a'-a'')^2}{(x-a')^4} - \frac{(a'-a'')^3}{(x-a')^5} + \dots\right)$$

$$= \frac{1}{a-a''}\left(\frac{a-a''}{(x-a')^3} + \frac{(a-a'')(a-2a'+a'')}{(x-a')^4} + \frac{(a-a'')((a-a')^2-(a-a')(a'-a'')+(a'-a'')^2)}{(x-a')^5} + \dots\right)$$

$$= \frac{1}{(x-a')^3} + \frac{a-2a'+a''}{(x-a')^4} + \dots$$

Unde posito $a = a' = a''$, fit

$$\left.\begin{array}{l} \frac{1}{a-a'.a-a''}\cdot\frac{1}{x-a} \\ +\frac{1}{a'-a.a'-a''}\cdot\frac{1}{x-a'} \\ +\frac{1}{a''-a.a''-a'}\cdot\frac{1}{x-a''} \end{array}\right\} = \frac{1}{(x-a')^3} = \frac{1}{(x-a)^3}.$$

Expulsis igitur impossibilitatis signis, coacta calculorum ad casus, ad quos non quadrant, applicatione introductis, restituitur expressio realis $\frac{1}{(x-a)^3}$, quæ in expressiones impossibiles fuerat resoluta.

Idem dicatur de resolutione impossibili fractionum $\frac{1}{(x-a)^n}\frac{1}{(xx-2ax\cos.\alpha+aa)^n}$. (a)

§. 158. Progredior ad nonnulla exempla, quibus differentiæ propositæ quantitates transcendentes comprehendunt; eaque desumta ex Euleri *Institutionum calculi differentialis* parte posteriori Cap. XV.

Exemplum primum. Sit functio $\frac{x}{x-1} - \frac{1}{\log. x}$, cujus termini $\frac{x}{x-1}$ & $\frac{1}{\log. x}$ fiunt $\frac{1}{0}$ seu ∞ facto $x = 1$.

Sit $\frac{x}{x-1} - \frac{1}{\log. x} = \frac{x\log. x - (x-1)}{x-1.\log. x} = \frac{P}{Q}$.

 Erit

(a) Applicationes inter huc usque traditorum notari inprimis meretur fractionum rationalium, quarum denominator factores habet tam simplices quam compositos, tam reales quam imaginarios, in alias resolutio a celeb. Eulero in *Institutionibus calculi differentialis* Cap. XVIII. exposita. Quamvis enim resolutio hæc methodis mere elementaribus (etiam absque methodo indeterminatarum) generaliter institui possit; propositiones capite hoc stabilitas mire eam juvare ac concinniorem reddere non est diffitendum.

Erit $\frac{dP}{dx} = \log.x + 1 - 1 = \log.x \quad = 0$, facto $x = 1$

$$\frac{dQ}{dx} = \log.x + \frac{x-1}{x} \qquad = 0,\ \text{facto } x = 1$$

$$\frac{ddP}{dx^2} = \frac{1}{x} \qquad = 1,\ \text{facto } x = 1$$

$$\frac{ddQ}{dx^2} = \frac{1}{x} + \frac{1}{x} - \frac{x-1}{xx} \qquad = 2,\ \text{facto } x = 1.$$

Hinc $\dfrac{1\left(\frac{ddP}{dx^2}\right)}{1\left(\frac{ddQ}{dx^2}\right)} = \dfrac{A}{B} = \frac{1}{2}.$

Exemplum secundum. Sit functio $\frac{1}{x(e^{2x}-1)} - \frac{1-x}{2xx}$, cujus quæritur valor casu, quo $x = 0$; & proinde uterque terminus impossibilis.

$$2\frac{P}{Q} = \frac{(x+1)-(1-x)(e^{2x})}{xx(e^{2x}-1)}.$$

$$\frac{dP}{dx} = 1 + (2x-1)e^{2x} \qquad = 0,\ \text{facto } x = 0$$

$$\frac{dQ}{dx} = 2x(e^{2x}-1) + 2xxe^{2x} \qquad = 0,\ \text{facto } x = 0$$

$$\frac{ddP}{dx^2} = 2xe^{2x} \qquad = 0,\ \text{facto } x = 0$$

$$\frac{ddQ}{dx^2} = 2(e^{2x}-1) + 8xe^{2x} + 4xxe^{2x} = 0,\ \text{facto } x = 0$$

$$\frac{d^3P}{dx^3} = 4e^{2x} + 4xe^{2x} \qquad = 4,\ \text{facto } x = 0$$

$$\frac{d^3Q}{dx^3} = 12e^{2x} + 24xe^{2x} + 8xxe^{2x} = 12,\ \text{facto } x = 0.$$

Hinc $\dfrac{0\left(\frac{d^3P}{dx^3}\right)}{0\left(\frac{d^3Q}{dx^3}\right)} = \dfrac{A}{B} = \frac{4}{12} = \frac{1}{3}.$

Aliter

Aliter $e^{2x}-1 = 2x + \frac{2^2}{1.2}x^2 + \frac{2^3}{1.2.3}x^3 + \frac{2^4}{1...4}x^4 + \ldots$

hinc $\frac{1}{x(e^{2x}-1)} - \frac{1-x}{2xx} = \frac{1}{2xx}\left\{\frac{1}{1+\frac{2}{1.2}x+\frac{2^2}{1.2.3}xx+\frac{2^3}{1...4}x^3+\ldots} - (1-x)\right.$

$$= \frac{1}{2xx}\left\{\frac{x(1+\frac{2}{1.2}x+\frac{2^2}{1.2.3}x^2+\frac{2^3}{1...4}x^3+\ldots) - 2x(\frac{1}{1.2}+\frac{2}{1.2.3}x+\frac{2^2}{1...4}x^2+\frac{2^3}{1...5}x^3+\ldots)}{1+\frac{2}{1.2}x+\frac{2^2}{1.2.3}x^2+\frac{2^3}{1...4}x^3+\ldots)}\right.$$

$$= \frac{1}{2xx}\left\{\frac{xx(\frac{1}{3}+(\frac{2^2}{1.2.3}-\frac{2^3}{1...4})x+(\frac{2^3}{1...4}-\frac{2^4}{1...5})x^2+\ldots}{1+\frac{2}{1.2}x+\frac{2^2}{1.2.3}x^2+\frac{2^3}{1...4}x^3+\ldots}\right.$$

$$= \frac{1}{2}\cdot\frac{\frac{1}{3}+(\frac{2^2}{1.2.3}-\frac{2^3}{1...4})x+(\frac{2^3}{1...4}-\frac{2^4}{1...5})x^2+\ldots}{1+\frac{2}{1.2}x+\frac{2^2}{1.2.3}x^2+\frac{2^3}{1...4}x^3+\ldots} = \frac{1}{6},$$

facto $x=0$.

Exemplum tertium. Sit $\frac{1}{xx} - \frac{1}{x\,\text{tang.}x}$ functio ex duobus terminis impossibilibus composita, facto $x=0$.

Ergo $\frac{P}{Q} = \frac{\text{tang.}x - x}{xx\,\text{tang.}x}$

$\frac{dP}{dx} = \text{sec.}^2x - 1 \qquad = 0$, facto $x=0$

$\frac{dQ}{dx} = 2x\,\text{tang.}x + xx\,\text{sec.}^2x \qquad = 0$, facto $x=0$

$\frac{ddP}{dx^2} = 2\,\text{sec.}^2x\,\text{tang.}x \qquad = 0$, facto $x=0$

$\frac{ddQ}{dx^2} = 2\,\text{tang.}x + 4x\,\text{sec.}^2x + 2xx\,\text{sec.}^2x\,\text{tang.}x \qquad = 0$, facto $x=0$

$\frac{d^3P}{dx^3} = 4\,\text{sec.}^2x\,\text{tang.}^2x + 2\,\text{sec.}^4x \qquad = 2$, facto $x=0$

$\frac{d^3Q}{dx^3} = 6\,\text{sec.}^2x + 12x\,\text{sec.}^2x\,\text{tang.}x + 4xx\,\text{sec.}^2x\,\text{tang.}^2x + 2xx\,\text{sec.}^4x \qquad = 6$, facto $x=0$

Unde $\frac{A}{B} = \frac{1}{3}$, facto $x=0$.

Aliter

Aliter $\frac{1}{\text{tang.}\,x} = \frac{\text{cof.}\,x}{\text{fin.}\,x} = \frac{1 - \frac{1}{1.2}x^2 + \frac{1}{1\ldots4}x^4 - \frac{1}{1\ldots6}x^6 + \ldots}{x - \frac{1}{1\ldots3}x^3 + \frac{1}{1\ldots5}x^5 - \frac{1}{1\ldots7}x^7 + \ldots}$

hinc $\frac{P}{Q} = \frac{1}{xx}\left(1 - \frac{1 - \frac{1}{1.2}xx + \frac{1}{1\ldots4}x^4 - \frac{1}{1\ldots6}x^6 + \ldots}{1 - \frac{1}{1\ldots3}xx + \frac{1}{1\ldots5}x^4 - \frac{1}{1\ldots7}x^6 + \ldots}\right)$

$$= \frac{\frac{1}{3} - \frac{4}{1\ldots5}xx + \frac{6}{1\ldots7}x^4 - \ldots}{1 - \frac{1}{1.2.3}xx + \frac{1}{1\ldots5}x^4 - \ldots} = \tfrac{1}{3},\ \text{facto}\ x = 0.$$

Exemplum quartum. Sit $\frac{1}{2x} - \frac{1}{x(e^x + 1)}$ functio ex duobus terminis impossibilibus composita, facto $x = 0$.

Igitur $2\frac{P}{Q} = \frac{e^x - 1}{x(e^x + 1)}$

$$\frac{dP}{dx} = e^x \qquad = 1,\ \text{facto}\ x = 0$$

$$\frac{dQ}{dx} = e^x + 1 + xe^x = 2,\ \text{facto}\ x = 0:$$

hinc $2\frac{P}{Q} = \frac{1}{2}$, facto $x = 0$, seu $\frac{P}{Q} = \frac{1}{4}$.

Aliter $e^x + 1 = 2 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \ldots$

hinc $\frac{1}{2x} - \frac{1}{x(e^x+1)} = \frac{1}{2x}\left[\frac{x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \ldots}{2 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \ldots}\right] = \frac{1}{2}\left[\frac{1 + \frac{x}{1.2} + \frac{x^2}{1.2.3} + \ldots}{2 + x + \frac{x^2}{1.2} + \ldots}\right] = \frac{1}{4}$, facto $x = 0$.

§. 159. Casibus explicatis, quibus quoti $\frac{A}{B}$ fiunt determinati; paucis perstringam casus (non satis hactenus observatos), quibus signum $\frac{0}{0}$ absolutam indicat indeterminationem. Quod ut luculentius faciam, ab exemplis quam simplicissimis ordiar.

Exem-

Exemplum primum. Sit $m : m'$ ratio data duarum quantitatum x, y; & ſit $n : n'$ ratio etiam data earundem, poſtquam quantitatibus datis a, b auctæ fuerunt.

$$\text{Erit } x = m \times \frac{n'a - nb}{nm' - n'm} \qquad x + a = n \times \frac{m'a - mb}{nm' - n'm}$$

$$y = m' \times \frac{n'a - nb}{nm' - n'm} \qquad x + b = n' \times \frac{m'a - mb}{nm' - n'm}.$$

Jam vero ſit $a : b = m : m'$.

Quoniam $x : y = m : m'$: erit $x + a : y + b = m : m'$; ideoque $n : n' = m : m'$.

$$\text{Ergo} \quad \begin{matrix} n'a = nb & x = a \times \frac{0}{0} & x + a = a \times \frac{0}{0} \\ m'a = mb & y = b \times \frac{0}{0} & y + b = b \times \frac{0}{0} \end{matrix}$$

Atqui ratione quantitatum a & b rationi quantitatum x & y æquali poſita, patet: quantitates x & y eſſe poſſe qualescunque, ſeu eas eſſe indeterminatas; proinde hoc caſu ſignum $\frac{0}{0}$ abſolutam indicat indeterminationem.

Caſibus, quibus quantitates x & y ſunt determinatæ, quantitas x (v. gr.) determinatur æquatione $mn'x + mn'a = nm'x + nmb$.

Atqui poſito $m : m' = n : n'$, eſt $mn' = m'n$; ideoque termini $mn'x$, $nm'x$ ſunt invicem æquales: unde debet eſſe $mn'a = nmb$, ſeu $n'a = nb$, & $a : b = n : n'$: duæ expreſſiones $mn'x + mn'a$, $nm'x + nmb$ ſunt ideo identicæ, & nihil ex illis deduci poteſt, quod ad valorem quantitatis incognitæ x, caſu, qûo $m : m' = n : n' = a : b$.

Exemplum ſecundum. Proponantur duæ æquationes $\begin{matrix} ax + by = p \\ a'x + b'y = p' \end{matrix}$: fit $x = \frac{pb' - p'b}{ab' - a'b}$, $y = \frac{pa' - p'a}{a'b - ab'}$. Quamdiu non eſt $ab' = a'b$, ſeu $a : a' = b : b$; duæ quantitates incognitæ x, y æquationibus præcedentibus determinantur: caſu autem, quo $a : a' = b : b'$, ſeu $a' = na$, & ſimul $b' = nb$; fit $\begin{matrix} ax + by = p \\ nax + nby = p' \end{matrix}$: unde $p' = np$, & $p'b = nbp = pb'$; fit ideo $x = \frac{0}{0}$, ſeu quantitas indeterminata: & ſi non eſſet $p' = np$; ſigni $x = \frac{pb' - p'b}{ab \cdot 0}$ introductio quæſtionis propoſitæ impoſſibilitatem nos doceret.

Eadem applicari poſſunt ad quæſtiones, in quibus tres aut plures quantitates incognitæ occurrunt.

Exemplum tertium. Exemplo primo prorſus analogum eſt hoc alterum.

 Detur

Detur ſumma a quantitatum x, x'; item ſumma b quantitatum y, y'; dentur etiam rationes tam quantitatum x & y, quam quantitatum x', y': & quærantur quatuor hæ quantitates. Si ratio ſummarum datarum a, b æqualis eſt rationi datæ quantitatum x, y: ratio quantitatum x', y' exinde ita determinatur, ut æqualis ſit priori rationi datæ quantitatum x, y; & hæ quantitates poſſunt eſſe quæcunque in ea ratione data, quæ indeterminatio ſymboli $\frac{0}{0}$ introductione indicatur.

Quantitates datæ numero pari a, b, c, d .... ſint ſummæ aut differentiæ quantitatum x, x', y, y', z, z', v, v'.... Dentur etiam rationes $x:y$, $y':z$, $z':v$, $v':x'$. Fieri poteſt, ut quæſtio ſit indeterminata.

Inter applicationes geometricas propoſitionis hujus ad indeterminationem ducentis tres ſequentes indicabo, quæ quaſi ſponte ſe mihi obtulerunt.

Figuræ rectilineæ poſitione ac magnitudine datæ inſcribenda ſit figura rectilinea cognominis perimetri omnium minimæ. Caſus, quo figuræ propoſitæ numerus laterum eſt par, ita eſt indeterminatus; ut, ſi problema propoſitum poſſibile fuerit, ſolutionum numerus nullum habeat limitem: & facile eſt indeterminationis hujus nexum cum caſu præcedente oſtendere. (Vid. *Relatio mutua capacitatis & terminorum figurarum.* Varſaviæ 1782.)

Pariter circulo dato inſcribenda ſit figura rectilinea, cujus latera per puncta poſitione data transeant. Problema hoc caſibus quibusdam etiam fit indeterminatum; uti oſtendi in diſſertatione, quam de hoc eleganti problemate ad Academiam Berolinenſem ante aliquot menſes transmiſi: & indeterminatio hæc ad eundem omnino fontem poteſt reduci.

Idem contingit, quando figura rectilinea cognomini figuræ rectilineæ ita inſcribenda eſt, ut latera ejus per puncta poſitione data transeant, aut ſint rectis poſitione datis parallela.

Sed miſſis hisce quæſtionibus magis arduis transeo ad exempla geometrica ſimplicitate ſua commendanda.

Sit circulus, cui inſcribenda ſit linea recta, quæ per punctum poſitione datum transeat.

Sit

Sit C centrum circuli, cujus radius ſit r; ſit P punctum poſitione datum, cujus diſtantia CP a centro ſit a; ſit XY recta magnitudine data inſcribenda $= 2b$; & ſit CZ ipſi XY perpendicularis. Omiſſa analyſi & conſtructione geometrica (quibus hic immorari a ſcopo foret alienum) exponam tantum calculum algebraicum, quo fit ſin. $CPZ = \frac{CZ}{CP} = \frac{\sqrt{(rr-bb)}}{a}$. Fiat autem $b=r$, & proinde recta inſcribenda transeat per centrum; fit ſin. $CPZ = \frac{0}{a}$: eodemque caſu punctum P propius propiusque ad centrum accedat, & cum eo tandem coincidat; ita fit ſin. $CPZ = \frac{0}{0}$: & cum hoc caſu unica ſit conditio, nempe ut recta ducenda per unum tantum punctum datum transeat; ſignum hoc indicat indeterminationem poſitionis lineæ magnitudine datæ. Fig. 42.

Obſervatio. Si fuiſſet tantum $a=0$, non autem $b=r$: foret ſin. $CPZ = \frac{\sqrt{(rr-bb)}}{0}$; & hoc ſymbolum eſſet ſignum impoſſibilitatis (Cap. IX.).

Alterum exemplum. Sint duo puncta poſitione data; ſit & tertium punctum poſitione datum, per quod ducenda ſit recta talis, ut perpendicula ex duobus prioribus punctis datis in eam demiſſa ſint inter ſe in ratione data. Fig. 43.

Sint A, B duo priora puncta data; ſit P tertium punctum datum: & quæratur recta PX, in quam demiſſa perpendicula Aa, Bb ſint inter ſe in ratione data. Agatur recta AB, & dividatur in C in ratione data: recta PC eſt recta quæſita. Ut problema ſit determinatum; neceſſe eſt, ut P & C puncta ſint diverſa: ſi ſecus eveniat, unicum datur punctum, per quod recta ducenda ſit, & proinde poſitio ejus eſt indeterminata; quod calculo etiam indicatur.

Sit $AC=a$, $PC=b$, $AP=c$; in triangulo APC ſit ſin. $C =$

$$= \frac{2\sqrt{\left(\frac{a+b+c}{2}\cdot\frac{a+b-c}{2}\cdot\frac{a-b+c}{2}\cdot\frac{-a+b+c}{2}\right)}}{ab}.$$

Ratione data manente eadem, ſeu eadem manente AC, evaneſcat PC: fiet $AP=AC$, ſeu $a=c$, & $b=0$; unde ſin. $C = 2\frac{0}{0}$: quod ſignum indicat, omnes rectas per C ductas conditioni propoſitæ ſatisfacere.

Hæc poſſent ad numerum quemcunque punctorum poſitione datorum ap-

plicari; & huc quoque revocatur indeterminatio, cui anſam præbet problema locale, quod expoſui in opuſculo meo inſcripto: *Polygonometrie*, pag. 72—93.

Indeterminatio hæc, ubi occurrit, affectiones quasdam generales quantitatum, quæ locum ei præbent, poteſt indicare. Sic v. gr. poſtremum exemplum intime connectitur cum proprietate centri gravitatis, de qua vide §. 125 ſq. Sufficiat paucis exemplis elementaribus aſſertum hoc iterum illuſtrare.

Circulo dato inſcribendum ſit quadrilaterum, cujus anguli dantur.

Fig. 44. Sit $ABA'B'$ quadrilaterum propoſitum, inſcribendum circulo, cujus centrum C & radius CA. Agantur radii CA, CB, CA', CB'.

$$\begin{array}{lcl} \text{Sit } CAB = CBA = x \\ \text{Erunt } CBA' = CA'B = B-x \\ CA'B' = CB'A' = A'-B+x \\ CB'A = CAB' = B'-A'+B-x = A-x. \end{array}$$

Unde angulus x fit indeterminatus: & ſimul $A+A' = B+B'$; ſeu diſcimus, ſummas angulorum alterne ſumtorum eſſe invicem æquales. Idem valet de quavis figura rectilinea numeri laterum paris, circulo inſcribenda. Huc etiam pertinet caſus indeterminatus problematis in Geodæſia tantopere utilis; quo, tribus punctis poſitione datis, quæritur quartum punctum, obſervatis ex hoc puncto angulis, ſub quibus mutuæ priorum punctorum diſtantiæ apparent.

Idem dicatur de ſummis laterum alterne ſumtorum figurarum circulo circumſcriptarum.

Sint A, B, C latera alicujus trianguli;

a, b, c anguli his lateribus oppoſiti: quamdiu latera A & B ſunt invicem inæqualia, ac proinde etiam anguli a & b invicem inæquales; eſt tang.$\frac{a+b}{2}$: tang.$\frac{a-b}{2}$ $= A+B : A-B$: data igitur ratione (inæqualitatis) laterum A & B, & data differentia angulorum a & b, determinatur tangens dimidiæ ſummæ horum angulorum per æquationem tang.$\frac{a+b}{2}$ = tang.$\frac{a-b}{2} \times \frac{A+B}{A-B}$.

Sint autem latera A, B invicem æqualia: fit tang.$\frac{a+b}{2}$ = tang.$\frac{a-b}{2} \times \frac{1}{0}$; & hoc foret ſignum impoſſibilitatis, niſi eſſet ſimul $a = b$, & tang.$\frac{a-b}{2} = 0$: unde

tang.

tang. $\frac{a+b}{2} = \frac{0}{0}$. Signum ideo indeterminationis $\frac{0}{0}$ nos monet: quod, ſi nulla ſit laterum differentia, etiam nulla ſit angulorum oppoſitorum differentia.

Idem dicatur de altera propoſitione, quæ unum eſt etiam ex præcipuis trigonometriæ planæ fundamentis. Sit B baſis alicujus trianguli, in quam ex vertice oppoſito b agatur recta perpendicularis; & ſint A', C' ſegmenta baſis lateribus A & C adjacentia. Quamdiu A & C ſunt invicem inæqualia, ſtat proportio $A+C : A-C = B : A'-C'$; unde $A+C = B \times \frac{A-C}{A'-C'}$: ſi vero $A'=C'$, & proinde $A=C$; fit $A+C = B \times \frac{0}{0}$, quæ eſt expreſſio indeterminata. Numerus nempe triangulorum æquicrurorum ſuper data baſi conſtruendorum eſt illimitatus.

Sed hæc ſufficiant de re ad elementa pertinentia: nec immoror oſtendendo nexui inter propoſitiones, ab antiquis Porismata nuncupatas, & ſignum $\frac{0}{0}$ caſu, quo indicat indeterminationem.

Caput decimum septimum.

De theoremate Tayloriano ad functiones duarum pluriumve variabilium extenſo; et de rationibus differentialibus atque integralibus earundem functionum.

§. 160.

Sit P functio duarum quantitatum mutabilium x, y. Valores, quos functio hæc recipit, quando quantitas x ſola, aut quantitas y ſola, mutationes Δx, Δy reſpective patiuntur, denotentur ſignis ${}^{x}P$, ${}^{y}P$. Valor, quem eadem functio recipit, quando x & y mutationes Δx & Δy ſucceſſive patiuntur, ponatur $\overset{x}{{}^{y}P}$; pariterque valor, quem functio hæc recipit, quando y & x mutationes Δy & Δx ſucceſſive patiuntur, deſignetur per $\overset{y}{{}^{x}P}$.

Functio $\overset{x}{{}^{y}P}$ ex functione ${}^{x}P$ oritur, ſi in ea $y + \Delta y$ loco y ſubſtituitur; pariterque functio $\overset{y}{{}^{x}P}$ ex functione ${}^{y}P$ oritur, ſi in ea $x + \Delta x$ loco x ſubſtituitur. Proinde duæ expreſſiones $\overset{x}{{}^{y}P}$, $\overset{y}{{}^{x}P}$ unum eundemque functionis P valorem deſignant:

gnant: eum nempe, quem obtinet, ſi quantitatum x & y loco quantitates $x+\Delta x$, $y+\Delta y$ ſimul ſubſtituantur.

Quoniam notatio a mathematicis uſurpata, qua exponentes differentiales functionum duarum pluriumve quantitatum mutabilium deſignant, minus commoda mihi videtur; liceat aliam proponere, calculo (mea quidem ſententia) aptiorem. Scilicet exponens differentialis functionis P duarum pluriumve variabilium, quatenus x ſola mutatur, denotetur ſigno ${}^{x}d'P$; & exponentes differentiales ſucceſſivi eodem modo ſumti ponantur ${}^{x}d''P$, ${}^{x}d'''P$, ${}^{x}d^{\text{iv}}P$ .... Tum exponens differentialis functionis ${}^{x}d'P$, quatenus y ſola mutatur, deſignetur ${}^{y}d'\,{}^{x}d'P$; & generatim exponens differentialis m^{ti} ordinis functionis ${}^{x}d^{N}P$, quatenus y ſola in hac functione mutatur, ſit ${}^{y}d^{M}\,{}^{x}d^{N}P$.

Quibus ſuppoſitis, paucis oſtendam: quomodo theorema Taylorianum, de functionibus unius tantum quantitatis mutabilis demonſtratum, ad functiones duarum pluriumve quantitatum mutabilium extendatur; a functionibus duarum quantitatum mutabilium x, y, quæ fiant $x+\Delta x$, $y+\Delta y$, ordiundo.

§. 161. Per theorema Taylorianum (Cap. III.) eſt

$$ {}^{x}P = P+\frac{\Delta x}{1}{}^{x}d'P+\frac{\Delta x^2}{1.2}{}^{x}d''P+\frac{\Delta x^3}{1.2.3}{}^{x}d'''P+\frac{\Delta x^4}{1...4}{}^{x}d^{\text{iv}}P+\frac{\Delta x^5}{1...5}{}^{x}d^{\text{v}}P+\dots $$

$$ {}^{y}P = P+\frac{\Delta y}{1}{}^{y}d'P+\frac{\Delta y^2}{1.2}{}^{y}d''P+\frac{\Delta y^3}{1.2.3}{}^{y}d'''P+\frac{\Delta y^4}{1...4}{}^{y}d^{\text{iv}}P+\frac{\Delta y^5}{1...5}{}^{y}d^{\text{v}}P+\dots $$

Proinde

$$ \begin{aligned} {}^{x}_{y}P = P &+\frac{\Delta y}{1}{}^{y}d'P+\frac{\Delta y^2}{1.2}{}^{y}d''P+\frac{\Delta y^3}{1.2.3}{}^{y}d'''P+\frac{\Delta y^4}{1...4}{}^{y}d^{\text{iv}}P+\frac{\Delta y^5}{1...5}{}^{y}d^{\text{v}}P+\dots \\ &+\frac{\Delta x}{1}\left({}^{x}d'P+\frac{\Delta y}{1}{}^{y}d'\,{}^{x}d'P+\frac{\Delta y^2}{1.2}{}^{y}d''\,{}^{x}d'P+\frac{\Delta y^3}{1.2.3}{}^{y}d'''\,{}^{x}d'P+\frac{\Delta y^4}{1...4}{}^{y}d^{\text{iv}}\,{}^{x}d'P+\dots\right) \\ &+\frac{\Delta x^2}{1.2}\left(\dots\dots\ {}^{x}d''P+\frac{\Delta y}{1}{}^{y}d'\,{}^{x}d''P+\frac{\Delta y^2}{1.2}{}^{y}d''\,{}^{x}d''P+\frac{\Delta y^3}{1.2.3}{}^{y}d'''\,{}^{x}d''P+\dots\right) \\ &+\frac{\Delta x^3}{1.2.3}\left(\dots\dots\dots\ {}^{x}d'''P+\frac{\Delta y}{1}{}^{y}d'\,{}^{x}d'''P+\frac{\Delta y^2}{1.2}{}^{y}d''\,{}^{x}d'''P+\dots\right) \\ &+\frac{\Delta x^4}{1...4}\left(\dots\dots\dots\dots\ {}^{x}d^{\text{iv}}P+\frac{\Delta y}{1}{}^{y}d'\,{}^{x}d^{\text{iv}}P+\dots\right) \\ &+\frac{\Delta x^5}{1...5}\left(\dots\dots\dots\dots\dots\ {}^{x}d^{\text{v}}P+\dots\right) \end{aligned} $$

Pari-

Pariter

$$\overset{y}{{}^{x}}P = P + \frac{\Delta x}{1}{}^{x}d'P + \frac{\Delta x^2}{1.2}{}^{x}d''P + \frac{\Delta x^3}{1.2.3}{}^{x}d'''P + \frac{\Delta x^4}{1 \ldots 4}{}^{x}d^{\text{iv}}P + \frac{\Delta x^5}{1 \ldots 5}{}^{x}d\;P + \ldots$$

$$+ \frac{\Delta y}{1}\left({}^{y}d'P + \frac{\Delta x}{1}{}^{x}d'\,{}^{y}d'P + \frac{\Delta x^2}{1.2}{}^{x}d''\,{}^{y}d'P + \frac{\Delta x^3}{1.2.3}{}^{x}d'''\,{}^{y}d'P + \frac{\Delta x^4}{1..4}{}^{x}d^{\text{iv}}\,{}^{y}d'P + \ldots\right)$$

$$+ \frac{\Delta y^2}{1.2}\left(\ldots\ {}^{y}d''P + \frac{\Delta x}{1}{}^{x}d'\,{}^{y}d''P + \frac{\Delta x^2}{1.2}{}^{x}d''\,{}^{y}d''P + \frac{\Delta x^3}{1.2.3}{}^{x}d'''\,{}^{y}d''P + \ldots\right)$$

$$+ \frac{\Delta y^3}{1.2.3}\left(\ldots\ldots\ {}^{y}d'''P + \frac{\Delta x}{1}{}^{x}d'\,{}^{y}d'''P + \frac{\Delta x^2}{1.2}{}^{x}d''\,{}^{y}d'''P + \ldots\right)$$

$$+ \frac{\Delta y^4}{1 \ldots 4}\left(\ldots\ldots\ldots\ {}^{y}d^{\text{iv}}P + \frac{\Delta x}{1}{}^{x}d'\,{}^{y}d^{\text{iv}}P + \ldots\right)$$

$$+ \frac{\Delta y^5}{1 \ldots 5}\left(\ldots\ldots\ldots\ldots\ {}^{y}d^{v}P + \ldots\right)$$

Atqui ${}^{y}\overset{x}{P} = {}^{x}\overset{y}{P}$; proinde æquando terminos harum expreſſionum mutationibus Δx & Δy eodem modo affectos, ſequentes oriuntur æquationes:

${}^{y}d'\,{}^{x}d'p = {}^{x}d'\,{}^{y}d'p$

${}^{y}d''\,{}^{x}d'p = {}^{x}d'\,{}^{y}d''p \quad {}^{y}d'\,{}^{x}d''p = {}^{x}d''\,{}^{y}d'p$

${}^{y}d'''\,{}^{x}d'p = {}^{x}d'\,{}^{y}d'''p \quad {}^{y}d''\,{}^{x}d''p = {}^{x}d''\,{}^{y}d''p \quad {}^{y}d'\,{}^{x}d'''p = {}^{x}d'''\,{}^{y}d'p$

${}^{y}d^{\text{iv}}\,{}^{x}d'p = {}^{x}d'\,{}^{y}d^{\text{iv}}p \quad {}^{y}d''\,{}^{x}d'''p = {}^{x}d'''\,{}^{y}d''p \quad {}^{y}d'''\,{}^{x}d''p = {}^{x}d''\,{}^{y}d'''p \quad {}^{y}d'\,{}^{x}d^{\text{iv}}p = {}^{x}d^{\text{iv}}\,{}^{y}d'p$

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Generatim ${}^{y}d^{N}\,{}^{x}d^{M}p = {}^{x}d^{M}\,{}^{y}d^{N}p$.

Exempla. 1°. Sit $p = xy$. ${}^{x}d'p = y \quad {}^{y}d'p = x$

${}^{x}d''p = 0 \quad {}^{y}d''p = 0$

${}^{y}d'\,{}^{x}d'p = 1 \quad {}^{x}d'\,{}^{y}d'p = 1.$

2°. Sit $p = x^4y^3$. ${}^{x}d'p = 4x^3y^3 \quad {}^{y}d'p = 3x^4y^2 \quad {}^{y}d'\,{}^{x}d'p = 4.3.x^3yy = {}^{x}d'\,{}^{y}d'p$

${}^{x}d''p = 4.3xxy^3 \quad {}^{y}d''p = 3.2x^4y \quad {}^{y}d''\,{}^{x}d'p = 4.3.2x^3y = {}^{x}d'\,{}^{y}d''p$

${}^{x}d'''p = 4.3.2xy^3 \quad {}^{y}d'''p = 3.2.1.x^4 \quad {}^{y}d'''\,{}^{x}d'p = 4.3.2.1x^3 = {}^{x}d'\,{}^{y}d'''p$

${}^{x}d^{\text{iv}}p = 4.3.2.1.y^3$

§. 162.

§. 162. Ex §. præcedente (insistendo vestigiis Capitum I. II.) deducitur determinatio mutationum simultanearum duarum quantitatum variabilium x, y, atque functionis ipsarum P; ac speciatim exponentis differentialis harum mutationum. Consideratis nimirum functione P & quantitatibus mutabilibus x, y, tanquam functionibus unius ejusdemque quantitatis mutabilis p;

erit $\frac{dP}{dp} = \frac{dx}{dp}\,{}^{x}d'P + \frac{dy}{dp}\,{}^{y}d'P$.

Exempla. 1°. Sit $P = xy$. Erit $\frac{dP}{dp} = \frac{dx}{dp}y + \frac{dy}{dp}x$ (ut notum §. 26.).

2°. Sit $P = x^4y^3$; itaque ${}^{x}d'P = 4x^3y^3$, ${}^{y}d'P = 3x^4yy$.

$$\text{Erit } \frac{dP}{dp} = 4x^3y^3\frac{dx}{dp} + 3x^4yy\frac{dy}{dp}.$$

Scilicet ut habeatur exponens differentialis $\frac{dP}{dp}$: sumantur exponentes differentiales ${}^{x}d'P$, ${}^{y}d'P$; qui multiplicentur per exponentes $\frac{dx}{dp}$, $\frac{dy}{dp}$ respective.

§. 163. Data igitur æquatione differentiali $\frac{dP}{dx} = \frac{dx}{dp}X + \frac{dy}{dp}Y$: ut æquationi huic respondeat æquatio integralis, terminis finitis expressa; oportet, sit $X = {}^{x}d'P$, & $Y = {}^{y}d'P$, proindeque ${}^{y}d'X = {}^{x}d'Y$.

Exempla. 1°. Sit $\frac{dP}{dp} = \frac{dx}{dp}x + \frac{dy}{dp}y$; igitur $X = x$, $Y = y$: cum ita sit $\begin{matrix} {}^{y}d'X = 0 \\ {}^{x}d'Y = 0 \end{matrix}$; contradictionem non involvit, ut æquationi differentiali propositæ respondeat æquatio integralis, quæ sit $2P = xx + yy$.

$$2°.\ \text{Sit } \frac{dP}{dp} = \frac{dx}{dp}(3xxyy + y^4) + \frac{dy}{dp}(2x^3y + 4xy^3 + 5y^4)$$
$$= \frac{dx}{dp}X + \frac{dy}{dp}Y.$$

Tum ${}^{y}d'X = 2.3xxy + 4y^3$, ${}^{x}d'Y = 2.3xxy + 4y^3$;

ac sit $P = x^3yy + xy^4 + y^5$.

3°. Sit

3°. Sit $\frac{\mathrm{d}P}{\mathrm{d}p} = \frac{\mathrm{d}x}{\mathrm{d}p}(y^4+x^3y) + \frac{\mathrm{d}y}{\mathrm{d}p}(xxyy+y^4)$

$$= \frac{\mathrm{d}x}{\mathrm{d}p}X \qquad + \frac{\mathrm{d}y}{\mathrm{d}p}Y.$$

Fiunt ${}^y\mathrm{d}'X = 4y^3+x^3$, ${}^x\mathrm{d}'Y = 2xyy$, et æquationi propoſitæ nulla reſpondet æquatio integralis terminis finitis expreſſa.

§. 164. Ex æquatione differentiali primi gradus fluunt æquationes differentiales graduum reliquorum: quod æquationis differentio-differentialis exemplo illuſtrare ſufficiat.

Quoniam $\frac{\mathrm{d}P}{\mathrm{d}p} = \frac{\mathrm{d}x}{\mathrm{d}p}{}^x\mathrm{d}'P + \frac{\mathrm{d}y}{\mathrm{d}p}{}^y\mathrm{d}'P$;

ſequitur $\frac{\mathrm{dd}P}{\mathrm{d}p^2} = \frac{\mathrm{dd}x}{\mathrm{d}p^2}{}^x\mathrm{d}'P + \left(\frac{\mathrm{d}x}{\mathrm{d}p}\right)^2 {}^y\mathrm{d}''P + \frac{\mathrm{d}x}{\mathrm{d}p}\cdot\frac{\mathrm{d}y}{\mathrm{d}p}{}^y\mathrm{d}'\,{}^x\mathrm{d}'P$

$$+\frac{\mathrm{d}y}{\mathrm{d}p}\cdot\frac{\mathrm{d}x}{\mathrm{d}p}{}^x\mathrm{d}'\,{}^y\mathrm{d}'P + \left(\frac{\mathrm{d}y}{\mathrm{d}p}\right)^2 {}^y\mathrm{d}''P + \frac{\mathrm{dd}y}{\mathrm{d}p^2}{}^y\mathrm{d}'P$$

$$= \frac{\mathrm{dd}x}{\mathrm{d}p^2}{}^x\mathrm{d}'P + \left(\frac{\mathrm{d}x}{\mathrm{d}p}\right)^2 {}^x\mathrm{d}''P + 2\frac{\mathrm{d}x}{\mathrm{d}p}\cdot\frac{\mathrm{d}y}{\mathrm{d}p}{}^y\mathrm{d}'\,{}^x\mathrm{d}'P + \left(\frac{\mathrm{d}y}{\mathrm{d}p}\right)^2 {}^y\mathrm{d}''P + \frac{\mathrm{dd}y}{\mathrm{d}p^2}{}^y\mathrm{d}'P$$

$$= \qquad \left(\frac{\mathrm{d}x}{\mathrm{d}p}\right)^2 {}^x\mathrm{d}''P + 2\frac{\mathrm{d}x}{\mathrm{d}p}\cdot\frac{\mathrm{d}y}{\mathrm{d}p}{}^y\mathrm{d}'\,{}^x\mathrm{d}'P + \left(\frac{\mathrm{d}y}{\mathrm{d}p}\right)^2 {}^y\mathrm{d}''P,$$ ſi exponentes differentiales $\frac{\mathrm{d}x}{\mathrm{d}p}$, $\frac{\mathrm{d}y}{\mathrm{d}p}$ ſupponuntur eſſe conſtantes.

§. 165. Ex formula pro duabus quantitatibus mutabilibus §. 161. expoſita, ſequitur formula pro functione trium quantitatum mutabilium. Fit enim

$$\overset{x}{\underset{z}{\overset{y}{}}}P = P + \frac{\Delta z}{1}{}^z\mathrm{d}'P + \frac{\Delta z^2}{1.2}{}^z\mathrm{d}''P + \frac{\Delta z^3}{1.2.3}{}^z\mathrm{d}'''P + \frac{\Delta z^4}{1\ldots4}{}^z\mathrm{d}^{\mathrm{iv}}P + \ldots.$$

$$+\frac{\Delta x}{1}\Big(\;{}^x\mathrm{d}'P + \frac{\Delta z}{1}{}^z\mathrm{d}'\,{}^x\mathrm{d}'P + \frac{\Delta z^2}{1.2}{}^z\mathrm{d}''\,{}^x\mathrm{d}'P + \frac{\Delta z^3}{1.2.3}{}^z\mathrm{d}'''\,{}^x\mathrm{d}'P + \ldots.\Big)$$

$$+\frac{\Delta y}{1}\Big(\;{}^y\mathrm{d}'P + \frac{\Delta z}{1}{}^z\mathrm{d}'\,{}^y\mathrm{d}'P + \frac{\Delta z^2}{1.2}{}^z\mathrm{d}''\,{}^y\mathrm{d}'P + \frac{\Delta z^3}{1.2.3}{}^z\mathrm{d}'''\,{}^y\mathrm{d}'P + \ldots.\Big)$$

$$+\frac{\Delta x^2}{1.2}\Big(\;{}^x\mathrm{d}''P + \frac{\Delta z}{1}{}^z\mathrm{d}'\,{}^x\mathrm{d}''P + \frac{\Delta z^2}{1.2}{}^z\mathrm{d}''\,{}^x\mathrm{d}''P + \ldots.\Big)$$

$$+2\frac{\Delta x\,\Delta y}{1.2}\Big(\;{}^x\mathrm{d}'\,{}^y\mathrm{d}'P + \frac{\Delta z}{1}{}^z\mathrm{d}'\,{}^x\mathrm{d}'\,{}^y\mathrm{d}'P + \frac{\Delta z^2}{1.2}{}^z\mathrm{d}''\,{}^x\mathrm{d}'\,{}^y\mathrm{d}'P + \ldots.\Big)$$

$$+\frac{\Delta y^2}{1.2}\Big(\qquad {}^{y}d''p \quad + \frac{\Delta z}{1}\,{}^{z}d'\,{}^{y}d''p + \ldots\ldots\Big)$$

$$+\frac{\Delta x^3}{1.2.3}\Big(\qquad {}^{x}d'''p \quad + \ldots\ldots\Big)$$

$$+3\frac{\Delta x^2.\Delta y}{1.2.3}\Big(\qquad {}^{x}d''\,{}^{y}d'p + \ldots\ldots\Big)$$

$$+3\frac{\Delta x.\Delta y^2}{1.2.3}\Big(\qquad {}^{x}d'\,{}^{y}d''p + \ldots\ldots\Big)$$

$$+\frac{\Delta y^3}{1.2.3}\Big(\qquad {}^{y}d'''p + \ldots\ldots\Big)$$

$$+ \quad - \quad - \quad - \quad - \quad - \quad -$$

$$+ \quad - \quad - \quad - \quad - \quad - \quad -$$

Si varietur ordo, juxta quem tres quantitates mutabiles x, y, z fucceffive poffunt variari; fex nempe modis, quibus inter fe permutantur; expreffiones ${}^{\substack{x\\y\\z}}P$, ${}^{\substack{x\\z\\y}}P$, ${}^{\substack{y\\x\\z}}P$, ${}^{\substack{y\\z\\x}}P$, ${}^{\substack{z\\x\\y}}P$, ${}^{\substack{z\\y\\x}}P$ erunt inter fe æquales. Unde varia confequuntur theoremata; & nominatim æqualitas terminorum, qui iisdem productis mutationum Δx, Δy, Δz afficiuntur. Sic v. gr. obtinentur æquationes fequentes: ${}^{z}d'\,{}^{x}d'\,{}^{y}d'p = {}^{z}d'\,{}^{y}d'\,{}^{x}d'p = {}^{x}d'\,{}^{y}d'\,{}^{z}d'p = {}^{x}d'\,{}^{z}d'\,{}^{y}d'p = {}^{y}d'\,{}^{x}d'\,{}^{z}d'p = {}^{y}d'\,{}^{z}d'\,{}^{x}d'p$. Ex quibus porro nectuntur corollaria analoga iis, quæ de functionibus duarum variabilium notavimus §. 162. 163.

Hinc facilis eft transitus ad functiones quatuor, quinque, & plurium quotcunque quantitatum mutabilium. Quare viam indicaffe fufficiat, qua tractatio hæc ab idea infiniti liberatur; & quod ad applicationes attinet, qui plura defideraverit, adeat Euleri *Calculum differentialem* Cap. VII. & VIII. partis prioris.

Caput decimum octavum.

De maximis et minimis; et de punctis flexus contrarii curvarum.

Quæftiones ad maxima & minima pertinentes adeo frequenter occurrunt, non in pura tantum mathefi, fed potiffimum quoque in applicata, quæ uberrimis utiliffimisque earum applicationibus anfam præbet; ut caput hoc fedulo evol-

evolvi mereatur. Quoniam autem intimus eſt ipſarum cum quibusdam curvarum ſymptomatibus nexus, quorum contemplatio magnam huic doctrinæ lucem affundit; utrumque argumentum ſimul complecti e re eſſe cenſeo.

§. 166. *Definitio* 1. Sit P functio quantitatis alicujus mutabilis x, quæ facta $x = a$ $\begin{matrix}\text{major}\\ \text{minor}\end{matrix}$ fiat, quam ſi quantitati x major minorve valor $x + \Delta x$, $x - \Delta x$ tribuatur, utut parum valores hi ab a differant: valor functionis P valori $x = a$ reſpondens vocatur $\begin{matrix}\textit{maximus}\\ \textit{minimus}\end{matrix}$.

Exempla. Numerus datus a in duas partes x, $x - a$ dividatur; & fiat productum $x(a - x)$ ex his partibus. Productum hoc functio eſt unius harum partium v. gr. x; qua creſcente a zero inde usque ad $\frac{1}{2}a$ creſcit etiam productum $x(a - x)$. Eadem autem parte creſcere pergente, idem productum decreſcit: ideoque productum hoc eſt omnium maximum, quando pars x æqualis eſt dimidio numero dividendo; ſeu quando ambæ partes ſunt invicem æquales.

Contra ſi fiant quadrata ex iisdem partibus; ſumma horum eſt omnium minima, quando ambæ partes ſunt invicem æquales.

Definitio 2. Sit M punctum quodpiam alicujus curvæ: ex quo ducantur recta curvam tangens TMT'; & recta MP axi ordinatim applicata, cui reſpondet abſciſſa AP. Tum $\begin{matrix}\text{aucta}\\ \text{imminuta}\end{matrix}$ abſciſſa, quæ fit $\begin{matrix}AP'\\ A'P\end{matrix}$, agantur $\begin{matrix}P'M'\\ 'P'M\end{matrix}$ rectæ axi ordinatim applicatæ, quæ curvæ in $\begin{matrix}M'\\ 'M\end{matrix}$ punctis, & tangenti in $\begin{matrix}N'\\ 'N\end{matrix}$ punctis occurrant. Si ſit $N'P' \gtrless M'P'$, utut parum abſciſſa AP' ab abſciſſa AP differat; arcus MM' dicitur $\begin{matrix}\textit{concavus}\\ \textit{convexus}\end{matrix}$ verſus axem AB: & ſi ſimul ſit $'N'P \gtrless {}'M'P$, totus arcus $'MMM$ verſus eundem axem eſt $\begin{matrix}\text{concavus}\\ \text{convexus}\end{matrix}$. Si vero ſimul ſit $N'P' \gtrless M'P'$ & $'N'P \lesseqgtr {}'M'P$, quo caſu arcus MM' $\begin{matrix}\text{concavus}\\ \text{convexus}\end{matrix}$ eſt verſus axem AB, dum arcus $'MM$ eſt verſus eundem axem $\begin{matrix}\text{convexus}\\ \text{concavus}\end{matrix}$; tum M punctum ſeparat arcum $\begin{matrix}\text{concavum}\\ \text{convexum}\end{matrix}$ curvæ ab ejus arcu $\begin{matrix}\text{convexo}\\ \text{concavo}\end{matrix}$, & dicitur *punctum flexus contrarii* curvæ. Fig. 45. 1°. 2°. Fig. 46.

§. 161. Hinc jam oſtendi poteſt nexus, qui ſymptomata inter curvarum, quibus verſus axem $\begin{matrix}\text{concavæ}\\ \text{convexæ}\end{matrix}$ ſunt, & doctrinam functionum, quæ $\begin{matrix}\text{maximæ}\\ \text{minimæ}\end{matrix}$ fiunt, intercedit.

Etenim

Etenim quando arcus totus est versus axem, ad quem refertur, $\substack{\text{concavus}\\ \text{convexus}}$: si recta duci potest arcum hunc tangens, eademque axi parallela; recta axi ordinatim applicata ex puncto contactus ducta $\substack{\text{major}\\ \text{minor}}$ est rectis axi ordinatim applicatis ad utramque ejus partem sitis; proinde ordinata hæc omnium $\substack{\text{maxima}\\ \text{minima}}$ est. Quare ordinatis curvæ sumtis proportionalibus functioni P quantitatis mutabilis, quæ ipsa per abscissas AP designetur; functio P omnium $\substack{\text{maxima}\\ \text{minima}}$ determinatur per rectam axi ordinatim applicatam a puncto curvæ, ubi hanc tangit recta axi parallela.

Quoniam autem, quando tangens axi est parallela, fit $\frac{dy}{dx}=0$, seu $\frac{dP}{dx}=0$ (§. 93.); $\substack{\text{maxima}\\ \text{minima}}$ determinantur, si fiat $\frac{dP}{dx}=0$.

§. 168. Si curva proposita, cujus ordinatæ sunt functioni P proportionales, unico constet ramo, qui totus sit versus axem $\substack{\text{concavus}\\ \text{convexus}}$: recta tangens axi parallela (si qua sit) est unica; & æquationis $\frac{dP}{dx}=0$ unica est radix realis. Si vero curva proposita sit undulatoria, ita ut alternis vicibus concava & convexa fiat versus axem, ad quem refertur: in unaquaque unda agi poterit recta tangens axi parallela; & proinde totidem erunt ordinatæ alternis vicibus maximæ aut minimæ, quot sunt undæ curvæ propositæ. Multitudo hæc $\substack{\text{maximorum}\\ \text{minimorum}}$ designatur multitudine radium realium æquationis $\frac{dP}{dx}=0$, quibus etiam $\substack{\text{maximi}\\ \text{minimi}}$ functionis P valores alternis vicibus respondent.

§. 169. In definitionibus præcedentibus (§. 166.) supposui: valorem a quantitatis mutabilis x, cui $\substack{\text{maximus}\\ \text{minimus}}$ functionis P valor respondet, ejusmodi esse, ut sumi possint quantitatis x valores ipsa a tam majores quam minores; seu (quod eodem redit) supposui: curvam, cujus ordinatæ sunt functioni P proportionales, ad utramque ordinatæ omnium $\substack{\text{maximæ}\\ \text{minimæ}}$ partem extendi. Fieri autem potest, ut valor a quantitatis mutabilis x, cui $\substack{\text{maximus}\\ \text{minimus}}$ functionis ejus valor respondet, sit simul & ipse $\substack{\text{maximus}\\ \text{minimus}}$ quantitatis mutabilis x valor: ita ut

non

non poſſint ſimul accipi valores $x+\Delta x$, $x-\Delta x$, ſeu abſciſſæ curvæ, his valoribus reſpondentes; & curva ceſſet ad alteram ordinatæ MP, abſciſſæ $x=a$ reſpondentis, partem. Quod ſi locum habeat: ordinatæ abſciſſis $x+\Delta x$ v. gr. reſpondentes fiunt imaginariæ; & proinde functio P radice $(x-a)^{\frac{2m+1}{2n}}$ vel $(a-x)^{\frac{2m+1}{2n}}$ afficitur.

Ut argumentum hoc, quantum fieri poteſt, dilucide pertractem; ſingulare hoc $\begin{matrix}\text{maximorum}\\ \text{minimorum}\end{matrix}$ genus ſeorſim perpendam: & de iis functionibus, quæ nonniſi poteſtates $(x-a)^n$, $(a-x)^n$, quarum exponens n eſt numerus integer poſitivus, involvunt, unice primum agam; ceteraſque aliis potentiis affectas demum conſiderabo, poſtquam, quæ ad priores pertinent, erunt declarata.

Et cum adeo arctus ſit nexus inter ſymptomata curvarum, quibus ſunt verſus axem $\begin{matrix}\text{concavæ}\\ \text{convexæ}\end{matrix}$, & ſymptomata functionum, quibus $\begin{matrix}\text{maximæ}\\ \text{minimæ}\end{matrix}$ fieri poſſunt; priorem determinationem præmittam.

§. 170. Sit M punctum curvæ, ad quod ducta eſt recta tangens $'TT'$. Sit M' aliud curvæ hujus punctum, puncto M utut proximum; a quo agatur recta $M'P'$ axi ordinatim applicata, quæ tangenti TT' in N' puncto occurrat. Tum ex puncto M agatur Mm' recta axi parallela, quæ rectæ $M'P'$ in m' puncto occurrat. Ad alteram ordinatæ MP partem agatur quoque recta $'M'P$, axi ordinatim applicata, quæ tangenti in $'N$ puncto, & rectæ Mm' in $'m$ puncto occurrat.

Sint $MP=y$, $M'P'=y'$, $'M'P='y$, $PP'='PP=\Delta x$.

Erit $y'=y+\frac{\Delta x}{1}\cdot\frac{dy}{1}+\frac{\Delta x^2}{1.2}\cdot\frac{ddy}{dx^2}+\frac{\Delta x^3}{1.2.3}\cdot\frac{d^3y}{dx^3}+\frac{\Delta x^4}{1\ldots4}\cdot\frac{d^4y}{dx^4}+\ldots\ldots$ (§. 32.)

$$'y=y-\frac{\Delta x}{1}\cdot\frac{dy}{dx}+\frac{\Delta x^2}{1.2}\cdot\frac{ddy}{dx^2}-\frac{\Delta x^3}{1.2.3}\cdot\frac{d^3y}{dx^3}+\frac{\Delta x^4}{1\ldots4}\cdot\frac{d^4y}{dx^4}-\ldots\ldots$$

ſeu $\begin{matrix}y'\\ 'y\end{matrix}=y\pm\frac{\Delta x}{1}\cdot\frac{dy}{dx}+\frac{\Delta x^2}{1.2}\cdot\frac{ddy}{dx^2}\pm\frac{\Delta x^3}{1.2.3}\cdot\frac{d^3y}{dx^3}+\frac{\Delta x^4}{1\ldots4}\cdot\frac{d^4y}{dx^4}\pm\ldots\ldots$

Atqui $\begin{matrix}N'P'\\ 'N'P\end{matrix}=y\pm\frac{\Delta x}{1}\cdot\frac{dy}{dx}$ (§. 40.)

 Ergo

Ergo casu concavitatis $\frac{N'M'}{'N'M} = -\frac{\Delta x^2}{1.2} \cdot \frac{ddy}{dx^2} \mp \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} - \frac{\Delta x^4}{1 \ldots 4} \cdot \frac{d^4y}{dx^4} \mp \ldots.$

casu convexitatis $\frac{N'M'}{'N'M} = +\frac{\Delta x^2}{1.2} \cdot \frac{ddy}{dx^2} \pm \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} + \frac{\Delta y^4}{1 \ldots 4} \cdot \frac{d^4y}{dx^4} \pm \ldots.$

Posito autem, functionem P alias factoris $x-a$, ipsi $x=a$ respondentis, potestates non continere, nisi quarum exponentes sunt numeri integri & positivi: exponentes differentiales successivi $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, ... aut factore $x-a$ non afficiuntur, aut non alias involvunt ejus potestates, nisi quarum exponentes sunt integri positivi; proinde facto $x=a$, exponentes hi differentiales aut finitam obtinent magnitudinem, aut evanescunt, neque impossibilis aut infiniti $\frac{1}{(x-a)^n}$ signo afficiuntur.

Quo posito, ut curva sit ad utramque puncti M partem concava; oportet, sit $-\left(\frac{\Delta x^2}{1.2} \cdot \frac{ddy}{dx^2} \pm \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} + \frac{\Delta x^4}{1 \ldots 4} \cdot \frac{d^4y}{dx^4} \pm \ldots\right)$ quantitas positiva, utut parvus sit ipsius Δx valor: proinde si $\frac{ddy}{dx^2}$ non evanescat; oportet, sit $\frac{ddy}{dx^2}$ quantitas negativa (§. 16.) propter Δx^2 semper positivum.

Casu autem convexitatis necesse est, ut sit $+\left(\frac{\Delta x^2}{1.2} \cdot \frac{ddy}{dx^2} \pm \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} + \frac{\Delta x^4}{1 \ldots 4} \cdot \frac{d^4y}{dx^4} \pm \ldots\right)$ quantitas positiva, utcunque exiguus sit ipsius Δx valor; quod (propter Δx^2 semper positivum) fieri nequit, nisi sit $\frac{ddy}{dx^2}$ quantitas positiva.

Proinde, exponente differentiali $\frac{ddy}{dx^2}$ non evanescente, curva versus axem $\begin{matrix}\text{concava}\\\text{convexa}\end{matrix}$ est, prouti exponens differentialis $\frac{ddy}{dx^2}$ est $\begin{matrix}\text{negativus}\\\text{positivus}\end{matrix}$.

§. 171. Quoniam autem $\frac{y'}{'y} = y \pm \frac{\Delta x}{1} \cdot \frac{dy}{dx} + \frac{\Delta x^2}{1.2} \cdot \frac{ddy}{dx^2} \pm \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} + \frac{\Delta x^4}{1..4} \cdot \frac{d^4y}{dx^4} \pm \ldots;$ casu, quo y est maxima, debet esse simul

$\mp \frac{\Delta x}{1} \cdot \frac{dy}{dx} - \frac{\Delta x^2}{1.2} \cdot \frac{ddy}{dx^2} \mp \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} - \frac{\Delta x^4}{1 \ldots 4} \cdot \frac{d^4y}{dx^4} \mp \ldots > 0$: quod juxta suppositio-

nem

nem (§. 169.) fieri nequit, nisi sit $\frac{dy}{dx} = 0$, & $\frac{ddy}{dx^2}$ negativa, seu curva versus axem concava.

Casu vero, quo y minima est, debet esse

$$\pm \frac{\Delta x}{1} \cdot \frac{dy}{dx} + \frac{\Delta x^2}{1.2} \cdot \frac{ddy}{dx^2} \pm \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} + \frac{\Delta x^4}{1 \ldots 4} \cdot \frac{d^4y}{dx^4} \pm \ldots > 0;$$

quod fieri nequit (juxta easdem suppositiones), nisi sit $\frac{dy}{dx} = 0$, & $\frac{ddy}{dx^2}$ positiva, seu curva versus axem convexa.

Posita igitur functione P hujus formæ $\frac{\varphi x(x-a)^n + \varphi' x}{\varphi x(a-x)^n + \varphi' x}$, ubi n est numerus integer positivus: facto $\frac{dP}{dx} = 0$, cui respondeat $x = a$; exponentibus differentialibus successivis $\frac{ddP}{dx^2}$, $\frac{d^3P}{dx^3}$, $\frac{d^4P}{dx^4}$, ... factorem impossibilem $\frac{1}{(x-a)^n}$ non involventibus, & exponente differentiali $\frac{ddy}{dx^2}$ non evanescente; functio P fit $\frac{\text{maxima}}{\text{minima}}$, prouti exponens differentialis $\frac{ddP}{dx^2}$, substitutione $x = a$ in ejus expressione facta, est $\frac{\text{negativus}}{\text{positivus}}$.

Exemplum primum. Sit $P = 2ax - xx$; ideoque $\frac{dP}{dx} = 2a - 2x$, $\frac{ddP}{dx^2} = -2$: facto $\frac{dP}{dx} = 0$, est $x = a$, & $\frac{ddP}{dx^2}$ est semper negativus; proinde valori $x = a$ respondet P maximum.

Exemplum secundum. Sit $P = xx + (a-x)^2$; igitur $\frac{dP}{dx} = 2(2x-a)$, $\frac{ddP}{dx^2} = +4$: facto $\frac{dP}{dx} = 0$, est $x = \frac{1}{2}a$, & $\frac{ddP}{dx^2}$ semper est positivus; proinde functio P, ipsi $x = \frac{1}{2}a$ respondens, est minima.

Exemplum tertium. Sit $P = xx - (a-x)^2$: erit $\frac{dP}{dx} = 2x + 2(a-x) = +2a$. Proinde $\frac{dP}{dx}$ nunquam fit zero; & functio $P = 2a(x-a)$ nunquam fit $\frac{\text{maxima}}{\text{minima}}$.

Exemplum quartum. Sit $P = x^3 - axx + bx + c$

$$\frac{dP}{dx} = 3xx - 2ax + b. \text{ Facto } \frac{dP}{dx} = 0, \text{ fit } x = \frac{a \pm \sqrt{(aa-3b)}}{3}$$

$$\frac{ddP}{dx^2} = 6x - 2a \qquad \frac{ddP}{dx^2} = \pm \sqrt{(aa-3b)}.$$

Pro-

Proinde posito $aa > 3b$, si $x = a + \sqrt{(aa-3b)}$, est P = minimum;
si $x = a - \sqrt{(aa-3b)}$, est P = maximum.

§. 172. Exponens differentialis $\frac{ddy}{dx^2}$ evanescat (sive $\frac{dy}{dx}$ simul evanescat, sive non). Hoc casu fit $\frac{N'M'}{'N'M} = \mp \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} - \frac{\Delta x^4}{1 \ldots 4} \cdot \frac{d^4y}{dx^4} \mp \frac{\Delta x^5}{1 \ldots 5} \cdot \frac{d^5y}{dx^5} - \ldots$ casu concavitatis

$$\frac{N'M'}{'N'M} = \pm \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} + \frac{\Delta x^4}{1 \ldots 4} \cdot \frac{d^4y}{dx^4} \pm \frac{\Delta x^5}{1 \ldots 5} \cdot \frac{d^5y}{dx^5} + \ldots \text{ casu convexitatis.}$$

Quare posito, exponentem differentialem $\frac{d^3y}{dx^3}$ non evanescere, & seriem exponentium differentialium $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, $\frac{d^5y}{dx^5} \ldots$ signo $\frac{1}{(x-a)^n}$ non affici; non potest simul esse $\begin{matrix} N'P' > M'P' \\ 'N'P > 'M'P \end{matrix}$ vel $\begin{matrix} N'P' < M'P' \\ 'N'P < 'M'P \end{matrix}$. Proinde curva non potest esse ad utramque puncti M partem $\begin{matrix}\text{concava}\\ \text{convexa}\end{matrix}$; sed punctum M concavum & convexum curvæ arcum invicem separat, seu est punctum flexus contrarii curvæ.

Quodsi autem simul fit $\frac{dy}{dx} = 0$, $\frac{ddy}{dx^2} = 0$, $\frac{d^3y}{dx^3} = 0$: ut curva sit versus axem $\begin{matrix}\text{concava}\\ \text{convexa}\end{matrix}$, debet esse $\frac{d^4y}{dx^4}$ $\begin{matrix}\text{negativus}\\ \text{positivus}\end{matrix}$; & proinde, ut functio P fiat omnium $\begin{matrix}\text{maxima}\\ \text{minima}\end{matrix}$, debet esse $\frac{d^4y}{dx^4}$ $\begin{matrix}\text{negativa}\\ \text{positiva}\end{matrix}$.

Exemplum primum. Sit $y = xx(a-x)$

$$\frac{dy}{dx} = x(2a-3x)$$

$$\frac{ddy}{dx^2} = 2a - 6x$$

$$\frac{d^3y}{dx^3} = -6$$

$\frac{dy}{dx}$ fit zero, si $\begin{matrix} x = 0 \\ 2a-3x = 0 \end{matrix}$, seu $\begin{matrix} x = 0 \\ x = \frac{2}{3}a \end{matrix}$. Tum vero

$\begin{matrix} \frac{ddy}{dx^2} = +2a \\ \frac{ddy}{dx^2} = -2a \end{matrix}$. Proinde abscissis $x = \begin{matrix} 0 \\ \frac{2}{3}a \end{matrix}$ respondent y $\begin{matrix}\text{minima}\\ \text{maxima}\end{matrix}$.

Sit

Sit autem $\frac{ddy}{dx^2} = 0$, & proinde $x = \frac{1}{3}a$; huic abſciſſæ reſpondet punctum flexus contrarii. Vide fig. 47. qua delineatur curſus curvæ, cujus æquatio eſt $y = xx(a - x)$.

Exemplum ſecundum. Sit

$$\begin{aligned} y &= x^3(a-x) \\ \frac{dy}{dx} &= xx(3a-4x) \\ \frac{ddy}{dx^2} &= 6x(a-2x) \\ \frac{d^3y}{dx^3} &= 6(a-4x) \\ \frac{d^4y}{dx^4} &= -24. \end{aligned}$$

Sit $\frac{dy}{dx} = 0$: erunt $x = \begin{matrix} 0 \\ 0 \\ \frac{3}{4}a \end{matrix}$; $\frac{ddy}{dx^2} = \begin{matrix} 0 \\ 0 \\ -\frac{9}{4}a \end{matrix}$, $\frac{d^3y}{dx^3} = \begin{matrix} +6a \\ +6a \\ -12a \end{matrix}$, $\frac{d^4y}{dx^4} = -24$.
Proinde abſciſſæ $x = 0$ reſpondet punctum flexus contrarii; & abſciſſæ $\frac{3}{4}a$ reſpondet maximum.

Sit autem $\frac{ddy}{dx^2} = 0$: erunt $x = \begin{matrix} 0 \\ \frac{1}{2}a \end{matrix}$, $\frac{d^3y}{dx^3} = \begin{matrix} +6a \\ -6a \end{matrix}$; quare iterum abſciſſis $\begin{matrix} 0 \\ \frac{1}{2}a \end{matrix}$ reſpondet punctum flexus contrarii.

In figura 48. delineatur curſus curvæ, cujus æquatio eſt $y = x^3(a-x)$.

§. 173. Simul ſint $\frac{ddy}{dx^2} = 0$, $\frac{d^3y}{dx^4} = 0$, $\frac{d^4y}{dx^4} = 0$.

Erit $\frac{N'M'}{'N'M} = \mp \frac{\Delta x^5}{1\ldots5} \cdot \frac{d^5y}{dx^5} - \frac{\Delta x^6}{1\ldots6} \cdot \frac{d^6y}{dx^6} \mp \frac{\Delta x^7}{1\ldots7} \cdot \frac{d^7y}{dx^7} - \ldots$ caſu concavitatis

$\frac{N'M'}{'N'M} = \pm \frac{\Delta x^5}{1\ldots5} \cdot \frac{d^5y}{dx^5} + \frac{\Delta x^6}{1\ldots6} \cdot \frac{d^6y}{dx^6} \pm \frac{\Delta x^7}{1\ldots7} \cdot \frac{d^7y}{dx^7} + \ldots$ caſu convexitatis.

Proinde poſito: exponentem differentialem $\frac{d^5y}{dx^5}$ non evaneſcere, & ſeriem exponentium differentialium $\frac{d^5y}{dx^5}$, $\frac{d^6y}{dx^6}$, $\frac{d^7y}{dx^7}$ ... factorem impoſſibilem $\frac{1}{(x-a)^n}$ non comprehendere; nequit pro omnibus utut parvis mutationis Δx valoribus ſimul eſſe $\frac{N'P'}{'N'P} > \frac{M'P'}{'M'P}$ caſu concavitatis, & $\frac{N'P'}{'N'P} < \frac{M'P'}{'M'P}$ caſu concavitatis: quare hoc caſu punctum M eſt punctum flexus contrarii.

Sint autem ſimul $\frac{dy}{dx} = 0$, $\frac{ddy}{dx^2} = 0$, $\frac{d^3y}{dx^3} = 0$, $\frac{d^4y}{dx^4} = 0$, $\frac{d^5y}{dx^5} = 0$.

Quoniam eſt $\overset{'}{\underset{y}{y}} = y \ldots + \frac{\Delta x^6}{1\ldots6} \cdot \frac{d^6y}{dx^6} + \frac{\Delta x^7}{1\ldots7} \cdot \frac{d^7y}{dx^7} + \frac{\Delta x^8}{1\ldots8} \cdot \frac{d^8y}{dx^8} \pm \ldots$

ut functio y fiat omnium $\begin{matrix}\text{maxima}\\\text{minima}\end{matrix}$, debet eſſe $\frac{d^6y}{dx^6}$ $\begin{matrix}\text{negativus}\\\text{poſitivus}\end{matrix}$.

§. 174. Porro omnes exponentes differentiales $\frac{ddy}{dx^2} \ldots \frac{d^6y}{dx^6}$ ſimul evaneſcant; unde $\frac{M'N'}{'M'N} = \ldots \mp \frac{\Delta x^7}{1\ldots7} \cdot \frac{d^7y}{dx^7} - \frac{\Delta x^8}{1\ldots8} \cdot \frac{d^8y}{dx^8} \mp \ldots$ caſu concavitatis,

$\frac{M'N'}{'M'N} = \ldots \pm \frac{\Delta x^7}{1\ldots7} \cdot \frac{d^7y}{dx^7} + \frac{\Delta x^8}{1\ldots8} \cdot \frac{d^8y}{dx^8} \pm \ldots$ caſu convexitatis.

Proinde $\frac{d^7y}{dx^7}$ non evaneſcente, non poteſt ſimul eſſe $\frac{M'N'}{'M'N} > \frac{M'P'}{'M'P}$ vel $\frac{M'N'}{'M'N} < \frac{M'P'}{'M'P}$; quare rurſus M eſt punctum flexus contrarii.

Sint autem $\frac{dy}{dx} = 0$, $\frac{ddy}{dx^2} = 0 \ldots \frac{d^7y}{dx^7} = 0$: quoniam eſt $\overset{'}{\underset{y}{y}} = y \ldots + \frac{\Delta x^8}{1\ldots8} \cdot \frac{d^8y}{dx^8} + \frac{\Delta x^9}{1\ldots9} \cdot \frac{d^9y}{dx^9} + \ldots$; ut y fiat omnium $\begin{matrix}\text{maxima}\\\text{minima}\end{matrix}$, debet eſſe $\frac{d^8y}{dx^8}$ $\begin{matrix}\text{negativus}\\\text{poſitivus}\end{matrix}$.

§. 175. Ex his ſatis ſuperque liquet methodus adhibenda, tam ut $\begin{matrix}\text{maxima}\\\text{minima}\end{matrix}$ functionis P (ſuppoſitionibus conſentaneæ), quam ut puncta flexus contrarii, ſi quæ occurrunt, determinentur. Et ſimul patet, eum eſſe utriusque hujus curvarum ſeu functionum ſymptomatis nexum mutuum, ut functiones ab uno ad alterum transeant; &, uno deficiente, alterum in locum ejus ſuccedat.

Generatim igitur regula hæc eſt. Si in ſerie exponentium differentialium $\frac{dy}{dx}$, $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3} \ldots \frac{d^my}{dx^m}$ impar horum exponentium numerus, a primo inde ordiundo, continue evaneſcit facto $x = a$; exponens autem par immediate ſequens non ſimul evaneſcit: functio propoſita omnium $\begin{matrix}\text{maxima}\\\text{minima}\end{matrix}$ eſt, prouti exponens hic par

par non evanescens est $\frac{\text{negativus}}{\text{positivus}}$. Quodsi vero, $\frac{dy}{dx}$ evanescente vel non evanescente posito $x=a$, impar exponentium differentialium, qui illum sequuntur, numerus simul evanescit, dum exponens differentialis sequens non evanescit simul; abscissae $x=a$ respondet punctum flexus contrarii.

Hinc posito m numero integro positivo, si sit $\frac{dP}{dx}=x^mQ$; abscissae $x=o$ respondet $\frac{\text{maximum}}{\text{minimum}}$ aut punctum flexus contrarii, prouti m impar est aut par.

1°. Sit $\frac{dP}{dx} = x^{2m-1}Q$

Erit
$$\frac{ddP}{dx^2} = x^{2m-2}Q'$$
$$\frac{d^3P}{dx^3} = x^{2m-3}Q''$$
$$\frac{d^4P}{dx^4} = x^{2m-4}Q'''$$
$$\vdots$$
$$\frac{d^{2M-II}P}{dx^{2m-2}} = x^2Q^{2M-III}$$
$$\frac{d^{2M-I}P}{dx^{2m-1}} = xQ^{2M-II}$$
$$\frac{d^{2M}P}{dx^{2m}} = Q^{2M-I}$$

Casu igitur, quo m est impar, numerus impar exponentium differentialium successivorum evanescit facto $x=o$, dum exponens differentialis sequens non evanescit: proinde functio P est $\frac{\text{maxima}}{\text{minima}}$.

2°. Sit $\frac{dP}{dx} = x^{2m}Q$

Erit
$$\frac{ddP}{dx^2} = x^{2m-1}Q'$$
$$\frac{d^3P}{dx^3} = x^{2m-2}Q''$$
$$\frac{d^4P}{dx^4} = x^{2m-2}Q'''$$
$$\vdots$$
$$\frac{d^{2M-II}P}{dx^{2m-2}} = x^3Q^{2M-III}$$
$$\frac{d^{2M-I}P}{dx^{2m-1}} = x^2Q^{2M-II}$$
$$\frac{d^{2M}P}{dx^{2m}} = xQ^{2M-I}$$
$$\frac{d^{2M+I}P}{dx^{2m+1}} = Q^{2M}.$$

Sed casu, quo m est par, numerus par exponentium differentialium successivorum evanescit facto $x=o$, dum exponens differentialis sequens non evanescit: abscissae igitur $x=o$ respondet punctum flexus contrarii.

§. 176.

§. 176. Ex æquationibus

$$\frac{dy}{dx} = 0 \quad \text{sequitur} \quad y = C$$

$$\frac{ddP}{dx^2} = 0 \qquad y = Cx + C'$$

$$\frac{d^3y}{dx^3} = 0 \qquad y = \frac{1}{1.2}Cxx + C'x + C''$$

$$\frac{d^4y}{dx^4} = 0 \qquad y = \frac{1}{1.2.3}Cx^3 + \frac{1}{1.2}C'xx + C''x + C'''$$

$$\frac{d^5y}{dx^5} = 0 \qquad y = \frac{1}{1.2.3.4}Cx^4 + \frac{1}{1.2.3}C'x^3 + \frac{1}{1.2}C''xx + C'''x + C''''$$

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Proinde æquatio curvæ, $\genfrac{}{}{0pt}{}{\text{maximam}}{\text{minimam}}$ ordinatam habentis, ab æquatione lineæ rectæ axi parallelæ, vel ab æquationibus ordinis paris parabolarum, quibus y per potentias integras abscissæ x exprimitur, eo minus differt, quo propius puncta curvæ ad $\genfrac{}{}{0pt}{}{\text{maximi}}{\text{minimi}}$ punctum curvæ accedunt. Et æquatio curvæ, punctum flexus contrarii habentis, ab æquatione lineæ rectæ (axi parallelæ aut obliquæ) vel ab æquationibus ordinis imparis parabolarum, quibus y per potestates integras ipsius x exprimitur, eo minus differt, quo propius puncta curvæ ad flexus contrarii punctum accedunt.

§. 177. Qui symptomata inter curvarum, quibus flexus contrarii puncta aut $\genfrac{}{}{0pt}{}{\text{maximas}}{\text{minimas}}$ ordinatas admittunt, intercedit nexus mutuus, sequenti etiam observatione illustratur.

Quæcunque de $\genfrac{}{}{0pt}{}{\text{maximo}}{\text{minimo}}$ functionis P dicuntur, vera sunt de exponente differentiali, quando flexus contrarii punctum existit. Proinde flexus contrarii determinatio reducitur ad determinationem $\genfrac{}{}{0pt}{}{\text{maximi}}{\text{minimi}}$ exponentis differentialis seu functionis $\frac{dy}{dx}$. Atqui (angulo coordinatarum posito recto) $\frac{dy}{dx}$ tangens est trigonometrica anguli, quem recta curvam contingens facit cum axe. Itaque in flexus contrarii puncto tangens hæc trigonometrica omnium $\genfrac{}{}{0pt}{}{\text{maxima}}{\text{minima}}$ est; proinde etiam angulus ipse fit omnium $\genfrac{}{}{0pt}{}{\text{maximus}}{\text{minimus}}$.

Pro-

Propositio hæc geometrice sic stabilitur. Sit AMM' curva ab A inde usque ad M versus axem concava, & ab M inde usque ad M' versus axem convexa; ita ut M sit punctum flexus contrarii. Ex M, M', $'M$ punctis agantur rectæ tangentes, quæ axi in T, T', $'T$ punctis occurrant; & tangens MT tangentibus $M'T'$, $'M'T$ in t' & $'t$ punctis occurrat. Casu hoc in triangulis $TT't'$, $T'T't$ angulus T sit internus, & minor angulis externis oppositis T', $'T$. Casu autem, quo curva est ab A versus M convexa, & ab M versus M' concava, in iisdem triangulis angulus T sit externus, & major angulis internis oppositis T', $'T$. Tandemque casu, quo exponens differentialis $\frac{dy}{dx}$ evanescit, seu tangens curvæ in flexus contrarii puncto axi est parallela; anguli, quos tangentes, per puncta ex utraque flexus contrarii parte sita ductæ, cum axe ad easdem ejus partes faciunt, simul sunt acuti, & continue ad evanescentiam accedunt. Fig. 36.

Casu, quo tam functio P, quam exponentes differentiales successivi $\frac{dy}{dx}$, $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$ ... signo impossibilis seu infiniti non afficiuntur, exposito; ad functiones progredior, quæ signo huic ansam præbent.

§. 178. Et primum quidem, si sit $P = \varphi'x + \frac{\varphi x}{(x-a)^n}$; facto $x = a$, signi $\frac{\varphi x}{o^n}$ introductione docemur, functionem P fieri impossibilem. Hoc quippe casu curva asymptotum habet abscissæ $x = a$ respondentem, & ordinata puncto huic respondens est impossibilis. (Cap. IX.) Exponentes differentiales successivi $\frac{dy}{dx}$, $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$ .... impossibilitatis symbolis $\frac{1}{(x-a)^{n+1}}$, $\frac{1}{(x-a)^{n+2}}$, $\frac{1}{(x-a)^{n+3}}$.... successive etiam afficiuntur: quibus monemur, contradictorium esse de maximis ac minimis, de concavitate aut convexitate curvæ in puncto impossibili disquirere. Omnes curvæ asymptoticæ, dum propius propiusque ad asymptotum accedunt, ad eum tendunt statum, ut tangentes earum fiant asymptoto parallelæ; & curvatura ipsarum continue ad evanescentiam tendit, quam tamen nunquam assequitur.

§. 179. Misso hoc casu, ad eos transeo, quibus functio P quidem impossibilis signum $\frac{1}{o}$ non involvit, sed exponentes ejus differentiales illo afficiuntur.

Ut hoc eveniat, functio P factorem $(x-a)^n$ seu $(a-x)^n$ continere debet, cujus exponens n est numerus positivus non integer; & magnitudo hujus exponentis determinat ordinem exponentis differentialis, qui primus impossibilis signum involvet.

Exempla. Sit n fractio vera, seu $n \gtrless {0 \atop 1}$: exponens $\frac{dP}{dx}$ casu $x = a$ afficietur signo impossibilis $\frac{1}{0^{1-n}}$: quo docemur: tangentem per punctum curvæ, quod abscissæ $x = a$ respondet, ductam fieri rectis axi ordinatim applicatis parallelam, seu tangentem hanc axi esse perpendicularem.

Sit n fractio spuria, & quidem $n \gtrless {1 \atop 2}$: tum exponens differentialis $\frac{ddP}{dx^2}$ impossibilis signo $\frac{1}{0^{2-n}}$ primus afficitur.

Sit $n \gtrless {2 \atop 3}$: exponens differentialis $\frac{d^3P}{dx^3}$ ille est, qui impossibilis signo $\frac{1}{0^{3-n}}$ primus afficitur.

Universim sit $n \gtrless {m-1 \atop m}$: erit exponens differentialis $\frac{d^mP}{dx^m}$ is, qui impossibilis signo $\frac{1}{0^{m-n}}$ primus afficitur.

§. 180. Ut, quæ ad casum hunc pertinent, eo distinctius tradam: a casibus ordiar simplicissimis, iis nempe, quibus $y = x^n$, seu a curvis parabolicis æquatione hac designatis.

Et primo quidem sit n numerus fractus verus $\frac{p}{q}$, ad simplicissimos redactus terminos, seu cujus termini p & q divisorem communem non habent (quod postea semper supponetur).

Tres occurrunt distinguendi casus. Potest quippe fractionis $\frac{p}{q}$ terminus unus esse par, quo casu alter erit impar; & tum par erit vel numerator p, vel denominator q: vel uterque terminus p, q potest esse impar.

Primus casus. Sit p numerus par, q impar.

Functionis $x^{\frac{p}{q}}$ idem est valor, seu x fiat negativa, seu positiva: proinde duabus abscissis æqualibus, quarum una positiva, altera negativa est, duæ respon-

dent

dent ordinatæ æquales poſitivæ; ideoque curva duobus conſtat ramis invicem congruentibus ad easdem axis partes, ſed ad diverſas verticis partes ſitis.

Quoniam $y = x^n$:

$$\text{ſunt } \frac{dy}{dx} = +\, nx^{n-1} = +n.\frac{1}{x^{1-n}}$$

$$\frac{ddy}{dx^2} = -n.1-n.\frac{1}{x^{2-n}}$$

$$\frac{d^3y}{dx^3} = +n.1-n.2-n.\frac{1}{x^{3-n}}$$

$$\frac{d^4y}{dx^4} = -n.1-n\ldots3-n.\frac{1}{x^{4-n}}$$

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Proinde, quamdiu x non eſt zero, curva verſus axem concava eſt, propter $\frac{ddy}{dx^2} = -n.1-n\frac{1}{x^{2-n}}$: & quoniam $y=x^n$; quo major eſt x, ſive poſitiva, ſive negativa, eo major eſt y: ideoque facta $x=0$, $y=0$ eſt omnium minima.

Factis $x=0$, $y=0$, eſt $\frac{dy}{dx}=\frac{1}{0}$; proinde communis duorum ramorum tangens in vertice eſt axi perpendicularis, ſeu rectis axi ordinatim applicatis parallela.

Duo igitur rami cuſpidem ad verticem formant, plus minusve acutam tum pro vario exponente n, tum & pro varia parametro, qua fit $y=p^{1-n}x^n$.

Figura 49. 1°. ſiſtit curvam, cujus æquatio eſt $y=x^{\frac{2}{5}}$; & fig. 49. 2°. curvam, cujus æquatio eſt $y=x^{\frac{4}{5}}$.

Caſus ſecundus. Sit p numerus impar, q par.

Hoc caſu x non poteſt eſſe negativa. Et quoniam $y=x^{\frac{p}{q}}$; ob q numerum parem, eidem abſciſſæ duæ reſpondent ordinatæ æquales, una poſitiva, altera negativa: unde curva duobus conſtat ramis invicem congruentibus, ad diverſas axis partes, ſed ad easdem verticis partes ſitis. Caſu hoc ordinata zero, abſciſſæ zero reſpondens, minor eſt rectis axi ordinatim applicatis in regione ordinatarum poſitivarum ſitis; major autem in calculo algebraico cenſetur applicatis in regione ordinatarum negativarum ſitis: itaque ordinata hæc neque maxima

xima neque minima judicatur. Sed abſciſſa zero minor eſt abſciſſis poſitivis; ideoque hoc caſu datur minimum abſciſſarum, ſed nullum $\begin{matrix}\text{maximum}\\ \text{minimum}\end{matrix}$ ordinatarum. Porro recta curvam in vertice contingens parallela eſt rectis axi ordinatim applicatis.

Quoniam $y = \pm x^n$: ſunt $\dfrac{dy}{dx} = \pm n \dfrac{1}{x^{1-n}}$

$$\frac{ddy}{dx^2} = \mp n.\overline{1-n}\frac{1}{x^{2-n}}.$$ Proinde ramus ad partes ordinatarum poſitivarum ſitus verſus axem concavus eſt; ſed ramus ad partes ordinatarum negativarum jacens verſus axem convexus eſt, quatenus ad easdem cum priore axis partes refertur; ſeu hi duo rami ſunt $\begin{matrix}\text{unus concavus}\\ \text{alter convexus}\end{matrix}$ verſus easdem partes cujusvis rectæ axi parallelæ. Vertex ideo ſpectari poſſet tanquam punctum flexus contrarii, quatenus curva ad rectam axi parallelam refertur. Sed cum duo rami ſimiliter flectantur verſus rectam axi perpendicularem; v. gr. verſus rectam, quæ curvam in vertice tangit: uſu receptum hoc caſu non eſt, verticem flexus contrarii punctum vocare; ſed potius vertex ut $\begin{matrix}\text{maximi}\\ \text{minimi}\end{matrix}$ punctum, quod ad abſciſſas, ſpectatur.

Fig. 50. $\begin{matrix}1^\circ.\\ 2^\circ.\end{matrix}$ curvam exhibet, cujus æquatio eſt $y \begin{matrix}= x^{\frac{3}{10}}\\ = x^{\frac{7}{10}}\end{matrix}.$

Ad hanc claſſem pertinet parabola conica, cujus æquatio eſt $y = x^{\frac{1}{2}}$.

Caſus tertius. Numeri p, q ambo ſint impares.

Hoc caſu abſciſſis æqualibus $\begin{matrix}\text{poſitivis}\\ \text{negativis}\end{matrix}$ reſpondent ordinatæ æquales etiam $\begin{matrix}\text{poſitivæ}\\ \text{negativæ}\end{matrix}$. Curva igitur duobus conſtat ramis invicem æqualibus, ad diverſas tam axis quam verticis partes ſitis; ideoque ordinata vertici reſpondens nec maxima nec minima eſt cenſenda.

Quoniam $y = x^n$, $\dfrac{dy}{dx} = n\dfrac{1}{x^{1-n}}$; proinde recta curvam in vertice contingens parallela eſt rectis axi ordinatim applicatis.

Porro $\dfrac{ddy}{dx^2} = -n.\overline{1-n}\dfrac{1}{x^{2-n}}$; & quoniam numeri p & q ambo ſunt impares, etiam fractionis $2-n$ ambo termini impares ſunt. Proinde ſumta x poſitiva, $\dfrac{ddy}{dx^2}$ eſt negativa; ideoque curva verſus axem concava eſt: ſumta autem x negativa,

gativa, $\frac{1}{x^{2-n}}$ pariter negativa eſt, proinde $\frac{ddy}{dx^2}$ fit poſitiva: quare in regione ordinatarum negativarum curva eſt convexa, quatenus easdem cum ramo priore partes axis reſpicit, ſeu quatenus refertur ad rectam aliquam axi parallelam. Proinde hoc caſu vertex curvæ eſt punctum flexus contrarii.

Figuris 51. N. $\begin{matrix}1.\\2.\end{matrix}$ delineantur ductus curvarum, quarum æquationes ſunt $y = \begin{matrix}x^{\frac{3}{5}}.\\x^{\frac{1}{5}}\end{matrix}$

Jam ſit n fractio ſpuria ſeu major unitate. Caſus hic ad præcedentem ſemper reducitur. Cum enim ſit n fractio ſpuria, $\frac{1}{n}$ eſt fractio vera: atqui quoniam $y = x^n$, eſt $y^{\frac{1}{n}} = x$: proinde, permutatis ordinatis, ſeu curva ad tangentem per verticem ductam tanquam axem relata, caſus hic ad priorem reducitur.

Exempla. Sit $y = x^{\frac{3}{2}}$: erit $x = y^{\frac{2}{3}}$

$y = x^{\frac{9}{5}}$: $\quad x = y^{\frac{5}{9}}$

$y = x^{\frac{6}{5}}$: $\quad x = y^{\frac{5}{6}}$.

§. 181. Enodato caſu hoc ſimpliciſſimo, facilior erit diſcuſſio ceterorum caſuum, quibus æquatio propoſita magis compoſita eſt.

Sit nempe $y = \varphi x . x^n$, n denotante numerum fractum, & φx functionem variabilis x, quæ facta $x = 0$ non evaneſcit.

Erunt
$$\frac{dy}{dx} = \frac{d\varphi x}{dx} . x^n + n\varphi x . x^{n-1}$$
$$\frac{ddy}{dx^2} = \frac{d^2\varphi x}{dx^2} . x^n + 2n . \frac{d\varphi x}{dx} . x^{n-1} + n . \overline{n-1}\varphi x . x^{n-2}$$
$$\frac{d^3y}{dx^3} = \frac{d^3\varphi x}{dx^3} . x^n + 3n . \frac{d^2\varphi x}{dx^2} . x^{n-1} + 3n . \overline{n-1} . \frac{d\varphi x}{dx} . x^{n-2} + n \ldots \overline{n-2}\varphi x . x^{n-3}$$

$$\frac{d^4y}{dx^4} = \frac{d^4\varphi x}{dx^4}.x^n$$
$$+ 4n.\frac{d^3\varphi x}{dx^3}.x^{n-1}$$
$$+ 6n.n-1.\frac{d^2\varphi x}{dx^2}.x^{n-2}$$
$$+ 4n\ldots n-2.\frac{d\varphi x}{dx}.x^{n-3}$$
$$+ n\ldots n-3\,\varphi x.x^{n-4}$$

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I°. Sit n numerus fractus verus: erit $\frac{dy}{dx} = \frac{d\varphi x}{dx}.x^n$ $+ n\varphi x\frac{1}{x^{1-n}}$. Proinde hoc casu, facta $x = 0$, fit $\frac{dy}{dx} = n\varphi x.\frac{1}{0^{1-n}}$; seu tangens ad verticem parallela est rectis axi ordinatim applicatis.

Tum quo minor est x, eo minus æquatio differentialis curvæ differt ab æquatione $\frac{dy}{dx} = n\varphi x.\frac{1}{x^{1-n}}$: & quoniam φx non involvit factorem x, functio hæc formam habet $A + Bx + Cx^2 + Dx^3 + \ldots$; & imminuta x limes ejus est A. Proinde æquatio differentialis $\frac{dy}{dx} = n\varphi x.\frac{1}{x^{1-n}}$ propius semper propiusque accedit ad æquationem differentialem parabolicam $\frac{dy}{dx} = nA\frac{1}{x^{1-n}}$; & curva ipsa ad parabolam, cujus æquatio est $y = Ax^n$: ideoque, facta $x = 0$, y minima est, si n sit numerus fractus, cujus numerator est par; y nec maxima nec minima est, si n sit numerus fractus, cujus denominator est par. Denique abscissæ $x = 0$ respondet punctum flexus contrarii, si n sit numerus fractus, cujus tam numerator quam denominator sunt numeri impares.

II°. Sit n numerus fractus spurius $\begin{matrix} > 1 \\ < 2 \end{matrix}$.

Hinc $\frac{dy}{dx} = \frac{d\varphi x}{dx}x^n$
$+ n\varphi x.x^{n-1}$.

Facta

Facta $x=0$, est $\frac{dy}{dx}=0$; proinde recta curvam in vertice contingens parallela est axi. Porro $\frac{ddy}{dx^2}=\frac{d^2\varphi x}{dx^2}x^n$

$$+\ 2n\frac{d\varphi x}{dx}x^{n-1}$$

$$+\ n.n-1\,\varphi x\frac{1}{x^{2-n}}.$$

Igitur, imminuta x, exponens differentialis secundi ordinis $\frac{ddy}{dx^2}$ propius semper propiusque accedit ad hunc valorem, ut fit $\frac{ddy}{dx^2}=n.n-1\,\varphi x\frac{1}{x^{2-n}}=n.n-1\,(A+Bx+Cx^2+Dx^3+\ldots).\frac{1}{x^{2-n}}$; & proinde æquatio curvæ propius semper propiusque accedit ad æquationem parabolæ, cujus æquatio differentialis secundi ordinis est $n.n-1.\frac{1}{x^{2-n}}$.

1°. Sit $n=1+\frac{2p+1}{2q}=\frac{2p+2q+1}{2q}$

$\frac{1}{n}=\frac{2q}{2p+2q+1}$; proinde forma hæc reducitur ad primum casum (§. 180.), atque x omnium $\begin{matrix}\text{maxima}\\\text{minima}\end{matrix}$ fit.

2°. Sit $n=1+\frac{2p}{2q+1}=\frac{2p+2q+1}{2q+1}$, $\frac{1}{n}=\frac{2q+1}{2p+2q+1}$; igitur (§. 180. cas. 3.) abscissæ $x=0$ respondet punctum flexus contrarii.

3°. Sit $n=1+\frac{2p+1}{2q+1}=\frac{2p+2q+2}{2q+1}$, $\frac{1}{n}=\frac{2q+1}{2p+2q+2}$; igitur (§. 180. cas. 2.) abscissæ $x=0$ respondet $\begin{matrix}\text{maximum}\\\text{minimum}\end{matrix}$ ordinatæ y.

III°. Sit n numerus fractus spurius $\begin{matrix}>2\\<3\end{matrix}$.

Hinc $\frac{dy}{dx}=\frac{d\varphi x}{dx}.x^n$

$$+\ n\varphi x.x^{n-1}$$

$$\frac{ddy}{dx^2}=\frac{d^2\varphi x}{dx^2}.x^n$$

$$+\ 2n\frac{d\varphi x}{dx}.x^{n-1}$$

$$+\ n.n-1\,\varphi x.x^{n-2}$$

$$\frac{d^3y}{dx^3} = \frac{d^3\varphi x}{dx^3} \cdot x^n$$
$$+ 3n\frac{d^2\varphi x}{dx^2} \cdot x^{n-1}$$
$$+ 3n \cdot n-1 \frac{d\varphi x}{dx} \cdot x^{n-2}$$
$$+ n \ldots n-2\varphi x \cdot \frac{1}{x^{3-n}}.$$

Proinde, imminuta x, exponens differentialis $\frac{d^3y}{dx^3}$ propius ſemper propiusque ad valorem $n . n-1 . n-2A\frac{1}{x^{3-n}}$ accedit: ideoque caſus hic reducitur ad contemplationem curvæ parabolicæ, cujus æquatio eſt $y = x^n$.

1°. Sit $n = 2 + \frac{2p}{2q+1} = \frac{4q+2p+2}{2q+1}$, $\frac{1}{n} = \frac{2q+1}{4q+2p+2}$: unde (§. 180. caſ. 2.) facta $x = 0$, y eſt omnium $\begin{matrix}\text{maxima}\\ \text{minima}\end{matrix}$.

2°. Sit $n = 2 + \frac{2p+1}{2q} = \frac{4q+2p+1}{2q}$, $\frac{1}{n} = \frac{2q}{4q+2p+1}$: unde (§. 180. caſ. 1.) facta $x = 0$, x eſt omnium $\begin{matrix}\text{maxima}\\ \text{minima}\end{matrix}$.

3°. Sit $n = 2 + \frac{2p+1}{2q+1} = \frac{4q+2p+3}{2q+1}$, $\frac{1}{n} = \frac{2q+1}{4q+2p+3}$: unde (§. 180. caſ. 3.) abſciſſæ $x = 0$ reſpondet punctum flexus contrarii.

IV. Sit n numerus fractus $\begin{matrix}> 3\\ < 4\end{matrix}$

$$\frac{d^4y}{dx^4} = \frac{d^4\varphi x}{dx^4} \cdot x^n$$
$$+ 4n\frac{d^3\varphi x}{dx^3} \cdot x^{n-1}$$
$$+ 6n \cdot n-1\frac{d^2\varphi x}{dx^2} \cdot x^{n-2}$$
$$+ 4n \ldots n-2\frac{d\varphi x}{dx} \cdot x^{n-4}$$
$$+ n \ldots n-3\varphi x\frac{1}{x^{4-n}}.$$

Quare, imminuta x, exponens differentialis $\frac{d^4y}{dx^4}$ propius ſemper propiusque accedit

cedit ad exponentem differentialem quarti gradus parabolæ, nempe $\frac{d^4y}{dx^4} = n \ldots n-3\, A \frac{1}{x^{n-4}}$.

1°. Sit $n = 3 + \frac{2p}{2q+1} = \frac{6q+2p+3}{2q+1}$, $\frac{1}{n} = \frac{2q+1}{6q+2p+3}$; proinde abscissæ $x = 0$ respondet punctum flexus contrarii (§. 180. cas. 3.).

2°. Sit $n = 3 + \frac{2p+1}{2q} = \frac{6q+2p+1}{2q}$, $\frac{1}{n} = \frac{2q}{6q+2p+1}$; proinde abscissa $x = 0$ est omnium $\begin{matrix}\text{maxima}\\ \text{minima}\end{matrix}$ (§. 180. cas. 1.).

3°. Sit $n = 3 + \frac{2p+1}{2q+1} = \frac{6q+2p+4}{2q+1}$, $\frac{1}{n} = \frac{2q+1}{6q+2p+4}$; proinde abscissæ $x = 0$ respondet ordinata omnium $\begin{matrix}\text{maxima}\\ \text{minima}\end{matrix}$.

V. Sit $n \begin{matrix}> 4\\ < 5\end{matrix}$: erit

$$\begin{aligned}\frac{d^5y}{dx^5} = {} & \frac{d^5\varphi x}{dx^5} . x^n \\ & + 5n \frac{d^4\varphi x}{dx^4} . x^{n-1} \\ & + 10 n . n-1 \frac{d^3\varphi x}{dx^3} . x^{n-2} \\ & + 10 n . n-1 . n-2 \frac{d^2\varphi x}{dx^2} . x^{n-3} \\ & + 5 n \ldots n-3 \frac{d\varphi x}{dx} . x^{n-4} \\ & + n \ldots n-4\, \varphi x . \frac{1}{x^{5-n}}.\end{aligned}$$

Proinde, imminuta x, æquatio curvæ per exponentem differentialem $\frac{d^5y}{dx^5}$ determinata propius semper propiusque accedit ad $\frac{d^5y}{dx^5} = n \ldots n-4\, A \frac{1}{x^{5-n}}$.

1°. Sit $n = 4 + \frac{2p}{2q+1} = \frac{8q+2p+4}{2q+1}$, $\frac{1}{n} = \frac{2q+1}{8q+2p+4}$; abscissæ $x = 0$ respondet ordinata y omnium $\begin{matrix}\text{maxima}\\ \text{minima}\end{matrix}$.

2°. Sit $n = 4 + \frac{2p+1}{2q} = \frac{8q+2p+1}{2q}$, $\frac{1}{n} = \frac{2q}{8q+2p+1}$; $x = 0$ est omnium $\begin{matrix}\text{maxima}\\ \text{minima}\end{matrix}$.

3°. Sit

3°. Sit $n = 4 + \frac{2p+1}{2q+1} = \frac{8q+2p+5}{2q+1}$, $\frac{1}{n} = \frac{2q+1}{8q+2p+5}$; abſciſſæ $x = 0$ reſpondet punctum flexus contrarii.

Ab exemplis his facilis eſt tranſitus ad regulam generalem, qua curvarum æquatione $y = \varphi x . x^n$ definitarum ſymptomata caſu $x = 0$ determinantur.

1°. Si $n = 2m + \frac{2p}{2q+1} = \frac{4qm+2m+2p}{2q+1}$, $\frac{1}{n} = \frac{2q+1}{4qm+2m+2p}$; y minima eſt caſu $x = 0$.

2°. Si $n = 2m + \frac{2p+1}{2q} = \frac{4qm+2p+1}{2q}$, $\frac{1}{n} = \frac{2q}{4qm+2p+1}$; $x \pm 0$ minima eſt.

3°. Si $n = 2m + \frac{2p+1}{2q+1} = \frac{4qm+2m+2p+1}{2q+1}$, $\frac{1}{n} = \frac{2q+1}{4qm+2m+2p+1}$; abſciſſæ $x = 0$ reſpondet punctum flexus contrarii.

4°. Si $n = 2m+1+\frac{2p}{2q+1} = \frac{4qm+2m+2q+2p+1}{2q+1}$, $\frac{1}{n} = \frac{2q+1}{4qm+2m+2q+2p+1}$; abſciſſæ $x = 0$ reſpondet punctum flexus contrarii.

5°. Si $n = 2m+1+\frac{2p+1}{2q} = \frac{4qm+2q+2p+1}{2q}$, $\frac{1}{n} = \frac{2q}{4qm+2q+2p+1}$; abſciſſa $x = 0$ minima eſt.

6°. Si $n = 2m+1+\frac{2p+1}{2q+1} = \frac{4qm+2m+2q+2p+2}{2q+1}$, $\frac{1}{n} = \frac{2q+1}{4qm+2m+2q+2p+2}$; y minima eſt caſu $x = 0$.

Eadem ratiocinia applicantur functionibus $y = \varphi' x \pm \varphi x . x^n$, in quibus n eſt numerus fractus poſitivus; & $\varphi' x$, φx functiones ſunt variabilis x non evaneſcentes caſu $x = 0$. Quoniam autem ſingulare hoc functionum genus in matheſi inprimis applicata quam rariſſime occurrit; ſufficiat generale, ad quod exigi debent, principium expoſuiſſe.

Huc usque tradita variis exemplis illuſtrabo.

§. 182. *Exemplum* 1. Sit $P = \sqrt{(aa+xx)} - \frac{p}{q} x$

$$\frac{dP}{dx} = \frac{x}{\sqrt{aa+xx}} - \frac{p}{q}$$

$$\frac{ddP}{dx^2} = \frac{1}{\sqrt{aa+xx}} - \frac{xx}{(aa+xx)^{\frac{3}{2}}} = + \frac{aa}{(aa+xx)^{\frac{3}{2}}}.$$

Facto

Facto $\frac{dP}{dx} = 0$, est $xx = \frac{pp}{qq}(aa+xx)$, $x = a \times \frac{p}{\sqrt{(qq-pp)}}$.

Ut problema possibile sit; oportet, sit $q > p$: & facto $x = a \cdot \frac{p}{\sqrt{(qq-pp)}}$ P est omnium minima, propter $\frac{ddP}{dx^2}$ semper positivum.

Exemplum 2. Sit $P = x^x$

$$\log. P = x \log. x$$

$$\frac{dP}{dx} \cdot \frac{1}{P} = \log. x + 1$$

$$\frac{ddP}{dx^2} \cdot \frac{1}{P} = \frac{1}{x} \text{ (propter } \frac{dP}{dx} = 0).$$

Facto $\log. x + 1 = 0$, est $\log. x = -1$, seu $x = e^{-1} = \frac{1}{e}$; $P = \left(\frac{1}{e}\right)^{\frac{1}{e}} = \sqrt[e]{\frac{1}{e}}$.

Exemplum 3. Sit $P = \text{fin.}^m x \, \text{fin.}^n (\varphi - x)$

$$\log. P = m \log. \text{fin.} x + n \log. \text{fin.} (\varphi - x)$$

$$\frac{dP}{dx} \cdot \frac{1}{P} = m \cot. x - n \cot. (\varphi - x)$$

$$\frac{ddP}{dx^2} \cdot \frac{1}{P} = -\tfrac{1}{2} m \, \text{cofec.} 2x - \tfrac{1}{2} n \, \text{cofec.} (2\varphi - 2x).$$

Proinde P maximum est, quando $m \cot. x = n \cot. (\varphi - x)$; unde $\text{fin.} (2x - \varphi) = \frac{n-m}{n+m} \text{fin.} \varphi$.

Exemplum 4. Sit $P = \text{tang.}^m x \, \text{tang.}^n (\varphi - x)$

$$\log. P = m \log. \text{tang.} x + n \log. \text{tang.} (\varphi - x)$$

$$\frac{dP}{dx} \cdot \frac{1}{P} = m \, \text{cofec.} x - n \, \text{cofec.} (\varphi - x)$$

$$\frac{ddP}{dx^2} \cdot \frac{1}{P} = -m \cot. x \, \text{cofec.} x - n \, \text{cofec.} (\varphi - x) \cot. (\varphi - x).$$

Proinde facto $m \, \text{cofec.} \varphi = n \, \text{cofec.} (\varphi - x)$, unde $\text{tang.} (\frac{1}{2}\varphi - x) = \frac{n-m}{n+m} \text{tang.} \frac{1}{2}\varphi$; functio P est omnium maxima.

Exemplum 4. Sit $P = x^m \text{fin.}^n x$

$$\log. P = m \log. x + n \log. \text{fin.} x$$

$$\frac{dP}{dx} \cdot \frac{1}{P} = m \cdot \frac{1}{x} + n \cot. x$$

$$\frac{ddP}{dx^2} \cdot \frac{1}{P} = -m \cdot \frac{1}{xx} - n \, \text{cofec.}^2 x.$$

Pro-

Proinde facto $\frac{m}{n} = -x\cot.x = x\cot.180^\circ - x$, seu $x = \frac{m}{n}\,\text{tang.}\,180^\circ - x$; P est omnium maxima.

Exemplum 6. Sit $P = \text{fin.}\,x\,\text{fin.}\,mx$

$$\log.P = \log.\text{fin.}\,x + \log.\text{fin.}\,mx$$

$$\frac{dP}{dx}\cdot\frac{1}{P} = \cot.x + m\cot.mx$$

$$\frac{ddP}{dx^2}\cdot\frac{1}{P} = -\text{cofec.}^2 x - mm\,\text{cofec.}^2 mx.$$

Proinde facto $\cot.180^\circ - x = m\cot.mx$, functio P est omnium maxima.

Exemplum 7. Sit $P = \frac{(a+x)^m}{(b+x)^n}$. Facta $x = -b$, nullus est functionis P limes quod ad magnitudinem.

$$\log.P = m\log.(a+x) - n\log.(b+x)$$

$$\frac{dP}{dx}\cdot\frac{1}{P} = m\frac{1}{a+x} - n\cdot\frac{1}{b+x}$$

$$\frac{ddP}{dx^2}\cdot\frac{1}{P} = -\frac{m}{(a+x)^2} + \frac{n}{(b+x)^2}.$$

1°. Sit $n = m$: facto $a+x = b+x$, omnes exponentes differentiales successivi functionis P evanescunt; prouti nullus est functionis hujus limes, tum quod ad magnitudinem, tum quod ad parvitatem.

2°. Sit $m > n$: facto $\frac{m}{a+x} = \frac{n}{b+x}$, est $\frac{m}{(a+x)^2} = \frac{nn}{m}\cdot\frac{1}{(b+x)^2}$, et $-\frac{m}{(a+x)^2} + \frac{n}{(b+x)^2} = \frac{n}{(b+x)^2}\left(1 - \frac{n}{m}\right) = +\frac{1}{(b+x)^2} \times \frac{n(m-n)}{m}$; proinde functio P est omnium minima.

3°. Sit $m < n$: tum $-\frac{m}{(a+x)^2} + \frac{n}{(b+x)^2} = -\frac{n}{(b+x)^2} \times \frac{n-m}{m}$; proinde functio P est omnium maxima.

§. 183. Hactenus suppofui, functionem P quantitatis mutabilis x ita exprimi per hanc quantitatem, ut functio hæc sola sit in uno æquationis membro, neque alterum ingrediatur. Fieri autem etiam potest, ut cognitio relationis functionis hujus ad quantitatem mutabilem x ab æquationis alicujus solutione pendeat, quo casu functio P fit multiformis.

Casus

Casus hujus difficultas unice refertur ad imperfectum æquationum doctrinæ statum. Etenim si solutio æquationum in promtu esset: reducta æquatione, a qua functionis P & quantitatis x relatio mutua pendet; radices ejus totidem exhiberent functionis P per variabilem x expressiones, & $\begin{matrix}\text{maximorum}\\ \text{minimorum}\end{matrix}$ ejus valorum determinatio reduceretur ad totidem functiones uniformes, quot æquatio proposita habet radices. Hinc functiones biformes, quæ ab æquatione secundi gradus pendent, facile reducuntur ad duas functiones uniformes.

Argumentum hoc exemplis quibusdam aptissime declarabitur.

Exemplum 1. Sit y functio triformis ipsius x, quæ exprimatur æquatione $y^3 - pxy + x^3 = 0$; & quæratur valor omnium $\begin{matrix}\text{maximus}\\ \text{minimus}\end{matrix}$ functionis y.

Quoniam $y^3 - pxy + x^3 = 0$;
$3yy\frac{dy}{dx} - px\frac{dy}{dx} - py + 3xx = 0$: facto $\frac{dy}{dx} = 0$, erit $py = 3xx$, & $y = \frac{3xx}{p}$. Valor hic in æquatione proposita substituatur: fiet $\frac{27x^6}{p^3} - 2x^3 = 0$, $27x^6 = 2p^3x^3$;

hinc $3xx = px\sqrt[3]{2}$, $x = \overset{0}{\tfrac{1}{3}p\sqrt[3]{2}}$, $y = \overset{0}{\tfrac{1}{3}p\sqrt[3]{4}}$.

Quoniam $3yy\frac{dy}{dx} - px\frac{dy}{dx} = py - 3xx$: facto $\frac{dy}{dx} = 0$,

est $3yy\frac{ddy}{dx^2} - px\frac{ddy}{dx^2} = -6x$; æquatio identica, factis $\begin{matrix}x\\ y\end{matrix} = 0$.

Porro $\begin{matrix}3yy\frac{d^3y}{dx^3}\\ -px\frac{d^3y}{dx^3}\end{matrix} - p\frac{ddy}{dx^2} = -6$: unde positis $\begin{matrix}x\\ y\end{matrix} = 0$, fit $-p\frac{ddy}{dx^2} = -6$,

$\frac{ddy}{dx^2} = +\frac{6}{p}$; quare $y = 0$ est omnium minima.

Sit autem $\begin{matrix}x = \tfrac{1}{3}p\sqrt[3]{2}\\ y = \tfrac{1}{3}p\sqrt[3]{4}\end{matrix}$; tum $\frac{ddy}{dx^2}(\tfrac{2}{3}pp\sqrt[3]{2} - \tfrac{1}{3}pp\sqrt[3]{2}) = -2p\sqrt[3]{2}$

seu $\frac{ddy}{dx^2} = -\frac{3}{p}$; proinde casu hoc y est omnium maxima.

Exemplum 2. Sit $y^4 - 4a^2xy + x^4 = 0$;
itaque $4y^3\frac{dy}{dx} - 4a^2x\frac{dy}{dx} - 4a^2y + 4x^3 = 0$: facto $\frac{dy}{dx} = 0$, est $y = \frac{x^3}{a^3}$.

 Hinc

Hinc $\frac{x^{12}}{a^8} - 3x^4 = 0$, $x^4\left(\frac{x^8}{a^8} - 3\right) = 0$, $x^4(x^8 - 3a^8) = 0$, $x^4(x^4 + a^4\sqrt{3})(x^4 - a^4\sqrt{3}) = 0$, $x^4(x^4 + a^4\sqrt{3})(xx + aa\sqrt[4]{3})(xx - aa\sqrt[4]{3}) = 0$; cujus æquationes radices reales sunt $x = \begin{cases} 0 \\ +a\sqrt[8]{3} \\ -a\sqrt[8]{3} \end{cases}$: unde $y = \begin{cases} 0 \\ a\sqrt[8]{27} \\ -a\sqrt[8]{27} \end{cases}$.

Quoniam $y^3\frac{dy}{dx} - a^2x\frac{dy}{dx} = a^2y - x^3$: posito $\frac{dy}{dx} = 0$, fit

$$y^3\frac{ddy}{dx^2} - a^2x\frac{ddy}{dx^2} = -3xx.$$

Fiat $y = +a\sqrt[8]{27}$: erit $2a^3\sqrt[8]{3} \times \frac{ddy}{dx^2} = -3a^3\sqrt[4]{3}$; proinde y est maxima.

Fiat $y = -a\sqrt[8]{27}$: erit $-4a^3\sqrt[8]{3} \times \frac{ddy}{dx^2} = -3a^3\sqrt[4]{3}$; igitur y est minima.

Ex his exemplis sequitur regula generalis. Differentietur æquatio proposita; facto $\frac{dy}{dx} = 0$, inde eliciatur (quoad fieri potest) valor ipsius y per x expressæ, qui in æquatione proposita substituatur. Æquatione hac reducta, notentur valores ipsius x inde eruti, & qui illis respondent valores ipsius y, qui exhibebunt $\frac{\text{maximum}}{\text{minimum}}$, si quod locum habet. Quod ut definiatur: ex æquatione differentiali eliciatur exponens differentialis $\frac{ddy}{dx^2}$ (omissis in altera differentiatione terminis exponente differentiali $\frac{dy}{dx} = 0$ affectis); ex quo, substitutis ipsarum x & y valoribus inventis, dijudicabitur (si non evanescat): utrum valoribus inventis $\frac{\text{maximum}}{\text{minimum}}$ respondeat? Quodsi autem exponens differentialis $\frac{ddy}{dx^2}$ evanescit; recurrendum est ad exponentes differentiales ulteriorum ordinum.

Hinc patet: quantopere methodus hæc ab æquationum solutione pendeat; & proinde quot difficultatibus possit casibus, præsertim complexis, esse obnoxia.

§. 184. Methodus huc usque tradita functionum alicujus quantitatis mutabilis $\frac{\text{maxima}}{\text{minima}}$ determinandi omnium est universalissima. Occurrunt tamen casus, quibus alium tenere modum minus algebraicum præstat, & qui in mathesi præcipue applicata felici successu frequenter usurpatur.

Metho-

Methodus hæc ſequenti nititur principio. Si functio quantitatis alicujus mutabilis $\substack{\text{maxima}\\ \text{minima}}$ ſit pro dato quodam variabilis x valore a; duo ejusdem functionis valores invicem æquales aſſignari poſſunt, quorum unus majori, alter minori quantitatis mutabilis x valori reſpondet.

Principium hoc apprime declarat theoria curvarum atque ordinatarum $\substack{\text{maximarum}\\ \text{minimarum}}$.

Primo tangens curvæ, per extremum ordinatæ omnium $\substack{\text{maximæ}\\ \text{minimæ}}$ ducta, ſit axi parallela; tangens hæc a puncto contactus inde, motu ſibi parallelo, aut axi propius admoveatur, aut ab ipſo recedat, prouti curva eſt verſus axem $\substack{\text{concava}\\ \text{convexa}}$: ita tangens hæc curvæ ex utraque ordinatæ omnium $\substack{\text{maximæ}\\ \text{minimæ}}$ parte occurret; ordinatisque æqualibus a punctis occurſus ductis æquales reſpondent functionis, per ordinatas curvæ deſignatæ, valores. Sit dein tangens curvæ, per extremum ordinatæ $\substack{\text{maximæ}\\ \text{minimæ}}$ ducta, ordinatis parallela: quoniam curva duobus conſtat ramis in hoc puncto ſe invicem contingentibus, & ad easdem axis partes ſitis; recta per hoc punctum axi parallela, & motu ſibi parallelo progreſſa a puncto illo inde verſus partes, ad quas curva jacet, utrique etiam ramo occurret; & punctis occurſus duæ reſpondebunt ordinatæ invicem æquales, ſeu duo functionis ordinatis curvæ proportionalis valores invicem æquales.

Applicationem principii hujus exemplis aliquot illuſtrabo, a ſimpliciſſimis ordiendo.

Exemplum 1. Sit AB recta in puncto Z in duas partes ſic dividenda, ut rectangulum $AZ \times ZB$ ſit omnium maximum. Fig. 52.

Sint X & X' duo puncta ad utramque puncti Z partem ſita, quibus reſpondeant rectangula $AX \times XB$, $AX' \times X'B$ invicem æqualia.

$$
\begin{aligned}
\text{Quoniam } AX \times XB = AX' \times X'B:\ \text{eſt}\quad AX : AX' &= X'B : XB \\
\text{hinc}\quad AX : XX' &= X'B : XX' \\
\text{et}\quad AX &= X'B \\
\text{unde}\quad \text{lim.}\, AX &= \text{lim.}\, X'B.
\end{aligned}
$$

Atqui poſito ſemper eſſe $AZ > AX$, $BZ > X'B$; $\text{lim.}\,AX = AZ$, & $\text{lim.}\,X'B = BZ$; proinde $AZ = BZ$.

Exemplum secundum. Summa quadratorum AZ, BZ debeat esse omnium minima.

$$
\begin{aligned}
\text{Sit}\quad AX^2 + BX^2 &= AX'^2 + BX'^2;\ \text{hinc}\\
AX'^2 - AX^2 &= BX^2 - BX'^2\\
XX'(AX' + AX) &= XX'(BX + BX')\\
AX' + AX &= BX + BX'\\
2AX + XX' &= 2BX' + XX'\\
AX &= BX'\\
\lim.AX &= \lim.BX';\ \text{hoc est } AZ = BZ.
\end{aligned}
$$

Exemplum tertium. Oporteat, sit $AZ^m \times BZ^n =$ maximum.

Fig. 53.
$$
\begin{aligned}
\text{Sit}\quad AX^m \times BX^n &= AX'^m \times BX'^n\\
\text{Erit}\quad AX^m : AX'^m &= BX'^n : BX^n\\
AX^m : AX'^m - AX^m &= BX'^n : BX^n - BX'^n
\end{aligned}
$$

$$AX^m : XX'(AX'^{m-1} + AX'^{m-2}.AX + AX'^{m-3}.AX^2 + \ldots + AX'.AX^{m-2} + AX^{m-1})$$
$$= BX'^n : XX'(BX^{n-1} + BX^{n-2}.BX' + BX^{n-3}.BX'^2 + \ldots + BX.BX'^{n-2} + BX'^{n-1})$$

unde

$$AX : BX' = \left(\left(\frac{AX'}{AX}\right)^{m-1} + \left(\frac{AX'}{AX}\right)^{m-2} + \ldots \frac{AX'}{AX} + 1\right) : \left(\left(\frac{BX}{BX'}\right)^{n-1} + \left(\frac{BX}{BX'}\right)^{n-2} + \ldots \frac{BX}{BX'} + 1\right).$$

Quare & limes prioris rationis æqualis est limiti rationis posterioris.

Quoniam autem $\lim.\frac{AX'}{AX} = 1$, & $\lim.\frac{BX'}{BX} = 1$, posterioris rationis limes est $m : n$; & limes rationis prioris est $AZ : BZ$: igitur $AZ : BZ = m : n$.

Exemplum quartum. Summa $a \times AZ^m + b \times BZ^m$ debeat esse omnium minima.

$$
\begin{aligned}
\text{Sit}\quad a \times AX^m + b \times BX^m &= a \times AX'^m + b \times BX'^m;\\
\text{erit}\quad b(BX^m - BX'^m) &= a(AX'^m - AX^m):
\end{aligned}
$$

unde $b.XX'(BX^{m-1} + BX^{m-2}.BX' + BX^{m-3}.BX'^2 + \ldots BX.BX'^{m-2} + BX'^{m-1})$

$$= a \times XX'(AX'^{m-1} + AX'^{m-2}.AX + AX'^{m-3}.AX^2 + \ldots AX'.AX^{m-2} + AX^{m-1})$$

$$b.BX^{n-1}\left(1 + \frac{BX'}{BX} + \left(\frac{BX'}{BX}\right)^2 + \ldots \left(\frac{BX'}{BX}\right)^{m-1}\right) = a AX'^{m-1}\left(1 + \frac{AX}{AX'} + \left(\frac{AX}{AX'}\right)^2 + \ldots \left(\frac{AX}{AX'}\right)^{m-1}\right)$$

$$b.BX^{m-1} : a.AX'^{m-1} = 1 + \frac{AX}{AX'} + \left(\frac{AX}{AX'}\right)^2 + \ldots \left(\frac{AX}{AX'}\right)^{m-1} : 1 + \frac{BX'}{BX} + \left(\frac{BX'}{BX}\right)^2 + \ldots \left(\frac{BX'}{BX}\right)^{m-1};$$

unde

unde & prioris rationis limes posterioris rationis limiti æqualis est; nempe $b.BZ^{m-1} : a.AZ^{m-1} = 1 : 1$, seu $BZ^{m-1} : AZ^{m-1} = a : b$.

Exemplum quintum. Sint A, B duo puncta positione data; & sit DD' curva specie ac positione data. Quæritur hujus curvæ punctum Z, ad quod ductis AZ, BZ rectis, sit summa $a \times AZ^m + b \times BZ^m$ omnium minima. Fig. 54.

Sint X & X' duo puncta ad utramque puncti Z partem sita, quibus æquales summæ propositæ respondent; scilicet $a \times AX^m + b \times BX^m = a \times AX'^m + b \times BX'^m$.

Per Z punctum acta concipiatur recta tangens TT'; centris A, B, radiis AX, BX' describantur arcus Xx', $X'x$, qui rectis AX', BX in x', x occurrant.

Quoniam $a \times AX^m + b \times BX^m = a \times AX'^m + b \times BX'^m$

$$b(BX^m - BX'^m) = a(AX'^m - AX^m):$$

hinc $b.Xx(BX^{m-1} + BX^{m-2}.BX' + BX^{m-3}.BX'^2 + \dots BX'^{m-1})$

$= a.X'x'(AX'^{m-1} + AX'^{m-2}.AX + AX'^{m-3}.AX^2 + \dots AX^{m-1})$

$$Xx : X'x' = aAX'^{m-1}\left(1 + \frac{AX}{AX'} + \frac{AX^2}{AX'^2} + \dots \left(\frac{AX}{AX'}\right)^{m-1}\right) : bBX^{m-1}\left(1 + \frac{BX'}{BX} + \left(\frac{BX'}{BX}\right)^2 + \dots \left(\frac{BX'}{BX}\right)^{m-1}\right);$$

ideoque rationum harum limites pariter sunt invicem æquales. Atqui X & X' versus Z punctum simul accedentibus est

$$\lim. Xx : XX' = \text{cos}.BZT' : 1$$

$$\lim. XX' : X'x' = 1 : \text{cos}.AZT$$

quare $\lim. Xx : X'x' = \text{cos}.BZT' : \text{cos}.AZT$: unde sit (ut in exemplis antecedentibus) $a.AZ^{m-1} : b.BZ^{m-1} = \text{cos}.BZT' : \text{cos}.AZT$,

seu $a.AZ^{m-1}\,\text{cos}.AZT + b.BZ^{m-1}\,\text{cos}.BZT = 0$.

Si plura sint puncta data, $A, B, C, D, E \dots$ & quæratur summa omnium minima $a \times AZ^m + b \times BZ^m + c \times CZ^m + d \times DZ^m + e \times EZ^m \dots$ Erit eodem modo $a.AZ^{m-1}\,\text{cos}.AZT + b.BZ^{m-1}\,\text{cos}.BZT + c.CZ^{m-1}\,\text{cos}.CZT + d.DZ^{m-1}\,\text{cos}.DZT + e.EZ^{m-1}\,\text{cos}.EZT + \dots = 0$.

Exemplum sextum. Sit P punctum intra curvam XAX' positione datum; per quod agenda sit recta ZPZ', quæ segmentum auferat ZAZ', cujus area sit omnium minima.

Sint XX', YY' duæ rectæ ad utramque rectæ ZZ' partem sitæ, quibus æquales areæ XAX', YAY' respondeant. Fig. 55.

Quo-

Quoniam $XAX' = \Upsilon A \Upsilon'$: ſublato ſegmento communi $\Upsilon AX'$, erit $XP\Upsilon = X'P\Upsilon'$. Centro P, radiis $P\Upsilon$, PX' deſcribantur arcus Yx, $X'y'$.

Quoniam $XPY = X'PY'$, ſeu $XPY : X'PY' = 1 : 1$

$$\begin{array}{rlcll}
 & \text{lim.}XPY & : \text{lim.}X'PY' & = & 1 : 1 \\
\text{ſed} & \text{lim.}XPY & : \text{lim.}YPx & = & 1 : 1 \\
 & \text{lim.}\Upsilon Px & : X'Py' & = & PZ^2 : PZ'^2 \\
 & \text{lim.}X'Py' & : X'PY' & = & 1 : 1 \\
\text{ergo} & \text{lim.}XPY & : X'PY' & = & PZ^2 : PZ'^2.
\end{array}$$

Proinde area ZPZ' poſita omnium minima, eſt $PZ = PZ'$.

Exemplum ſeptimum. Omnibus ut prius poſitis, recta ZZ' debeat eſſe omnium minima. Iisdem, quæ in exemplo ſexto, factis; per puncta Z, Z' actæ concipiantur rectæ tangentes ZT, $Z'T'$: & ſit $\Upsilon\Upsilon' = XX'$.

Quoniam $XX' = \Upsilon\Upsilon'$: eſt $Xx = Y'y'$;

$$\begin{array}{rlcccl}
\text{et} & \text{lim.}Xx & : Y'y' & = & 1 & : 1 \\
\text{Atqui} & \text{lim.}Xx & : Yx & = & 1 & : \text{tang.}PZT \\
 & \text{lim.}Yx & : X'y' & = & PZ & : PZ' \\
 & \text{lim.}X'y' & : Y'y' & = & \text{tang.}PZ'T' & : 1 \\
\text{Ergo} & \text{lim.}Xx & : Y'y' & = & PZ\,\text{tang.}PZ'T' & : PZ'\,\text{tang.}PZT.
\end{array}$$

Proinde recta ZZ' poſita omnium minima, erit $PZ : PZ' = \text{tang.}PZT : \text{tang.}PZ'T'$.

Exemplum octavum. Arcus ZAZ' debeat eſſe omnium minimus.

Sint arcus XAX', $\Upsilon A \Upsilon'$ invicem æquales: erit ideo $X\Upsilon = X'\Upsilon'$, & lim. $X\Upsilon : X'\Upsilon' = 1 : 1$.

$$\begin{array}{rlcccl}
\text{Atqui} & \text{lim.}XY & : Yx & = & 1 & : \text{ſin.}PZT \\
 & \text{lim.}Yx & : X'y' & = & PZ & : PZ' \\
 & \text{lim.}X'y' & : X'Y' & = & \text{ſin.}PZ'T' & : 1 \\
\text{Ergo} & \text{lim.}XY & : X'Y' & = & PZ\,\text{ſin.}PZ'T' & : PZ'\,\text{ſin.}PZT.
\end{array}$$

Inde arcu ZAZ' poſito omnium minimo, eſt $PZ : PZ' = \text{ſin.}PZT : PZ'T'$.

Exem-

Exempla hæc oſtendunt, methodum poſteriorem (quatenus quantitatum, quæ $\begin{matrix}\text{maximæ}\\ \text{minimæ}\end{matrix}$ reddi debent, evolutionem non requirit) priore methodo generali poſſe eſſe faciliorem ac breviorem.

§. 185. In exemplis præcedentibus relatio finita inter quantitates incognitas ita fuit poſt differentiationem determinata, ut ipſæ quantitates incognitæ (quoad imperfecta æquationum doctrina permittit) poſſent aſſignari. Transeo ad exempla, quibus quæſtio propoſita indeterminata eſſe videtur, quoniam quantitatum incognitarum numerus videtur æquationum numerum ſuperare: cum tamen quæſtio propoſita reapſe ſit determinata, ſi natura ejus attentius perpendatur. Hoc quæſtionum genus ut facilius intelligatur, nonnulla proponam exempla, a ſimplicioribus ordiendo.

Exemplum primum. Summa propoſita a dividenda ſit in tres partes, ſic ut ſumma quadratorum ex iisdem factorum ſit omnium minima.

Hoc caſu tres occurrunt quantitates incognitæ; & tamen prima fronte duæ tantum proponi conditiones videntur, quarum una ad ſummam datam trium partium refertur, altera ad ſummam omnium minimam quadratorum earundem. Quare problema propoſitum poteſt videri indeterminatum: quod tamen, re accuratius perpenſa, reapſe eſt determinatum. Etenim, parte qualibet eadem manente, duæ reliquæ partes debent eſſe invicem æquales, ut quadratorum ſumma (poſita unius partis conſtantia) ſit omnium minima. Proinde partes propoſitæ binæ ſumtæ debent eſſe invicem æquales; unde tres partes debent eſſe invicem æquales. Ratiocinium hoc applicatur ad numerum quemcunque partium, in quas ſumma data eſt dividenda, ut partium quadratorum ſumma ſit omnium minima. Transeo ad computum.

Sit S ſumma data; & ſint x, y, z partes quæſitæ.

Erit ideo $x + y + z = S$

$xx + yy + zz =$ minimo. Sit v quantitas quædam mutabilis, cujus mutationes pro conſtantibus ſumantur.

Erit

$$\text{Erit}\quad \frac{dx}{dv} + \frac{dy}{dv} + \frac{dz}{dv} = 0$$
$$x\frac{dx}{dv} + y\frac{dy}{dv} + z\frac{dz}{dv} = 0.$$

Hinc posita z constante $\begin{array}{l} \frac{dx}{dv} + \frac{dy}{dv} = 0 \\ x\frac{dx}{dv} + y\frac{dy}{dv} = 0 \end{array}$; ideoque $x\frac{dx}{dv} - y\frac{dx}{dv} = 0$, & $x = y$.

Eodem modo, posita y constante, $\begin{array}{l} \frac{dx}{dv} + \frac{dz}{dv} = 0 \\ x\frac{dx}{dv} + z\frac{dz}{dv} = 0 \end{array}$; quare $x\frac{dx}{dv} - z\frac{dx}{dv} = 0$, & $x = z$,

Pariter, posita x constante, $\begin{array}{l} \frac{dy}{dv} + \frac{dz}{dv} = 0 \\ y\frac{dy}{dv} + z\frac{dz}{dv} = 0 \end{array}$; hinc $y\frac{dy}{dv} - z\frac{dy}{dv} = 0$, & $y = z$.

Unde tres partes quæsitæ debent esse invicem æquales.

Sit pariter $\begin{array}{l} q + x + y + z = S \\ aqq + bxx + cyy + ezz = \text{min}. \end{array}$

ideoque $\frac{dq}{dv} + \frac{dx}{dv} + \frac{dy}{dv} + \frac{dz}{dv} = 0$

$$aq\frac{dy}{dv} + bx\frac{dx}{dv} + cy\frac{dy}{dv} + ez\frac{dz}{dv} = 0.$$

Sint y & z constantes: erit $\frac{dq}{dv} + \frac{dx}{dv} = 0$

$aq\frac{dq}{dv} + bx\frac{dx}{dv} = 0$: unde $aq = bx$. Eodem modo ostenditur, debere esse $aq = cy = ez$. Proinde $\begin{array}{l} q + x + y + z = S \\ aq = bx = cy = ez \end{array}$; unde res ad elementa reducitur, & fit $q = S \times \frac{bce}{abc + abe + ace + bce}$.

Sit pariter $\begin{array}{l} q^m + x^m + y^m + z^m + \ldots = \text{dato} \\ aq^n + bx^n + cy^n + ez^n + \ldots = \text{min}. \end{array}$

Erit

Erit $mq^{m-1}\frac{dq}{dv}+mx^{m-1}\frac{dx}{dv}+my^{m-1}\frac{dy}{dv}+mz^{m-1}\frac{dz}{dv}+\ldots=0$

$$anq^{n-1}\frac{dq}{dv}+bnx^{n-1}\frac{dx}{dv}+cny^{n-1}\frac{dy}{dv}+enz^{n-1}\frac{dz}{dv}+\ldots=0:$$

unde, omnibus quantitatibus mutabilibus præter q & x poſitis conſtantibus, fiunt

$$mq^{m-1}\frac{dq}{dv}+mx^{m-1}\frac{dx}{dv}=0;\quad \text{quare}\quad \frac{q^{m-1}}{x^{m-1}}=-\frac{dx}{dq}$$

$$anq^{n-1}\frac{dq}{dv}+bnx^{n-1}\frac{dx}{dv}=0\qquad \frac{aq^{n-1}}{bx^{n-1}}=-\frac{dx}{dq};$$

hinc $aq^{n-m}=bx^{n-m}=cy^{n-m}=ez^{n-m}\ldots\ldots$

ſeu $\sqrt[n-m]{a}\times q=\sqrt[n-m]{b}\times x=\sqrt[n-m]{c}\times y=\sqrt[n-m]{e}\times z\ldots\ldots$

Exemplum ſecundum. Sit $q+x+y+z\ldots=\text{dato}=S$

$$qxyz\ldots\ldots=\text{max.}$$

Igitur $q+x+y+z+\ldots=S$

$$\log.q+\log.x+\log.y+\log.z+\ldots=\text{max.}$$

Hinc $\frac{dq}{dv}+\frac{dx}{dv}+\frac{dy}{dv}+\frac{dz}{dv}+\ldots=0$

$$\frac{dq}{dv}.\frac{1}{q}+\frac{dx}{dv}.\frac{1}{x}+\frac{dy}{dv}.\frac{1}{y}+\frac{dz}{dv}.\frac{1}{z}+\ldots=0.$$

Poſitis omnibus quantitatibus conſtantibus, præter duas q, x; fiunt

$$\left.\begin{array}{l}\frac{dq}{dv}+\frac{dx}{dv}=0\\ \frac{dq}{dv}.\frac{1}{q}+\frac{dx}{dv}.\frac{1}{x}=0\end{array}\right\};\ \text{unde}\ \frac{1}{q}=\frac{1}{x},\ \text{ſeu}\ q=x:\ \text{quare}\ q=x=y=z\ldots.$$

Si eſſet $q+x+y+z+\ldots=\text{dato}$

$q^m.x^n.y^r.z^s.\ldots=\text{max.}$, ſeu $m\log.q+n\log.x+r\log.y+s\log.z+\ldots=\text{max.}$:

forent $\frac{dq}{dv}+\frac{dx}{dv}+\frac{dy}{dv}+\frac{dz}{dv}+\ldots.=0$

$m\frac{dq}{dv}.\frac{1}{q}+n\frac{dx}{dv}.\frac{1}{x}+r\frac{dy}{dv}.\frac{1}{y}+s\frac{dz}{dv}.\frac{1}{z}+\ldots.=0.$ Hinc, omnibus quantitatibus præter q & x poſitis conſtantibus, fiunt

$$\frac{dq}{dv} + \frac{dx}{dv} = 0$$

$m\frac{dq}{dv}.\frac{1}{q}+n\frac{dx}{dv}.\frac{1}{x} = 0$: igitur $m.\frac{1}{q}=n.\frac{1}{x}$; & proinde $m.\frac{1}{q}=n.\frac{1}{x}=r.\frac{1}{y}=s.\frac{1}{z}\ldots$

unde res ad calculum vulgarem reducitur per æquationes $\begin{array}{l} q + x + y + z + \ldots = S \\ m.\frac{1}{q} = n.\frac{1}{x} = r.\frac{1}{y} = s.\frac{1}{z} \ldots \end{array}$

Obſervatio. Hinc determinantur tam $\frac{\text{maxima}}{\text{minima}}$ relativa ſub datis quibusdam conditionibus, quam $\frac{\text{maxima}}{\text{minima}}$ abſoluta, ſeu $\frac{\text{maxima maximorum}}{\text{minima minimorum}}$. Scilicet una pluribusve quantitatibus mutabilibus poſitis conſtantibus, determinantur relationes mutuæ reliquarum, ut $\frac{\text{maximum}}{\text{minimum}}$ ad has conditiones relatum locum habeat: remotis autem his conditionibus, determinatur $\frac{\text{maximum}}{\text{minimum}}$ abſolutum, ſeu $\frac{\text{maximum maximorum}}{\text{minimum minimorum}}$, ſtabiliendo nempe relationes mutuas omnium quantitatum mutabilium, ut $\frac{\text{maximum maximorum}}{\text{minimum minimorum}}$ locum habeat. Sic uno latere trianguli alicujus magnitudine dato, & ſumma reliquorum magnitudine data; triangulum æquicrurum, in quo latera hæc ſunt invicem æqualia, maximum eſt triangulorum ſub priore conditione conſtructorum: ut vero ſub data perimetro triangulum ſit omnium maximum, ſeu maximum maximorum, tria ejus latera debent eſſe æqualia.

Methodum præcedentem exemplis ad geometriam pertinentibus illuſtrare e re eſſe cenſeo.

§. 186. *Problema.* Inter parallelepipeda rectangula, eadem ſuperficie integra comprehenſa, quæritur illud, cujus capacitas eſt omnium maxima.

Sint x, y, z acies parallelepipedi quæſiti.

Fit ideo $\begin{array}{l} xy+xz+yz = \text{dato} \\ xyz = \text{max.} \end{array}$ ſeu $\begin{array}{l} xy + xz + yz = \text{dato} \\ \log.x+\log.y+\log.z = \text{max.} \end{array}$

hinc $\frac{dx}{dv}(y+z)+\frac{dy}{dv}(x+z)+\frac{dz}{dv}(x+y) = 0$

$$\frac{dx}{dv}.\frac{1}{x} + \frac{dy}{dv}.\frac{1}{y} + \frac{dz}{dv}.\frac{1}{z} = 0.$$

Sit

Sit z conſtans: erit $\left.\begin{array}{l}\frac{dx}{dv}(y+z)+\frac{dy}{dv}(x+z)=0\\ \frac{dx}{dv}\cdot\frac{1}{x}+\frac{dy}{dv}\cdot\frac{1}{y}=0\end{array}\right\}$; quare $\begin{array}{l}y+z:x+z=\frac{1}{x}:\frac{1}{y}=y:x\\ \text{et}\quad z:z=y:x\end{array}$

unde $y=x$. Eodem modo $y=z$. Proinde $x=y=z$.

Problema. Omnia latera figuræ alicujus rectilineæ præter unum dantur magnitudine. Quæritur figura omnium maxima, lateribus datis & latere non dato comprehenſa.

Lemma. Area figuræ cujusvis rectilineæ ſemiſſis eſt ſummæ rectangulorum omnium laterum binorum ſumtorum, uno excepto, ductorum in ſinus ſummæ angulorum externorum, qui inter latera hæc comprehenduntur. Vid. Opuſculum inſcriptum: *Polygonometrie*, Genève 1791.

Exemplum 1. Figura propoſita ſit quadrilatera.

Sint A, B, C tria latera magnitudine data; & ſint y, z anguli figuræ externi, inter latera A & B, B & C reſpective comprehenſi.

Erit $\left.\begin{array}{l}AB\,\text{ſin.}y\\ +AC\,\text{ſin.}(y+z)+BC\,\text{ſin.}z\end{array}\right\}=$ maximo.

Hinc $\left.\begin{array}{l}AB\frac{dy}{dv}\,\text{coſ.}y\\ +AC\left(\frac{dy}{dv}+\frac{dz}{dv}\right)\text{coſ.}(y+z)+BC\frac{dz}{dv}\,\text{coſ.}z\end{array}\right\}=0.$

Ponatur z conſtans: erit $\left.\begin{array}{l}AB\,\text{coſ.}y\\ +AC\,\text{coſ.}(y+z)\end{array}\right\}=0$; unde $B\,\text{coſ.}y=C\,\text{coſ.}(180°-(y+z))$

Pariter facto y conſtante $\left.\begin{array}{l}AC\,\text{coſ.}(y+z)\\ +BC\,\text{coſ.}z\end{array}\right\}=0$: unde $B\,\text{coſ.}z=A\,\text{coſ.}(180°-(y+z))$.

Exemplum 2. Figura propoſita ſit pentagona.

Sint latera data A, B, C, D;
anguli externi quæſiti x, y, z.

Ideoque (Lemma) $\left.\begin{array}{lll}AB\,\text{ſin.}x & & \\ +AC\,\text{ſin.}(x+y) & +BC\,\text{ſin.}y & \\ +AD\,\text{ſin.}(x+y+z) & +BD\,\text{ſin.}(y+z) & +CD\,\text{ſin.}z\end{array}\right\}=\text{max.}$

hinc

Hinc $AB\dfrac{dx}{dv}\cos.x$

$$+AC\left(\frac{dx}{dv}+\frac{dy}{dv}\right)\cos.(x+y) \qquad +BC\frac{dy}{dv}\cos.y \qquad = 0.$$
$$+AD\left(\frac{dx}{dv}+\frac{dy}{dv}+\frac{dz}{dv}\right)\cos.(x+y+z)+BD\left(\frac{dy}{dv}+\frac{dz}{dv}\right)\cos.(y+z)+CD\frac{dz}{dv}\cos.z$$

Positis y & z constantibus, fit $\quad B\cos.x + C\cos.(x+y) + D\cos.(x+y+z) = 0$

positis x & z constantibus, fit $\quad AC\cos.(x+y)+AD\cos.(x+y+z)+BC\cos.y+BD\cos.(y+z) = 0.$

denique positis x & y constantibus, fit $\quad A\cos.(x+y+z) + B\cos.(y+z) + C\cos.z = 0$

Proinde quæstio proposita ad alteram reducitur mere geometricam (aut algebraicam), qua quantitates quæsitæ x, y, z per tres æquationes datas determinantur.

Investigationis methodum eandem esse, quicunque sit numerus laterum figuræ propositæ, ultro patet.

Non immoror demonstrando formularum harum consensui cum altera palmaria figuræ propositæ omnium maximæ proprietate, qua scilicet figura hæc semicirculo inscribitur, cujus diameter est latus quæsitum. (Vide inter alia Opuscula inscripta: *De relatione mutua capacitatis & terminorum figurarum*, Varsav. 1782. *Abregé d'Isoperimetrie elementaire*, Genève 1791.)

Quæri etiam potest figura omnium maxima, lateribus magnitudine datis comprehensa. Investigationis hujus methodum paucis etiam exemplis illustrabo; a casu determinato incipiens, quo figura proposita est quadrilatera.

Sint A, B, C, D latera data quadrilateri, quod omnium maximum fieri debeat. Ducatur recta diagonalis, quæ propositum quadrilaterum in duo triangula dividat. Sint A, B crura unius horum triangulorum; & sit x angulus inter ea comprehensus. Sint C, D crura alterius trianguli; & sit y angulus illis comprehensus.

Erit $AA-2AB\cos.x+BB = CC-2CD\cos.y+DD$

$AB\sin.x+CD\sin.y = \text{maximo.}$

Hinc

Hinc $AB\,\text{fin.}x\frac{dx}{dv} = CD\,\text{fin.}y\frac{dy}{dv}$

$AB\text{cof.}x\frac{dx}{dv}+CD\text{cof.}y\frac{dy}{dv} = 0$, feu $AB\,\text{cof.}x\frac{dx}{dv} = CD\text{cof.}(180^\circ - y)\frac{dy}{dv}$.

Ergo $AB\,\text{fin.}x : CD\,\text{fin.}y = AB\,\text{cof.}x : CD\,\text{cof.}(180^\circ - y)$

$\text{tang.}x = \text{tang.}(180^\circ - y)$; $x = 180^\circ - y$.

Maximum igitur quadrilaterum lateribus datis comprehenfum circulo poteft infcribi.

Exemplum fecundum. Quæratur pentagonum, cujus latera A, B, C, D, E dantur magnitudine, & quod fit omnium maximum.

Ducatur diagonalis, quæ pentagonum propofitum dividat in triangulum, cujus crura fint D, E; & in quadrilaterum, cujus latera reliqua fint A, B, C.

Sit x angulus externus figuræ lateribus A, B interjacens

y - - - - - - - B, C - - -

z - - - - - - - D, E - - -

Erit (*Polygonometrie*) $\begin{matrix} AA+2AB\text{cof.}x \\ +2AC\text{cof.}(x+y)+BB+2BC\text{cof.}y+CC \end{matrix} = DD+2DE\text{cof.}z+EE$

$$\begin{matrix} 2AB\,\text{fin.}x \\ +2AC\,\text{fin.}(x+y)+BC\,\text{fin.}y+DE\text{fin.}z \end{matrix} = \text{maximo}.$$

Hinc $$\left.\begin{matrix} AB\,\text{fin.}x\frac{dx}{dv} \\ +AC\text{fin.}(x+y)\left(\frac{dx}{xv}+\frac{dy}{dv}\right) \\ +BC\,\text{fin.}y\frac{dy}{dv} \end{matrix}\right\} = DE\,\text{fin.}z\frac{dz}{dv}$$

$$\left.\begin{matrix} AB\text{cof.}x\frac{dx}{dv} \\ +AC\text{cof.}(x+y)\left(\frac{dx}{dv}+\frac{dy}{dv}\right)+BC\,\text{cof.}y\frac{dy}{dv}+DE\,\text{cof.}z\frac{dz}{dv} \end{matrix}\right\} = 0.$$

Hinc $$\frac{dx}{dv}\begin{pmatrix} AB\,\text{fin.}x \\ +AC\,\text{fin.}(x+y) \end{pmatrix} + \frac{dy}{dv}\begin{pmatrix} AC\,\text{fin.}(x+y) \\ +BC\,\text{fin.}y \end{pmatrix} - \frac{dz}{dv}DE\text{fin.}z = 0$$

$$\frac{dx}{dv}\begin{pmatrix} AB\,\text{cof.}x \\ +AC\,\text{cof.}(x+y) \end{pmatrix} + \frac{dy}{dv}\begin{pmatrix} AC\,\text{cof.}(x+y) \\ +BC\,\text{cof.}y \end{pmatrix} + \frac{dz}{dv}DE\,\text{cof.}z = 0.$$

Fiat

Fiat z conſtans: erit

$$B\sin.x+C\sin.(x+y):B\cos.x+C\cos.(x+y)=A\sin.(x+y)+B\sin.y:A\cos.(x+y)+B\cos.y.$$

Fiat y conſtans: erit

$$B\sin.x+C\sin.(x+y)\ :\ B\cos.x+C\cos.(x+y)\ =\ \sin.z\ :\ \cos.(180^\circ-z).$$

Pariter, poſita x conſtante, fit

$A\sin.(x+y)+B\sin.y\ :\ A\cos.(x+y)+B\cos.y\ =\ \sin.z\ :\ \cos.(180^\circ-z)$; quæ proportio jam reſultat ex duabus præcedentibus.

Duæ iſtarum proportionum cum æquatione

$$\begin{matrix}AA+2AB\cos.x \\ +2AC\cos.(x+y)+BB+2BC\cos.y+CC\end{matrix}=DD+2DE\cos.z+EE$$

combinatæ tres exhibent æqnationes, quibus quæſtio propoſita ad problema algebraicum determinatum reducitur.

Tertium exemplum. Quæratur hexagonum lateribus A, B, C, D, E, F magitudine datis comprehenſum, cujus area ſit omnium maxima.

Angulus externus inter latera A, B comprehenſus ſit x

$$\begin{array}{rl} B,\ C & \text{- - - - -}\ y \\ D,\ E & \text{- - - - - -}\ x' \\ E,\ F & \text{- - - - -}\ y'. \end{array}$$

Erit
$$\begin{matrix}AA+2AB\cos.x \\ +2AC\cos.(x+y)+BB+2BC\cos.y+CC\end{matrix}=\begin{matrix}DD+2DE\cos.x' \\ +2DF\cos.(x'+y')+EE+2EF\cos.y'+FF\end{matrix}$$

et
$$\begin{matrix}AB\sin.x & +\ DE\sin.x' \\ +AC\sin.(x+y)+BC\sin.y & +\ DF\sin.(x'+y')+EF\sin.y'\end{matrix}=\text{maximo.}$$

Differentietur utraque æquatio; & omnes quantitates mutabiles x, y, x', y' ponantur ſucceſſive conſtantes, duabus exceptis: hinc eruentur tres relationes diverſæ quantitatum harum mutabilium; quæ cum priori æquatione combinatæ totidem ſuppeditabunt æquationes, quot ſunt quantitates incognitæ.

Theorematibus, quibus formulæ, ad quas pervenitur, anſam præbent, evolvendis non immoror.

Problemata hæc propoſui tanquam exempla univerſalitatis ſolutionum calculo differentiali innixarum: eadem vero aptiſſima ſunt ad oſtendendum, quanto [illegible]es & lucidiores eſſe poſſint ſolutiones mere elementares iis caſibus, in quos

quos quadrant; cum facillime & univerſaliter demonſtretur: figuram lateribus magnitudine datis comprehenſam omnium maximam eam eſſe, quæ circulo poteſt inſcribi. (Vide v. gr. *Opuſcula mea* jam nominata.)

Iisdem principiis inſiſtendo determinari poteſt figura omnium maxima, cujus anguli & perimeter dantur.

Problema. Inter pyramides triangulares æquealtas, baſi ſpecie & magnitudine datæ inſiſtentes, ea quæritur, cujus ſuperficies eſt omnium minima.

Altitudo pyramidis dicatur h; latera baſis ſint A, A', A''. Anguli (quæſiti), ſub quibus facies pyramidis lateribus his reſpective adjacentes ad baſin inclinantur, ſint x, x', x'' reſpective. Puncti, in quo altitudo pyramidis baſi occurrit, a lateribus baſis diſtantiæ ſunt reſpective $h\,\text{cot}.x$, $h\,\text{cot}.x'$, $h\,\text{cot}.x''$.

Altitudines facierum pyramidis ſunt $h\,\text{coſec}.x$, $h\,\text{coſec}.x'$, $h\,\text{coſec}.x''$.

Duplum areæ baſis eſt $h(A\,\text{cot}.x + A'\text{cot}.x' + A''\text{cot}.x'')$.

Dupla ſumma facierum eſt $h(A\,\text{coſec}.x + A'\text{coſec}.x' + A''\text{coſec}.x'')$.

Ideoque
$$A\,\text{cot}.x + A'\text{cot}.x' + A''\text{cot}.x'' = \text{dato}$$
$$A\,\text{coſec}.x + A'\text{coſec}.x' + A''\text{coſec}.x'' = \text{minimo}.$$

Hinc
$$A\frac{dx}{dv}\text{coſec}.^2x + A'\frac{dx'}{dv}\text{coſec}.^2x' + A''\frac{dx''}{dv}\text{coſec}.^2x'' = 0$$
$$A\frac{dx}{dv}\text{cot}.x\,\text{coſec}.x + A'\frac{dx'}{dv}\text{cot}.x'\text{coſec}.x' + A''\frac{dx''}{dv}\text{cot}.x''\text{coſec}.x'' = 0.$$

Fiat x'', conſtans: erit
$$A\frac{dx}{dv}\text{coſec}.^2x + A'\frac{dx'}{dv}\text{coſec}.^2x' = 0$$
$$A\frac{dx}{dv}\text{cot}.x\,\text{coſec}.x + A'\frac{dx'}{dv}\text{cot}.x'\text{coſec}.x' = 0.$$

Hinc
$$A\frac{dx}{dv} : -A'\frac{dx'}{dv} = \text{coſec}.^2x' : \text{coſec}.^2x$$
$$A\frac{dx}{dv} : -A'\frac{dx'}{dv} = \text{cot}.x'\text{coſec}.x' : \text{cot}.x\,\text{coſec}.x$$

Proinde
$$\text{coſec}.^2x' : \text{coſec}.^2x = \text{cot}.x'\text{coſec}.x' : \text{cot}.x\,\text{coſec}.x$$
$$\text{coſec}.x' : \text{coſec}.x = \text{cot}.x' : \text{cot}.x$$
$$1 : 1 = \text{coſ}.x' : \text{coſ}.x;$$ ideoque $x = x'$.

Eodem modo infertur $x = x''$. Pyramidum igitur triangularium æquealtarum, baſi

basi datæ insistentium, ea terminatur superficie minima, cujus facies ad basin æqualiter inclinantur. Vide Opuscula modo commemorata, in quibus propositio hæc methodo mere elementari adstruitur, & uberes ejus consequentiæ evolvuntur.

§. 187. In quæstionibus postremo loco tractatis quantitas, quæ $\genfrac{}{}{0pt}{}{\text{maxima}}{\text{minima}}$ reddi debet, functio est plurium quantitatum mutabilium, a se invicem ita independentium, ut determinatio unius reliquas non determinet. Criteria, quibus dijudicari potest, utrum ejusmodi functio $\genfrac{}{}{0pt}{}{\text{maxima}}{\text{minima}}$ esse possit, nec ne? primus omnium universalissime constituit celeb. DE LA GRANGE (*Miscellanea Societatis Taurinensis*, Tom. I.). Sufficiat hoc loco, functiones duarum variabilium considerare, atque criteria hæc ex formulis capite præcedenti traditis deducere.

Quoniam est
$$
{}^{x}_{y}P = P \begin{array}{llll}
+\frac{\Delta x}{1}\,{}^{x}\mathrm{d}'P & +\frac{\Delta x^2}{1.2}\,{}^{x}\mathrm{d}''P & +\frac{\Delta x^3}{1.2.3}\,{}^{x}\mathrm{d}'''P & \\
+\frac{\Delta y}{1}\,{}^{y}\mathrm{d}'P & +2\frac{\Delta x.\Delta y}{1.2}\,{}^{y}\mathrm{d}'\,{}^{x}\mathrm{d}'P & +3\frac{\Delta x^2.\Delta y}{1.2.3}\,{}^{y}\mathrm{d}'\,{}^{x}\mathrm{d}''P & \\
 & +\frac{\Delta y^2}{1.2}\,{}^{y}\mathrm{d}''P & +3\frac{\Delta x.\Delta y^2}{1.2.3}\,{}^{y}\mathrm{d}''\,{}^{x}\mathrm{d}'P & + \ldots \\
 & & +\frac{\Delta y^3}{1.2.3}\,{}^{y}\mathrm{d}'''P &
\end{array}
$$

ut functio hæc fiat $\genfrac{}{}{0pt}{}{\text{maxima}}{\text{minima}}$, debet esse $\begin{array}{l} {}^{x}\mathrm{d}'P = 0 \\ {}^{y}\mathrm{d}'P = 0 \end{array}$

Est autem

$$
\begin{array}{l}
\Delta x^2\,{}^{x}\mathrm{d}''P \\
+2\Delta x.\Delta y\,{}^{y}\mathrm{d}'\,{}^{x}\mathrm{d}'P \\
+\Delta y^2\,{}^{y}\mathrm{d}''P
\end{array}
= {}^{x}\mathrm{d}''P\left[\Delta x^2 + 2\Delta x.\Delta y\frac{{}^{y}\mathrm{d}'\,{}^{x}\mathrm{d}'P}{{}^{x}\mathrm{d}''P} + \Delta y^2\frac{{}^{y}\mathrm{d}''P}{{}^{x}\mathrm{d}''P}\right]
$$

$$
= {}^{x}\mathrm{d}''P\left[\Delta x^2 + 2\Delta x.\Delta y\frac{{}^{y}\mathrm{d}'\,{}^{x}\mathrm{d}'P}{{}^{x}\mathrm{d}''P} + \Delta y^2\left[\frac{{}^{y}\mathrm{d}'\,{}^{x}\mathrm{d}'P}{{}^{x}\mathrm{d}''P}\right]^2\right]
$$

$$
-\Delta y^2.\,{}^{x}\mathrm{d}''P\left[\left[\frac{{}^{y}\mathrm{d}'\,{}^{x}\mathrm{d}'P}{{}^{x}\mathrm{d}''P}\right]^2 - \frac{{}^{y}\mathrm{d}''P}{{}^{x}\mathrm{d}''P}\right]
$$

$$
= {}^{x}\mathrm{d}''P\left[\Delta x + \Delta y\frac{{}^{y}\mathrm{d}'\,{}^{x}\mathrm{d}'P}{{}^{x}\mathrm{d}''P}\right]^2 + \Delta y^2\,{}^{x}\mathrm{d}''P\left[\frac{{}^{y}\mathrm{d}''P}{{}^{x}\mathrm{d}''P} - \left[\frac{{}^{y}\mathrm{d}'\,{}^{x}\mathrm{d}'P}{{}^{x}\mathrm{d}''P}\right]^2\right]
$$

$$
= {}^{x}\mathrm{d}''P\left[\Delta x + \Delta y\frac{{}^{y}\mathrm{d}'\,{}^{x}\mathrm{d}'P}{{}^{x}\mathrm{d}''P}\right]^2 + \Delta y^2\left[{}^{y}\mathrm{d}''P - \frac{({}^{y}\mathrm{d}'\,{}^{x}\mathrm{d}'P)^2}{{}^{x}\mathrm{d}''P}\right]
$$

Pro-

Proinde quoniam quadrata $\left[\Delta x + \Delta y \frac{{}^{y}d'{}^{x}d'P}{{}^{x}d''P}\right]^2$, Δy^2 semper sunt positiva, functio P maxima est, si sit tam ${}^{x}d''P$, quam ${}^{y}d''P - \frac{({}^{y}d'{}^{x}d'P)^2}{{}^{x}d''P}$ negativa: & posita ${}^{x}d''P$ negativa, ac proinde $-\frac{({}^{y}d'{}^{x}d'P)^2}{{}^{x}d''P}$ positiva; necesse est, ut sit ${}^{y}d''P$ negativa; prætereaque (signis omissis) debet esse ${}^{y}d''P > \frac{({}^{y}d'{}^{x}d'P)^2}{{}^{x}d''P}$, seu ${}^{x}d''P \times {}^{y}d''P > ({}^{y}d'{}^{x}d'P)^2$.

Contra functio P minima fit, si tam ${}^{x}d''P$, quam ${}^{y}d''P - \frac{({}^{y}d'{}^{x}d'P)^2}{{}^{x}d''P}$ est positiva: & posita ${}^{x}d''P$ positiva; oportet, fit ${}^{y}d''P$ positiva, eaque talis, ut ${}^{y}d''P . {}^{x}d''P > ({}^{y}d'{}^{x}d'P)^2$.

Exemplum hoc functionis duarum variabilium docet: criteria, quibus $\begin{smallmatrix}\text{maxima}\\\text{minima}\end{smallmatrix}$ functionum plurium variabilium discernuntur, nonnisi summa adhibita cautione posse determinari; nisi ex ipsa quæstionis natura immmediate judicari possit, utrum quæstio ad $\begin{smallmatrix}\text{maxima}\\\text{minima}\end{smallmatrix}$ referenda sit, aut nulli limiti ansam præbeat?

Exemplum primum. Sit $P = xx + xy + yy - ax - by$

$$\begin{array}{ll} {}^{x}d'P = 2x + y - a & {}^{x}d''P = +2 \quad {}^{y}d'{}^{x}d'P = 1; \\ {}^{y}d'P = x + 2y - b & {}^{y}d''P = +2 \end{array}$$

proinde ${}^{x}d''P . {}^{y}d''P (= 4) > ({}^{y}d'{}^{x}d'P)^2$.

Fiat $\begin{array}{l} {}^{x}d'P (= 2x + y - a) = 0 \\ {}^{y}d'P (= x + 2y - b) = 0 \end{array}$. Erit $\begin{array}{l} x = \frac{2a - b}{3} \\ y = \frac{2b - a}{3} \end{array}$; & functio P his valoribus respondens, nempe $-\frac{aa - ab + bb}{3}$, est omnium minima. (a)

Exem-

(a) Irrepsit error typographicus in pag. 649. Calculi differentialis Euleri, ubi functio hæc minima dicitur esse $-\frac{aa + ab - bb}{3}$, loco $\frac{-aa + ab - bb}{3}$.

Exemplum secundum. Sit

$$\begin{array}{lll} P = x^3 - 3axy + y^3 & {}^y\mathrm{d}'P = -3ax + 3yy & \\ {}^x\mathrm{d}'P = 3xx - 3ay & & {}^y\mathrm{d}'\,{}^x\mathrm{d}'P = -3a. \\ {}^x\mathrm{d}''P = 2.3x & {}^y\mathrm{d}''P = +2.3y & \end{array}$$

Fiat $\begin{matrix} xx - ay = 0 \\ -ax + yy = 0 \end{matrix}$: erit $\begin{matrix} y = \frac{xx}{a} \\ -ax + \frac{x^4}{aa} = 0; \quad x = {0 \atop a} \quad y = {0 \atop a}. \end{matrix}$

Quoniam factis ${x=0 \atop y=0}$, est ${}^x\mathrm{d}''P \times {}^y\mathrm{d}''P < ({}^y\mathrm{d}'\,{}^x\mathrm{d}'P)^2$; functio P his valoribus respondens non est omnium $\frac{\text{maxima}}{\text{minima}}$.

Fiant ${x=a \atop y=a}$: erunt ${{}^x\mathrm{d}''P = 6a \atop {}^y\mathrm{d}''P = 6a}$; proinde ${}^x\mathrm{d}''P.{}^y\mathrm{d}''P > ({}^y\mathrm{d}'P.{}^x\mathrm{d}'P)^2$, & functio fit $\frac{\text{maxima}}{\text{minima}}$. Atqui ${{}^x\mathrm{d}''P = +6a \atop {}^y\mathrm{d}''P = +6a}$; igitur functio $P = -a^3$ minima est.

§. 188. Quæstiones ad $\frac{\text{maxima}}{\text{minima}}$ pertinentes magnam cum ductu tangentium a puncto extra curvam dato habent affinitatem.

Per punctum enim datum agatur recta, ad quam tanquam axem curva referatur. Si curva versus axem hunc concava est; angulus, quem recta curvam tangens ab hoc puncto ducta facit cum axe, major est angulis, quos rectæ curvam secantes ab eodem puncto ductæ faciunt cum eodem axe: contra prior angulus minor est posterioribus, si curva est versus axem convexa. Problema igitur, a puncto extra curvam dato rectam ducere, quæ curvam contingat, ad alterum reducitur, quo ab eodem puncto recta duci jubetur curvæ occurrens, quæ $\frac{\text{maximum}}{\text{minimum}}$ angulum cum axe comprehendat. Subtangens curvæ dicatur t; ordinata rectangula per punctum contactus acta dicatur y: erit $\frac{y}{t}$ tangens trigonometrica anguli, quem recta curvam contingens cum axe facit; proinde $\frac{y}{t}\,\frac{\text{maximum}}{\text{minimum}}$: unde $\frac{t\frac{dy}{dx} - y\frac{dt}{dx}}{tt} = 0$; $t\frac{dy}{dx} = y\frac{dt}{dx}$. Atqui $\frac{dt}{dx} = 1$, sive abscissarum origo in ipso puncto dato ponatur, sive in alio puncto intervallo dato a priori distante. Ergo $t \times \frac{dy}{dx} = y$, & $t = y \times \frac{dx}{dy}$; quod consentit cum §. 40.

Rela-

Relatio hæc non effugit ſuperioris ſeculi mathematicos, quorum molimina utrumque problema de ductu tangentium & de maximis ac minimis ſolvendi ad generales calculi differentialis regulas viam ſtraverunt; quos inter FERMATIUM, ROBERVALLIUM, PASCALIUM, BARROWIUM nominare ſufficiat.

§. 189. Quamvis quæſtiones ad $\frac{\text{maxima}}{\text{minima}}$ pertinentes inde ab eo tantum tempore cenſeri debeant perfecte ſolutæ, quo generales calculi differentialis regulæ iis applicatæ fuerunt: abunde tamen, quæ ſuperſunt, monumenta docent, veteres geometras hujus generis quæſtionibus operam haud inanem impendiſſe. Ipſa Elementa EUCLIDIS propoſitiones nonnullas ad $\frac{\text{maxima}}{\text{minima}}$ pertinentes ſiſtunt, ex. gr. in 5ta & 9na Libri ſecundi. Luculentiſſime autem hoc comprobant libri antiquorum ad analyſin geometricam pertinentes, a PAPPO recenſi, & partim ſuperſtites, partim a recentioribus mathematicis reſtituti; præſertim Tractatus *de Sectione rationis & ſpatii*, *de Sectione determinata*, & *de Inclinationibus*, atque inprimis Liber V. *Sectionum conicarum* APOLLONII.

Quando problema aliquod geometrice ſolvitur; defectus occurſus mutui linearum in conſtructione ducendarum monet de propoſitæ quæſtionis impoſſibilitate: algebra autem impoſſibilitatem hanc indicat per introductionem quantitatum (quas vocant) imaginariarum, ſeſe invicem non deſtruentium. Sed imperfecta ſolutionis æquationum conditio obſtat, quominus algebra hoc reſpectu ſufficiat. Cum vero æquationum ſecundi tantum gradus ſolutio plana ſit; functionum ejusdem ordinis $\frac{\text{maximæ}}{\text{minimæ}}$ facillime per algebram elementarem determinantur, quod paucis exemplis declarabo.

Primum exemplum. Sit $2ax - xx = x(2a - x)$ functio ſecundi ordinis, cujus maximum quæritur. Sit $2ax - xx = p$: erit $x - a = \pm \sqrt{(aa - p)}$. Ut problema ſit poſſibile; requiritur, non ſit p major quam aa: proinde maximus ipſius p valor eſt aa; & tunc $x - a = 0$, $x = a$, $2a - x = a$; ſeu duo termini producti $x(2a - x)$ ſunt invicem æquales.

Exemplum ſecundum. Sit numerus datus a dividendus in duas partes, ſic ut ſumma productorum quadratorum ipſarum per numeros datos m, n ſit omnium minima. Sint duæ partes x, $a - x$; & ponatur $mxx + n(a - x)^2 = p$:

 erit

erit $x - a\frac{n}{m+n} = \frac{\pm\sqrt{(p(m+n)-aamn)}}{m+n}$. Ut problema possibile sit, debet esse $p(m+n) \overset{=}{>} aamn$; seu minimus ipsius p valor est $aa\frac{mn}{m+n}$: & casu, quo p est omnium minima, fit $x = \frac{an}{m+n}$, $a - x = \frac{am}{m+n}$.

Exemplum tertium. Sit $\sqrt{(aa+xx)} - \frac{m}{n}x = b$

$$\text{unde} \quad aa + xx = bb + \frac{2m}{n}bx + \frac{mm}{nn}xx.$$

1°. Sit $m = n$: prodit æquatio primi gradus, quæ $\begin{matrix}\text{maximo}\\\text{minimo}\end{matrix}$ locum non præbet.

2°. Sit $m > n$: $x + b\frac{mn}{mm-nn} = \pm\sqrt{\left(aa\frac{nn}{mm-nn} + bb\frac{n^4}{(mm-nn)^2}\right)}$; unde functio proposita nullum admittit limitem.

3°. Sit $m < n$: $x - b\frac{mn}{nn-mm} = \pm\sqrt{\left(bb\frac{n^4}{(nn-mm)^2} - aa\frac{nn}{nn-mm}\right)}$. Itaque, posito $m < n$, valor omnium minimus ipsius b talis est, ut sit

$$b = a\frac{\sqrt{(nn-mm)}}{n};$$

$$\text{\& tum est} \quad x = b \times \frac{mn}{nn-mm} = a \times \frac{m}{\sqrt{(nn-mm)}}$$

$$\sqrt{(aa+xx)} = a \times \frac{n}{\sqrt{(nn-mm)}}.$$

Methodum hanc elementarem investigationi minimi ceræ apium cellularum feliciſſime applicuit ſagaciſſimus DN. LE SAGE; qui ita celebre hoc tam apud mathematicos quam apud rerum naturalium ſtudioſos problema primus ad elementa reduxit, & quidem ad inveſtigationem minimi functionis $\sqrt{(aa+xx)} - \frac{m}{n}x$ casu, quo $m < n$ (uti in poſteriori exemplo). Solutionem hanc amiciſſime mecum communicavit, quo tempore ſtudia mea mathematica paterno amore regebat; pariter ac plurimas obſervationes ad hoc caput pertinentes, quæ extant in *Commentariis Academiæ Berolinenſis* ad annum 1782: *Sur le minimum de cire des alveoles des abeilles, & en particulier ſur un minimum minimorum relatif à cette matiere.* Vid. etiam *Relatio mutua &c.*

CAPUT

Caput decimum nonum.

De radiis curvaturæ, et de curvis evolutione genitis.

§. 190.

Sint M, M' duo puncta alicujus curvæ; per quæ agantur duæ rectæ normales MN, $M'N'$, quæ axi in punctis N, N' occurrant. Distantia NN' punctorum horum est $PP' - PN + P'N' = \Delta x - y\frac{dy}{dx} + y'\frac{dy'}{dx}$.

Quoniam

$$y'y' = yy + 2\frac{\Delta x}{1}y\frac{dy}{dx} + 2\frac{\Delta x^2}{1.2}\left[\left(\frac{dy}{dx}\right)^2 + y\frac{ddy}{dx^2}\right] + 2\frac{\Delta x^3}{1.2.3}\left[3\frac{dy}{dx}\cdot\frac{ddy}{dx^2} + y\frac{d^3y}{dx^3}\right] + 2\frac{\Delta x^4}{1\ldots4}\left[3\left(\frac{ddy}{dx^2}\right)^2 + 4\frac{dy}{dx}\cdot\frac{d^3y}{dx^3} + y\frac{d^4y}{dx^4}\right] + \cdots$$

fit

$$y'\frac{dy'}{dx} = y\frac{dy}{dx} + \frac{\Delta x}{1}\left[\left(\frac{dy}{dx}\right)^2 + y\frac{ddy}{dx^2}\right] + \frac{\Delta x^2}{1.2}\left[3\frac{dy}{dx}\cdot\frac{ddy}{dx^2} + y\frac{d^3y}{dx^3}\right] + \frac{\Delta x^3}{1.2.3}\left[3\left(\frac{ddy}{dx^2}\right)^2 + 4\frac{dy}{dx}\cdot\frac{d^3y}{dx^3} + y\frac{d^4y}{dx^4}\right] + \frac{\Delta x^4}{1\ldots4}\left[10\frac{ddy}{dx^2}\cdot\frac{d^3y}{dx^3} + 5\frac{dy}{dx}\cdot\frac{d^4y}{dx^4} + y\frac{d^5y}{dx^5}\right] + \cdots$$

Unde

$$NN' = \Delta x\left[1 + \left(\frac{dy}{dx}\right)^2 + y\frac{ddy}{dx^2}\right] + \frac{\Delta x^2}{1.2}\left[3\frac{dy}{dx}\cdot\frac{ddy}{dx^2} + y\frac{d^3y}{dx^3}\right] + \frac{\Delta x^3}{1.2.3}\left[3\left(\frac{ddy}{dx^2}\right)^2 + 4\frac{dy}{dx}\cdot\frac{d^3y}{dx^3} + y\frac{d^4y}{dx^4}\right] + \frac{\Delta x^4}{1\ldots4}\left[10\frac{ddy}{dx^2}\cdot\frac{d^3y}{dx^3} + 5\frac{dy}{dx}\cdot\frac{d^4y}{dx^4} + y\frac{d^5y}{dx^5}\right] + \cdots$$

§. 191. Duæ normales MN, $M'N'$ sibi invicem (si fieri possit) occurrant in Z puncto. Punctum hoc occursus sic determinatur.

Fig. 56. In triangulo NZN' eſt ſin.NZN' : ſin.N' $= NN' : NZ$

$$\begin{array}{llcl}
\text{ſeu} & \text{ſin.}(M'N'P'-MNP) & : \text{ſin.}M'N'P' & = NN' : NZ \\
\text{unde} & \text{coſ.}N-\text{ſin.}N\cot.N' & : \quad 1 & = NN' : NZ \\
 & \text{coſ.}N-\text{ſin.}\frac{dy'}{dx} & : \quad 1 & = NN' : NZ \\
 & NP-MP\frac{dy'}{dx} & : \quad MN & = NN' : NZ \\
\text{et} & NP-MP\frac{dy'}{dx} & : \quad NN' & = MN : NZ
\end{array}$$

Atqui $\frac{dy'}{dx} = \frac{dy}{dx} + \frac{\Delta x}{1}\cdot\frac{ddy}{dx^2} + \frac{\Delta x^2}{1.2}\cdot\frac{d^3y}{dx^3} + \frac{\Delta x^3}{1.2.3}\cdot\frac{d^4y}{dx^4} + \dots$

Igitur $NP-MP\frac{dy'}{dx} = -y\left(\frac{\Delta x}{1}\cdot\frac{ddy}{dx^2} + \frac{\Delta x^2}{1.2}\cdot\frac{d^3y}{dx^3} + \frac{\Delta x^3}{1.2.3}\cdot\frac{d^4y}{dx^4} + \dots\right)$

$$\begin{aligned}
NZ : MN = {} & 1 + \left(\frac{dy}{dx}\right)^2 + y\frac{ddy}{dx^2} \qquad : -y\left[\frac{ddy}{dx^2} + \frac{\Delta x}{1.2}\cdot\frac{d^3y}{dx^3} + \frac{\Delta x^2}{1.2.3}\cdot\frac{d^4y}{dx^4} + \dots\right] \\
& + \frac{\Delta x}{1.2}\left[3\frac{dy}{dx}\cdot\frac{ddy}{dx^2} + y\frac{d^3y}{dx^3}\right] \\
& + \frac{\Delta x^2}{1.2.3}\left[3\left(\frac{ddy}{dx^2}\right)^2 + 4\frac{dy}{dx}\cdot\frac{d^3y}{dx^3} + y\frac{d^4y}{dx^4}\right] \\
& + - - - - - - \\
& + - - - - - -
\end{aligned}$$

$$\begin{aligned}
\text{et} \quad MZ : MN = {} & 1 + \left(\frac{dy}{dx}\right)^2 \qquad : -y\left[\frac{ddy}{dx^2} + \frac{\Delta x}{1.2}\cdot\frac{d^3y}{dx^3} + \frac{\Delta x^2}{1.2.3}\cdot\frac{d^4y}{dx^4} + \dots\right] \\
& + \frac{\Delta x}{1.2}\left[3\frac{dy}{dx}\cdot\frac{ddy}{dx^2} + y\frac{d^3y}{dx^3}\right] \\
& + \frac{\Delta x^2}{1.2.3}\left[3\left(\frac{ddy}{dx^2}\right)^2 + 4\frac{dy}{dx}\cdot\frac{d^3y}{dx^3} + y\frac{d^4y}{dx^4}\right] \\
& + - - - - - - \\
& + - - - - - -
\end{aligned}$$

Ut conſequentiæ ex hac proportione necti poſſint, duo diſtinguendi ſunt caſus, uti in §. 170. 179.

§. 192.

§. 192. *Primus casus.* Exponentes differentiales successivi $\frac{dy}{dx}$, $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, . . . signo impossibilis non afficiantur.

Casu hoc posterioris rationis limes æqualis est rationi $1+\left(\frac{dy}{dx}\right)^2 : -y\frac{ddy}{dx^2}$; proinde & rationis prioris limes æqualis est eidem rationi $1+\left(\frac{dy}{dx}\right)^2 : -y\frac{ddy}{dx^2}$, seu $\lim. MZ : MN = 1+\left(\frac{dy}{dx}\right)^2 : -y\frac{ddy}{dx^2}$, & $\lim. MZ = MN \times \frac{1+\left(\frac{dy}{dx}\right)^2}{-y\frac{ddy}{dx^2}}$. Atqui $MN = y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}$ (§. 110.) Ergo $\lim. MZ = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{-\frac{ddy}{dx^2}}$.

Observatio prima. Quamvis hæc expressio formam habeat negativam, reapse fit positiva, quando curva versus axem concava est.

Observatio secunda. Ex Z puncto agatur ZQ ipsi MP perpendicularis. Est $MQ = MZ \times \frac{MP}{MN} = MZ \times \frac{1}{\sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)}}$: unde $\lim. MQ = \frac{1+\left(\frac{dy}{dx}\right)^2}{-\frac{ddy}{dx^2}}$.

Observatio tertia. Quoniam $\text{sin}. NMP = \frac{\frac{dy}{dx}}{\sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)}}$; est $\frac{d.\,\text{sin}. NMI}{dx} = \frac{\frac{ddy}{dx^2}}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}$. Hinc $\lim. \frac{1}{MZ} = -\frac{d.\,\text{sin}. NMP}{dx} = -\frac{d.\,\text{cos}. PMT}{dx}$.

Æquatio hæc immediate sic potest demonstrari. Recta $M'P'$ ipsi ZQ occurrente in p, sunt $\text{sin}. NMP = \frac{ZQ}{ZM}$, $\text{sin}. N'M'P' = \frac{Zp}{ZM'}$: hinc $\text{sin}. NMP - \text{sin}. N'M'P' = \frac{ZQ}{ZM} - \frac{Zp}{ZM'} = \frac{PP'}{ZM} + Zp\left(\frac{1}{ZM} - \frac{1}{ZM'}\right)$; $-\frac{\Delta.\text{sin}. NMP}{\Delta x} = \frac{1}{ZM} + \frac{Zp}{\Delta x} \times \frac{ZM' - ZM}{ZM \,.\, ZM'}$, & $-\frac{d.\text{sin}. NMP}{dx} = \frac{1}{ZM}$.

Obser-

Observatio quarta. $M'P'$ recta occurrat in t tangenti MT per M punctum ductæ. $M't = -\left(\frac{\Delta x^2}{1.2} \cdot \frac{ddy}{dx^2} + \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} + \frac{\Delta x^4}{1 \ldots 4} \cdot \frac{d^4y}{dx^4} + \ldots\right)$ (§. 170.)

et $Mt^2 = \Delta x^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)$.

Proinde $\frac{Mt^2}{M't} = \frac{1 + \left(\frac{dy}{dx}\right)^2}{-\left(\frac{1}{1.2} \cdot \frac{ddy}{dx^2} + \frac{\Delta x}{1.2.3} \cdot \frac{d^3y}{dx^3} + \frac{\Delta x^2}{1 \ldots 4} \cdot \frac{d^4y}{dx^4} + \ldots\right)}$

et $\lim. \frac{Mt^2}{M't} = \frac{1 + \left(\frac{dy}{dx}\right)^2}{-\frac{1}{1.2} \cdot \frac{ddy}{dx^2}} = 2\,\frac{1 + \left(\frac{dy}{dx}\right)^2}{-\frac{ddy}{dx^2}}$.

Atqui $\lim. MQ = \frac{1 + \left(\frac{dy}{dx}\right)^2}{-\frac{ddy}{dx^2}}$. Ergo $2 \lim. MQ = \lim. \frac{Mt^2}{M't}$.

§. 193. Sumta $Mq = 2MQ = \lim. \frac{Mt^2}{M't} = \frac{1 + \left(\frac{dy}{dx}\right)^2}{-\frac{ddy}{dx^2}}$, fiat semper $tq' = \frac{Mt^2}{M't}$; & describatur curva per puncta q, q' hoc modo determinata transiens. Tum centro Z radio $MZ = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{-\frac{ddy}{dx^2}}$ describatur circumferentia circuli; quæ proinde per M & q transit, atque rectam MT, ac proinde etiam curvam MM' in M tangit. Circumferentia hæc occurrat rectæ $M'P'$ in R & R' punctis.

Fig. 56. 1°. *Primus casus.* Sit Mq quantitatis $\frac{Mt^2}{M't}$ limes quod ad parvitatem; ideoque semper sit $tq' > tR'$, & arcus qq' extra arcum qR'. Quoniam $Mt^2 = R't \times tR = M't \times tq'$, & $tq' > tR'$; ergo $tR > M't$: proinde arcus circuli MR cadit intra arcum curvæ MM'; ac tanto magis quælibet circumferentia rectam MT in M tangens & radio minore quam MZ descripta, utpote circulum ipsum MR intus tangens, intra curvam MM' cadit. Tum radio MZ' majore quam MZ describatur circulus, qui rectæ MP in $'q$ puncto, & rectæ $M'P'$ in r & r' punctis

ctis occurrat. Erit ideo $M'q > Mq$, & punctum $'q$ extra curvam qq'; pariterque arcus $'qr'$ extra arcum qR': quare, puncto M' ad punctum M accedente, arcus curvæ qq' cadit inter arcus circulares qR', $'qr'$; itaque erit $tq' < tr'$. Atqui $Mt^2 = M't \times tq' = rt \times tr'$: igitur $M't > rt$; & circumferentia radio $Z'M > ZM$ descripta cadit inter arcum MM' & tangentem MT. Proinde hoc casu arcus curvæ MM' jacet inter arcus circulares MR, Mr, sic ut nullus circulus, curvam in puncto M tangens, inter arcus MM' & MR cadat.

Secundus casus. Sit Mq quantitatis $\frac{Mt^2}{M't}$ limes quod ad magnitudinem; ita ut semper sit $tq' < tR'$, & arcus qq' intra arcum qR'. Quoniam $Mt^2 = Rt \times tR' = M't \times tq'$, & $tq' < tR'$: erit $Rt < M't$; proinde arcus MR cadit inter tangentem & arcum MM'; ac tanto magis quælibet circumferentia radio majore quam MZ descripta, quippe circulum MR extus contingens, inter tangentem MT & arcum MM' cadit. Tum radio $MZ' < MZ$ describatur circulus, qui rectæ $M'P$ in r & r' punctis, & rectæ MQ in $'q$ puncto occurrat. Erit ideo $M'q < Mq$, $tr' < tR'$, & arcus qq' jacebit inter arcus qR', $'qr'$. Proinde puncto M' ad punctum M accedente, fiet $tq' > tr'$; ideoque $tM' < tr$, & arcus MM' cadet inter tangentem MT & arcum Mr. Quare hoc etiam casu nullus circulus, curvam in M tangens, inter arcus MM' & MR transit. Fig. 56. 2°.

Nullus itaque circulus curvam in M tangens, radio sive majore sive minore quam MZ descriptus, cadit inter arcum curvæ MM' & arcum circularem MR, ad partes puncto M utrinque vicinas. Circumferentia igitur radio MZ descripta arcum curvæ MM' strictius tangit, quam ulla alia circumferentia, sive major sive minor priore. Hinc circumferentia hæc curvam osculari, & radius ZM *radius osculator* dicitur. Curvatura arcus MM' ad curvaturam arcus circularis MR propius accedit, quam ad curvaturam cujusvis alius arcus circularis; unde radius ZM vocatur *radius curvaturæ* arcus MM' in puncto M.

Formula ideo radii curvaturæ est $\frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{-\frac{ddy}{dx^2}}$; cujus applicationem nonnullis exemplis illustrabo.

§. 194.

§. 194. *Exempla.* 1°. Sit parabola conica, cujus æquatio eſt $yy = 2px$. Fit $y\frac{dy}{dx} = p$, $y\frac{ddy}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$, $y\frac{ddy}{dx^2} = -\frac{pp}{yy}$, $\frac{ddy}{dx^2} = -\frac{pp}{y^3}$, $1+\left(\frac{dy}{dx}\right)^2 = \frac{pp+yy}{yy}$. $MZ = \frac{(pp+yy)^{\frac{3}{2}}}{pp}$, & $MQ = \frac{y(pp+yy)}{pp}$.

Obſervatio. Facto $y = 0$, fit $MZ = p$; & hic eſt limes (quod ad parvitatem) normalium parabolæ.

2°. Sit ellipſis conica, cujus æquatio eſt $yy = \frac{bb}{aa}(aa - xx)$. Hinc $y\frac{dy}{dx} = -\frac{bb}{aa}x$, $\left(\frac{dy}{dx}\right)^2 + y\frac{ddy}{dx^2} = -\frac{bb}{aa}$, $y\frac{ddy}{dx^2} = -\frac{b^4}{aa}\cdot\frac{1}{yy}$, $\frac{ddy}{dx^2} = -\frac{b^4}{aa}\cdot\frac{1}{y^3}$. $MZ = \frac{((aa-bb)yy+b^4)^{\frac{3}{2}}}{ab^4}$. Fiat $y = 0$: erit $MZ = \frac{bb}{a}$; & hic eſt limes (quod ad parvitatem) normalium ellipſis. Sit $a = b$: erit $MZ = \frac{a^6}{a^5} = a$, ſeu conſtans; quo caſu ellipſis in circulum abiit.

3°. Eodem modo determinatur radius curvaturæ hyperbolæ conicæ ex æquatione $yy = \frac{bb}{aa}(xx - aa)$; qua fit $MZ = \frac{((aa+bb)yy + b^4))^{\frac{3}{2}}}{ab^4}$.

4°. Sit cyclois vulgaris, cujus æquatio eſt $y = \sqrt{(2rx - xx)} + r\,\text{arc.ſin.v.}\frac{x}{r}$. Fit $\frac{dy}{dx} = \sqrt{\frac{2r-x}{x}}$, $\frac{ddy}{dx^2} = -\frac{r}{x\sqrt{(2rx-xx)}}$, $1+\left(\frac{dy}{dx}\right)^2 = \frac{2r}{x}$; $MZ = 2r\sqrt{(2r(2r-x))}$.

§. 195. Hucusque tradita nituntur ſuppoſitione, dari limitem quoti $\frac{Mt^2}{M't}$; quo caſu radius curvaturæ poteſt determinari. Ut *alterum caſum*, quo nullus eſt quoti hujus limes, diſtinctius evolvam; a curvis ordiar, quarum æquatio ſimpliciſſima eſt, nempe $y = p^{1-n}x^n$.

Deſcribatur quicunque circulus curvam in vertice contingens, cujus radius ſit r; & rectæ in hoc circulo axi ordinatim applicatæ ſint y'.

Quoniam $y = p^{1-n}x^n$, eſt $yy = p^{2-2n}x^{2n}$.

Atqui $y'y' = x(2r - x)$

Ergo $yy : y'y' = p^{2-2n}x^{2n} : x(2r - x)$.

1°. Sit

1°. Sit $2n = 1$, seu $n = \frac{1}{2}$: erit $yy : y'y' = p^{2-2n} : 2r - x = p : 2r - x$. Proinde, imminuta x, rationis posterioris limes est ratio $p : 2r$; & ratio hæc est ratio æqualitatis, si $p = 2r$, seu $r = \frac{1}{2}p$.

2°. Sit $2n > 1$; ideoque $yy : y'y' = p^{2-2n}x^{2n-1} : 2r - x$. Imminuta x, nullus est limes (quod ad parvitatem) posterioris rationis; itaque nullus etiam est limes (quod ad parvitatem) rationis prioris: quælibet igitur circuli circumferentia, curvam in vertice contingens, cadit extra curvam ad partes vertici vicinas, utcunque parvus sit radius ejus r.

3°. Sit $2n < 1$; $yy : y'y' = p^{2-2n} : x^{1-2n}(2r - x)$. Imminuta x, nullus est posterioris rationis limes, quod ad magnitudinem; ideoque nullus etiam est prioris rationis limes, quod ad magnitudinem. Proinde quælibet circuli circumferentia curvam in vertice contingens cadit intra curvam ad partes vertici vicinas, utut magnus sit ejus radius r.

Ideoque parabolas inter, æquatione $y = p^{1-n}x^n$ determinatas, parabola conica sola est, cujus curvatura in vertice cum curvatura circuli conferri possit.

Idem consequitur ex applicatione formulæ generalis radii osculi ad casum, in quem quadrare non potest: eum nempe, quo nullus est quoti $\frac{Mt^2}{M't}$ limes, sive quod ad magnitudinem, sive quod ad parvitatem; seu quo formula radii curvaturæ $R = \dfrac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{-\frac{ddy}{dx^2}}$ fit aut zero, aut impossibilis.

Etenim posito $y = p^{1-n}x^n$, fit $\dfrac{dy}{dx} = np^{1-n}x^{n-1}$

$$\frac{ddy}{dx^2} = n \cdot \overline{n-1}\, p^{1-n}x^{n-2}$$

$$\text{unde} \quad R = \frac{(1 + nnp^{2-2n}x^{2n-2})^{\frac{3}{2}}}{-n \cdot \overline{n-1}\, p^{1-n}x^{n-2}}.$$

1°. Sit $n = 2$: erit $R = -\frac{1}{2}p \times \left(1 + 4\frac{xx}{pp}\right)^{\frac{3}{2}}$; & facta $x = 0$, $R = -\frac{1}{2}p$.

2°. Sit

2°. Sit $n > 2$: erit $R = \frac{\left(1 + nn\frac{x^{2n-2}}{p^{2n-2}}\right)^{\frac{3}{2}}}{-n.n-1\frac{x^{n-2}}{p^{n-1}}}$; & facta $x = 0$, $R = -\frac{p^{n-1}}{n(n-1)0^{n-2}}$, quod est signum impossibilis. Nullus est circulus, utut magno radio describatur, curvam in vertice contingens, qui extra curvam cadat ad partes vertici vicinas.

3°. Sit $n \begin{smallmatrix} < 2 \\ > 1 \end{smallmatrix}$; erit $R = -p^{n-1}x^{2-n} \times \frac{\left(1 + nn\frac{x^{2n-2}}{p^{2n-2}}\right)^{\frac{3}{2}}}{n.n-1}$: proinde facta $x = 0$, est $R = 0$; seu nullus est circulus, radio utut parvo descriptus, qui curvam in vertice ita contingat, ut ad partes vertici vicinas intra curvam cadat.

4°. Sit $n = 1$: erit $R = \frac{2^{\frac{3}{2}}}{0} \times x$. Hoc casu linea proposita est recta: proinde contradictorium est, de curvatura ejus dicere; quod monemur signo impossibilitatis $\frac{1}{0}$, ab abscissa x independente. Idem contingit, si sit $n = 0$; quo casu linea recta parallela est axi, ad quem refertur.

5°. Sit $n \begin{smallmatrix} < 1 \\ > 0 \end{smallmatrix}$: permutatis coordinatis, casus hic ad tres priores reducitur.

6°. Sit n negativa, seu curva proposita sit hyperbolica.

Tunc $R = -\frac{(x^{2+2n} + nnp^{2+2n})^{\frac{3}{2}}}{n.n+1p^{1+n}} \times \frac{1}{x^{2n+1}}$. Et quoniam contradictoria est suppositio $x = 0$; introductione signi $\frac{1}{0^{2n+1}}$ monemur, contradictorium esse, loqui de curvatura curvæ in puncto per absurdum ficto, ubi asymptoto occurrat.

Quo minor est x, eo magis radii curvaturæ valor accedit ad $R = -\frac{nn}{n+1}\frac{p^{2n+2}}{x^{2n+1}}$; & quoniam nullus est abscissæ x limes quod ad parvitatem, nullus etiam est radii R limes quod ad magnitudinem.

Sit $ppy = xx(a-x)$. Erit $pp\frac{dy}{dx} = x(2a-3x)$

$$pp\frac{ddy}{dx^2} = 2(a-3x) \qquad R = \frac{pp}{2(3x-a)}\left(1 + \frac{xx(2a-3x)^2}{p^4}\right)^{\frac{3}{2}}$$

Fiat

Fiat $3x=a$: erit $R=\frac{pp}{2a.0}\left(1+\frac{a^4}{9p^4}\right)^{\frac{3}{2}}$. Huic abſciſſæ reſpondet punctum flexus contrarii. (Vid. Fig. 47.)

Sit $p^3y=x^3(a-x)$. Tunc $p^3\frac{dy}{dx}=xx(3a-4x)$

$$p^3\frac{ddy}{dx^2}=2x(3a-6x) \qquad R=\frac{p^3}{2x(6x-3a)}\left(1+\frac{xx(3a-4x)^2}{p^6}\right)^{\frac{3}{2}}.$$

Utrique abſciſſæ $\begin{matrix}x=0\\x=\frac{1}{2}a\end{matrix}$ reſpondent puncta flexus contrarii, quibus nullus reſpondet limes quoti $\frac{Mt^2}{M't}$. (Vid. Fig. 48.)

In puncto flexus contrarii ob $\frac{ddy}{dx^2}=0$ eſt $\frac{dy}{dx}=C$, $y=C'+Cx$. Proinde æquatio curvæ eo propius accedit ad æquationem lineæ rectæ, quo puncta curvæ ad punctum flexus contrarii propius accedunt.

§. 196. Conſideratio curvaturæ curvarum ad focum aliquem relatarum in matheſi, mixta præſertim, frequenter magni momenti applicationibus inſervit; quare breviter eam explicabo.

Sit F focus, ad quem curva MM' refertur per radios vectores FM, FM', atque angulos AFM, AFM'. Per M agatur recta tangens MT, cui FM' in t occurrat. Curva eſt verſus focum F $\begin{matrix}\text{concava}\\\text{convexa}\end{matrix}$, prouti $Ft \gtrless FM'$. Fig. 57.

Sit $FM=y$, $FM'=y'$, $AFM=x$, $MFM'=\Delta x$.

Itaque $y'=y+\frac{\Delta x}{1}\cdot\frac{dy}{dx}+\frac{\Delta x^2}{1.2}\cdot\frac{ddy}{dx^2}+\frac{\Delta x^3}{1.2.3}\cdot\frac{d^3y}{dx^3}+\frac{\Delta x^4}{1\ldots4}\cdot\frac{d^4y}{dx^4}+\ldots$

$$Ft=FM\frac{\text{ſin.}FMT}{\text{ſin.}(FMT-\Delta x)}=y\times\frac{1}{\text{coſ.}\Delta x-\text{cot.}FMT\,\text{ſin.}\Delta x}$$

$$=y\,\text{ſec.}\Delta x(1+\text{cot.}FMT\,\text{tang.}\Delta x+\text{cot.}^2FMT\,\text{tang.}^2\Delta x+\text{cot.}^3FMT\,\text{tang.}^3\Delta x+\ldots)$$

$$=y(1+\tfrac{1}{2}\text{tang.}^2\Delta x-\tfrac{\frac{1}{2}}{1}\cdot\tfrac{\frac{1}{2}}{2}\text{tang.}^4\Delta x\ldots)(1+\text{cot.}FMT\,\text{tang.}\Delta x+\text{cot.}^2FMT\,\text{tang.}^2\Delta x+\ldots)$$

$$=y+y\,\text{cot.}FMT\,\text{tang.}\Delta x+\tfrac{1}{2}y\,\text{tang.}^2\Delta x \qquad +\tfrac{1}{2}y\,\text{cot.}FMT\,\text{tang.}^3\Delta x+\ldots$$
$$+y\,\text{cot.}^2FMT\,\text{tang.}^2\Delta x+y\,\text{cot.}^3FMT\,\text{tang.}^3\Delta x\ldots$$

Atqui $\text{cot.}FMT=\frac{1}{y}\frac{dy}{dx}$ (§. 49.). Ergo, curva poſita verſus focum concava, eſt

 $M't$

$$M't = \frac{dy}{dx}(\text{tang.}\Delta x\text{-}1) + \tfrac{1}{2}y\,\text{tang.}^2\Delta x \qquad + \tfrac{1}{2}\frac{dy}{dx}.\,\text{tang.}^3\Delta x + \ldots$$
$$+\frac{1}{y}.\left(\frac{dy}{dx}\right)^2\text{tang.}^2\Delta x + \frac{1}{yy}.\left(\frac{dy}{dx}\right)^2.\,\text{tang.}^3\Delta x$$
$$-\frac{1}{1.2}.\frac{ddy}{dx^2}.\Delta x^2 \qquad - \frac{1}{1.2.3}.\frac{d^3y}{dx^3}.\Delta x^3$$

Proinde quoti $\frac{M't}{\Delta x^2}$ limes (si quis detur) est $\frac{1}{2}y + \frac{1}{y}.\left(\frac{dy}{dx}\right)^2 - \frac{1}{1.2}.\frac{ddy}{dx^2}$.

Scilicet curva versus focum est $\begin{matrix}\text{concava}\\\text{convexa}\end{matrix}$, si $\frac{1}{2}y + \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{1}{1.2}.\frac{ddy}{dx^2} \begin{matrix}>\\<\end{matrix} 0$. Et posito curvam ad utramque puncti M partem extendi, M est punctum flexus contrarii, si sit $\frac{1}{2}y + \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{1}{1.2}.\frac{ddy}{dx^2} = 0$.

Ab radio vectore MF abscindatur $Mq = \lim.\frac{Mt^2}{M't} = \frac{yy + \left(\frac{dy}{dx}\right)^2}{\frac{1}{2}y + \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{1}{1.2}\frac{ddy}{dx^2}}$; & ex q puncto erigatur perpendiculum qR, quod normali per M ductæ in R occurrat: erit MR diameter curvaturæ in puncto M (quod demonstratur, uti §.193.). Sed $MR = Mq\,\text{sec.}\,qMR = Mq\,\text{cosec.}\,FMT = \frac{Mq}{\text{sin.}\,FMT} = Mq \times \frac{\sqrt{\left(yy + \left(\frac{dy}{dx}\right)^2\right)}}{y}$. Proinde

$MR = \frac{\left(yy + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{1}{2}yy + \left(\frac{dy}{dx}\right)^2 - \frac{1}{2}y\frac{ddy}{dx^2}}$; & recta MR bifariam divisa in Z, fit radius curvaturæ $MZ = \frac{\left(yy + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{yy + 2\left(\frac{dy}{dx}\right)^2 - y\frac{ddy}{dx^2}}$.

Formulas has variis exemplis illustrabo.

Exemplum primum. Sit $y = r$ quantitas constans: tunc $MZ = \frac{y^3}{yy} = y$, quæ est æquatio circuli.

Exemplum secundum. Sit $y = p\,\text{sec.}^2\tfrac{1}{2}x$, quæ est æquatio focalis parabolæ conicæ.

$$\frac{dy}{dx} = p\,\text{sec.}^2\tfrac{1}{2}x\,\text{tang.}\tfrac{1}{2}x = y\,\text{tang.}\tfrac{1}{2}x$$
$$\frac{ddy}{dx^2} = p(\text{sec.}^2x\,\text{tang.}^2\tfrac{1}{2}x + \tfrac{1}{2}\text{sec.}^4\tfrac{1}{2}x) \qquad \lim.\frac{M't}{\Delta x^2} = \tfrac{1}{4}.\frac{yy}{p}$$

Mq

$$Mq = \frac{yy + yy\,\text{tang.}^2\tfrac{1}{2}x}{\tfrac{1}{4}\frac{yy}{p}} = 4p\,\text{sec.}^2\tfrac{1}{2}x = 4y.$$

$$MZ = \frac{y^3\,\text{sec.}^3\tfrac{1}{2}x}{\tfrac{1}{2}\frac{y^3}{p}} = 2p\,\text{sec.}^3\tfrac{1}{2}x = 2y\,\text{sec.}\tfrac{1}{2}x = 2y\sqrt{\frac{y}{p}}.$$

Exemplum tertium. Sit $y = \frac{bb}{a+e\,\text{cos.}x}$, quæ est æquatio ellipsis conicæ ad focum relatæ. Fit $\frac{dy}{dx} = \frac{bbe\,\text{sin.}x}{(a+e\,\text{cos.}x)} = e\,\text{sin.}x \times \frac{yy}{bb}$ $\quad \lim.\frac{M't}{\Delta x^2} = \tfrac{1}{2}a.\frac{yy}{bb}.$

$$\frac{ddy}{dx^2} = \frac{bbe\,\text{cos.}x}{(a+e\,\text{cos.}x)^2} + \frac{2bbee\,\text{sin.}^2x}{(a+e\,\text{cos.}x)^3}$$

$$Mq = \frac{yy + ee\,\text{sin.}^2x\frac{y^4}{b^4}}{\tfrac{1}{2}a\frac{yy}{bb}} = 2\frac{y(2a-y)}{a}.$$

$$MZ = \frac{\sqrt{\left(yy+\left(\frac{dy}{dx}\right)^2\right)}}{y} \times \frac{y(2a-y)}{a} = \sqrt{\left(1 - \frac{y(2a-y)}{bb}\right)} \times \frac{y(2a-y)}{a}.$$

Eædem formulæ ad hyperbolam quoque applicantur.

Exemplum quartum. Sit $y = r^{\frac{x}{\varphi}}$, quæ est æquatio spiralis logarithmicæ.

Fit $\frac{dy}{dx} = \frac{1}{\varphi}y.\frac{1}{\varphi}\log.r$ $\quad \lim.\frac{M't^2}{\Delta x^2} = \tfrac{1}{2}y\left(1 + \frac{1}{\varphi\varphi}\log.^2r\right)$

$\frac{ddy}{dx^2} = y.\frac{1}{\varphi\varphi}\log.^2r$ $\quad \lim.\frac{Mt^2}{M't} = 2y$ $\quad$ Proinde limites hi pariter crescunt in progressione geometrica.

$MZ = y\sqrt{\left(1 + \frac{1}{\varphi\varphi}\log.^2r\right)}$. Radii igitur curvaturæ etiam in progressione geometrica crescunt.

Exemplum quintum. Sit $y = a\,\text{sec.}x \pm b$, quæ est æquatio conchoidis.

$$\frac{dy}{dx} = a\,\text{sec.}x\,\text{tang.}x$$

$$\frac{ddy}{dx^2} = a\,\text{sec.}x\,\text{tang.}^2x + a\,\text{sec.}^3x = a\,\text{sec.}x(\text{tang.}^2x + \text{sec.}^2x)$$

$$MZ = \frac{(aa\,\text{sec.}^4x \pm 2ab\,\text{sec.}x + bb)^{\frac{3}{2}}}{\mp 2ab\,\text{sec.}x\,\text{tang.}^2x \pm ab\,\text{sec.}x + bb}.$$

1°. Sit

1°. Sit $b = 0$: linea, cujus æquatio $y = a\,\mathrm{sec.}x$, recta est; de cujus curvatura dicere contradictorium est, quod monet introductio signi impossibilis $\frac{1}{0}$.

2°. Sit $b = a$: est $MZ = a\dfrac{(\mathrm{sec.}^4x \pm 2\mathrm{sec.}x + 1)^{\frac{3}{2}}}{\mp 2\mathrm{sec.}x(\mathrm{tang.}^2x \mp 1) + 1}$.

Fiat $x = 0$: erit $MZ = \dfrac{\frac{2}{3}a}{\frac{0}{3}a}$. Nempe conchois inferior casu, quo $b = a$ & $x = a$, in polo cuspidem habet; & dum ad polum accedit, nullus est radii curvaturæ limes quod ad parvitatem. Vid. Fig. 58.

3°. Conchois superior habet punctum flexus contrarii, quod determinatur per æquationem $2a\,\mathrm{sec.}x\,\mathrm{tang.}^2x = a\,\mathrm{sec.}x + b$. Fiat $b = a$: erit $2\mathrm{sec.}x\,\mathrm{tang.}^2x = a(\mathrm{sec.}x + 1)$; unde $2\mathrm{sec.}x(\mathrm{sec.}x - 1) = 1$, & $\mathrm{sec.}x = \begin{cases} \dfrac{\sqrt{3}+1}{2} \\ -\dfrac{\sqrt{3}-1}{2} \end{cases}$, quorum valorum prior tantum proposito satisfacit.

4°. Facto $x = 90°$, fit $MZ = a \times \dfrac{\mathrm{sec.}^6 90°}{\mathrm{sec.}90°\,\mathrm{tang.}^2 90°} = a\,\mathrm{sec.}^3 90° = a \times \infty^3$. Signum hoc impossibilis monet: contradictorium esse de curva loqui casu, quo $x = 90°$; nempe conchois curva est asymptotica, cujus asymptota est axi perpendicularis.

§. 197. Cum doctrina curvaturæ arctissime connectitur, quam primus geometrica methodo tradidit celeb. HUGENIUS, theoria curvarum evolutione genitarum; strictim itaque hoc loco adhuc exponenda.

Fig. 59. Curvæ AMM' filum seu linea flexilis circumplicata intelligatur; &, manente una extremitate illi affixa, altera extremitas, stylo ex. gr. illi annexo, sensim ita abduci concipiatur, ut pars fili MN, quæ soluta est, semper in directum extensa sit, & curvam tangat in puncto M, ubi illam deserit. Curva ANN', quam stylus motu hoc describit, dicitur *evolutione* curvæ AMM' *genita*; ipsa vero AMM' dicitur *evoluta*.

Observatio. Filum curvæ AMM' applicatum potest in A terminari; quo casu curva evolutione genita transit per punctum A. Sed fieri etiam potest, ut filum ultra curvæ punctum A protendatur juxta rectam AA' positione & magnitudine datam, quæ scilicet curvam AMM' in puncto A contingit. Quo casu curva evolutione genita transit per punctum A'.

Centro

Centro M radio MN deſcribatur circulus, & agatur recta NT ipſum in N tangens. Dico: rectam NT in eodem puncto N contingere curvam evolutione deſcriptam.

1°. Sit N' punctum curvæ NN' remotius a principio A, quam eſt punctum N. Sit $N'M'$ ſitus fili, quo extremum ejus pervenit in N'; & ſit M' punctum curvæ evolutæ AMM', ubi filum $N'M'$ eam contingit: rectæ NM, $N'M'$ ſibi invicem occurrant in puncto m', & recta $M'N'$ occurrat in n' tangenti NT circuli. Fig. 59. 1°.

In triangulo mixtilineo $MM'm'$ eſt $Mm' + m'M' > MM'$; proinde addito utrinque arcu $MA = MN$, fit $Mm' + m'M' + MN > M'N'$:

unde $m'N \quad > m'N'$. Atqui, propter angulum rectum N, $m'n' > m'N$: ergo (a fortiori) $m'n' > m'N'$; proinde punctum n' tangentis NT eſt extra curvam NN'.

2°. Sit N' punctum inter puncta A & N. Sit $N'M'$ ſitus fili, quo extremum ejus eſt in N'. Agatur recta MN', quæ tangenti NT in n' puncto occurrat. In triangulo mixtilineo $MM'N'$ eſt $MM' + M'N' > MN'$, hoc eſt $MN > MN'$; atqui propter angulum rectum N eſt $Mn' > MN$: ergo a fortiori $Mn' > MN'$; ideoque punctum n' eſt extra curvam NN'. Fig. 59. 2°.

Circulus igitur centro M radio MN deſcriptus & curva evolutione genita NN' habent in N puncto tangentem communem; proindeque circulus hic & curva NN' ſeſe invicem contingunt in N.

Porro autem circulus hic & curva NN' ſeſe invicem in N puncto ita contingunt, ut prior poſteriorem oſculetur, ſeu ut nullus alius circulus, curvam in N contingens, inter hanc curvam & priorem circulum tranſire poſſit. Etenim

1°. Sit M' punctum quodvis curvæ AMM' ultra punctum M ſitum; ac ſit $M'N'$ ſitus fili, quo curvam MM' in M' tangit, eodemque ſitu filum in puncto R' occurrat circumferentiæ radio MN deſcriptæ. In triangulo mixtilineo Fig. 59. 1°.

$M'MR$ eſt $M'M + MR' > M'R'$
ſeu $\quad M'N' \quad > M'R'$; proinde circumferentia centro M radio MN deſcripta cadit intra curvam NN'; & nulla circumferentia, radio minore quam MN deſcripta, atque curvam in N contingens, cadit inter arcum curvæ NN'

 &

& arcum circularem NR'. Sit autem Nm' radius major quam NM; atque filum per m' transiens curvis MM', NN', NR' in M', N', R' punctis respective occurrat. Quoniam $Mm' + m'M' > MM'$

est $Nm' + m'M' > M'N'$:

unde $m'N > m'N'$; ideoque circulus centro m', radio $m'N$ descriptus extra curvam NN' cadit.

Fig. 59. 2°. 2°. Sit M' punctum quodvis curvæ AM inter puncta A & M situm; sit rursus $M'N'$ situs fili, quo curvam MM' tangit; & hoc situ filum occurrat in R' circumferentiæ NR', & curvæ NN' in N' puncto.

In triangulo mixtilineo $MM'R'$ est $MM' + M'R' > MR' > MN > MM' + M'N'$:

unde $M'R' > M'N'$;

proinde circumferentia NR' cadit extra curvam NN': & a fortiori nulla circumferentia curvam NN' contingens in N, & radio majore quam MN descripta, inter arcus NN', NR' transire potest. Sit autem Nm' radius minor quam NM; & sit $m'M'N'$ situs fili, quo per m' punctum transit. Quoniam

$Mm' + m'M' > MM'$; addito utrinque arcu $M'A$, est

$Mm' + m'N' > MM'N' > MN$:

unde $m'N' > m'N$. Proinde circulus centro m' radio $m'N$ descriptus cadit intra curvam NN'.

Circumferentia igitur circuli, centro M, radio MN descripti, curvam osculatur in N puncto.

§. 198. Curva itaque AMM' locus est centrorum circulorum curvam ANN' osculantium. Si curva NN' in aliquo puncto nullum habet radium curvaturæ, ideo quod nullus sit radii hujus limes quod ad parvitatem; curvæ MM', NN' hoc puncto sibi invicem occurrunt: si vero curva NN' in aliquo puncto nullum habet radium curvaturæ, ideo quod nullus sit radii hujus limes quod ad magnitudinem; nullus etiam est limes distantiæ, qua curva MM' ab curva NN' removetur. Quando autem aliquis est radii curvaturæ limes quod ad parvitatem, curva MM curvæ NN' non occurrit; & curva MM' intra limites quosdam continetur, si simul aliquis est radii curvaturæ curvæ NN' limes quod ad magnitudinem.

Sit

Sit Z centrum circuli ejusdem cum curva SM curvaturæ in puncto M; ex quo demittatur in axem SP recta perpendicularis ZX. Data æquatione curvæ SM, determinatur etiam æquatio curvæ, quæ est locus punctorum Z, per coordinatas SX, ZX. Etenim sit ZQ ipsi MP perpendicularis. Erunt Fig. 56.

$$ZX = MQ - PQ = \frac{1+\left(\frac{dy}{dx}\right)^2}{-\frac{ddy}{dx^2}} - y$$

$$SX = SP + ZQ = SP + MQ\,\text{tang.}ZMQ = x + \frac{dy}{dx} \times \frac{1+\left(\frac{dy}{dx}\right)^2}{-\frac{ddy}{dx^2}}.$$

Exemplum primum. Sit $yy = rr - xx$: $\frac{dy}{dx} = -\frac{x}{y}$;

$$\frac{ddy}{dx^2} = -\frac{1}{y} + \frac{x\frac{dy}{dx}}{yy} = -\frac{1}{y} - \frac{xx}{y^3} = -\frac{rr}{y^3};$$

$$ZX = \frac{1+\frac{xx}{yy}}{\frac{rr}{y^3}} - y = y - y = 0;$$

$SX = x - \frac{x}{y} \times x = x - x = 0.$ Proinde puncta Z ad punctum unicum reducuntur; quod consentit cum palmaria circumferentiæ circuli proprietate.

Exemplum secundum. Sit $y = \sqrt{(2rx - xx)} + r\,\text{arc.sin.v.}\frac{x}{r}$:

$$\text{erit } \frac{dy}{dx} = \sqrt{\frac{2r-x}{x}}$$

$$\frac{ddy}{dx^2} = -\frac{r}{x} \times \frac{1}{\sqrt{(2rx-xx)}}$$

$$ZX = -\left(r\,\text{arc.sin.v.}\frac{x}{r} - \sqrt{(2rx-xx)}\right)$$

$$SX = 4r - x.$$

Unde facile deducitur: cycloidis vulgaris evolutam pariter esse cycloidem vulgarem, eamque cum priore congruentem.

Exemplum tertium. Sit $yy = 2px$: $\frac{dy}{dx} = \frac{p}{y}$; $\frac{ddy}{dx^2} = -\frac{p\frac{dy}{dx}}{yy} = -\frac{pp}{y^3}$.

Hinc $ZX = \frac{y^3}{pp}$, $SX = p + 3x$.

$$ZX^2 = \frac{y^6}{p^4} = \frac{8p^3x^3}{p^4} = 8\frac{x^3}{p} = 8\frac{\left(\frac{SX-p}{3}\right)^3}{p}\text{ ; } p.ZX^2 = \tfrac{8}{27}(SX-p)^3.$$ Parabolae igitur conicae evoluta & ipſa parabola eſt æquatione $pyy = x^3$ determinata.

Contra determinatio curvæ evolutione genitæ requirit rectificationem curvæ evolutæ. Et cum curvæ propoſitæ unica reſpondeat evoluta: uni eidemque curvæ reſpondent innumeræ curvæ evolutione hujus genitæ, quatenus variatur fili evolventis principium.

§. 199. Quamdiu curva evoluta non habet punctum flexus contrarii, curva evolutione ejus genita continue verſus easdem partes progreditur. Quodſi autem curva evoluta punctum aliquod flexus contrarii habet; dum filum abducitur a punctis curvæ ſenſu oppoſito flexæ, ſtylus ad partes oppoſitas regreditur: unde curva evolutione genita duobus conſtat ramis, in puncto regreſſus in cuſpidem coëuntibus, quorum uterque verſus easdem partes $\begin{matrix}\text{concavus}\\ \text{convexus}\end{matrix}$ eſt. Quare doctrina evolutionis curvarum apta eſt ad dirimendam controverſiam de hoc curvarum genere inter mathematicos agitatam, & ab ill. Eulero in *Commentariis Acad. Berol. ad ann.* 1749. ſolide enodatam. Hic breviſſime argumentum illud attingere ſufficiet.

Sit n numerus fractus ſpurius poſitivus, cujus denominator eſt numerus par. Sit $py = \varphi'x \pm \varphi x . x^n$; ita ut functiones $\varphi'x$, φx factorem x non comprehendant. Per hypotheſin x non poteſt eſſe negativa.

Sint

$$\varphi'x = A' + B'x + C'x^2 + D'x^3 + \ldots$$
$$\varphi x = A + Bx + Cx^2 + Dx^3 + \ldots$$

Igitur

$$py = A' + B'x + C'x^2 + D'x^3 + \ldots \pm (Ax^n + Bx^{n+1} + Cx^{n+2} + Dx^{n+3} + \ldots)$$
$$p\frac{dy}{dx} = B' + 2C'x + 3D'x^2 + \ldots \pm (nAx^{n-1} + (n+1)Bx^n + (n+2)Cx^{n+1} + (n+3)Dx^{n+2} + \ldots)$$
$$p\frac{ddy}{dx^2} = 2C' + 3.2D'x + \ldots \pm (n.\overline{n-1}Ax^{n-2} + \overline{n+1}.nBx^{n-1} + \overline{n+2}.\overline{n+1}Cx^n + \overline{n+3}.\overline{n+2}Dx^{n+1} + \ldots)$$

Poſito

Posito $n > 2$; ramus uterque $\begin{smallmatrix}\text{concavus}\\ \text{convexus}\end{smallmatrix}$ eſt verſus axem ad partes abſciſſæ $x = 0$ proximas, prouti C' eſt numerus $\begin{smallmatrix}\text{negativus}\\ \text{poſitivus}\end{smallmatrix}$. Et ordinata abſciſſæ $x = 0$ reſpondens $\begin{smallmatrix}\text{major}\\ \text{minor}\end{smallmatrix}$ eſt ordinatis ipſi proximis utroque ramo terminatis, prouti B' numerus eſt $\begin{smallmatrix}\text{negativus}\\ \text{poſitivus}\end{smallmatrix}$ (poſito A' poſitivo).

Caput vicesimum.

De problematibus, quæ vocantur, isoperimetricis.

§. 200.

Problemata, quibus caput hoc deſtinatur, præcipuos tam vergentis ſuperioris, quam præſentis ſeculi mathematicos occuparunt; & nonnulla eorum materiam ipſis ſuggeſſere provocationum, quibus vires ſuas mutuo tentarunt. Poſt Joh. Bernoullium (a) cel. Fontaine jam inde ab anno 1732. ad methodum aliquam generalem problemata hæc ſolvendi eniti allaboravit. (b). Ill. Eulerus eximium de illis anno 1744. edidit tractatum, inſcriptum: *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes.* Sagaciſſimus de la Grange ſolutionem ipſorum ad methodum univerſalem & mere analyticam reduxit, quæ *methodus variationum* nominari conſuevit. (c) Quamvis autem diverſi poſt eum ſcriptores methodum hanc tradiderint vel transcripſerint; difficile tamen etiamnum mihi videtur principia ejus ita explanare, ut rigori mathematico aſſueti tirones nullibi hæreant.

Cum tamen argumentum hoc inter mathematicos celebre magni ſit in matheſi, inprimis mixta, momenti; præcipuam ſaltem ejus partem diſtincte & accurate exponere, atque ad inſtar ceterarum calculi differentialis & integralis applicationum captui tironum accommodare conatus ſum. Nec diffiteor, me, pro hujus ſcopi ratione, argumentum illud eadem univerſalitate amplexum non

 eſſe,

(a) Vid. *Memoires de Paris* pour 1706.
(b) Vid. *Memoires de Paris* pour 1734. & *Memoires preſentés à l'Academie de Paris* par Mr. Fontaine. Paris 1764.
(c) *Memoires de Turin.* T. II.

esse, qua ab EULERO & DE LA GRANGE fuit pertractatum. Veruntamen satis me ostendisse spero, fieri posse, ut ad simplicia & solida methodi limitum principia reducatur.

Expositionem meam ita instituam, ut singularia primum evolvam problemata; tum a casibus particularibus ad generaliores paulatim annitar.

Fig. 60. §. 201. *Problema.* Sint A & A' duo puncta positione data; sit etiam PM recta positione data; ac sint F, F' duo factores magnitudine dati: quæritur punctum M tale, ut $F\times AM + F'\times MA'$ sit omnium minimum.

A punctis A, A' in rectam MP demittantur perpendicula Aa, $A'a'$; & sint $Aa = b$, $A'a' = b'$, $aa' = a$. Sint etiam $AM = z$, $A'M = z'$, $aM = y$, $a'M = y'$.

Quoniam $Fz + F'z' =$ minimo;

oportet, sit $F\dfrac{dz}{dv} + F'\dfrac{dz'}{dv} = 0$.

Atqui $zz = bb + yy$ ergo $\dfrac{dz}{dv} = \dfrac{y}{z}\cdot\dfrac{dy}{dv} = \dfrac{dy}{dv}\cos.AMa$

$z'z' = b'b' + y'y'$ ergo $\dfrac{dz'}{dv} = \dfrac{y'}{z'}\cdot\dfrac{dy'}{dv} = \dfrac{dy'}{dv}\cos.A'Ma'$.

Quare $F\dfrac{dy}{dv}\cos.AMa + F'\dfrac{dy'}{dv}\cos.A'Ma' = 0$.

Atqui $y + y'(=a)$ datur magnitudine. Igitur $\dfrac{dy}{dv} + \dfrac{dy'}{dv} = 0$;

& hinc $F\cos.AMa = F'\cos.A'Ma'$.

Proinde summa $Fz + F'z'$ est omnium minima, quando $F\cos.AMa = F'\cos.A'Ma'$.

Exemplum. Sit $F = F'$: casu igitur, quo $z + z'$ est omnium minima, fit $\cos.AMa = \cos.A'Ma'$; proinde $AMa = A'Ma'$; & puncta A, M, A' jacent in directum.

Corollarium. Per puncta data A, A' agantur duæ rectæ AB, $A'B'$, positione datæ MP parallelæ: erit $F\cos.MAB = F'\cos.A'MP$, casu, quo summa $Fz + F'z'$ est omnium minima.

Fig. 61. §. 202. Ducatur quæcunque recta, quæ (v. gr.) sit rectis AB, $A'B'$ perpendicularis, atque ipsis in punctis B, B' occurrat. Dividatur BB' in partes quot-

quotcunque æquales in punctis P, P', P'', P''', P^{iv}.... Per puncta divisionum agantur totidem rectæ ipsi BB' perpendiculares. Sint F, F', F'', F''', F^{iv}.... coefficientes dati; & determinanda sint in perpendiculis istis puncta M, M', M'', M''', M^{iv}.... ita posita, ut ductis rectis

AM, MM', $M'M''$, $M''M'''$, $M'''M^{iv}$.... quæ dicantur respective

z, z', z'', z''', z^{iv} .... summa

$Fz + F'z' + F''z'' + F'''z''' + F^{iv}z^{iv} + \ldots$ sit omnium minima.

Ut huic conditioni satisfiat, debet esse $F\,\text{cos}.MAB = F'\,\text{cos}.M'MP$

$$= F''\,\text{cos}.M''M'P'$$
$$= F'''\,\text{cos}.M'''M''P''$$
$$= F^{iv}\,\text{cos}.M^{iv}M'''P'''$$
$$= \text{- - - -}$$

Pro vertice igitur quocunque M, inter duos vertices $'M$, M' posito, erit $F\,\text{cos}.M'MP$ quantitas constans.

Sit S punctum aliquod in axe BB' (producto v. gr.); & sint

F, F', F'', F''', F^{iv}.... functiones quædam similes abscissarum

SP, SP', SP'', SP''', SP^{iv}.... Laterum figuræ, seu axis BB' partium, numero manente eodem; pariter est $F\,\text{cos}.M'MP$ quantitas constans.

Proinde, aucto partium axis numero, in curva etiam, quæ limes est polygonorum $AMM'M''M'''M^{iv}$.... quantitatis $F\,\text{cos}.M'MP$ limes erit quantitas aliqua constans, quæ dicatur C. Itaque $F\frac{dy}{dz} = C$. Hinc $\frac{dz}{dy} = \frac{F}{C}$;

$1 + \left(\frac{dx}{dy}\right)^2 = \frac{FF}{CC}$; $\left(\frac{dx}{dy}\right)^2 = \frac{FF-CC}{CC}$; $\frac{dy}{dx} = \frac{C}{\sqrt{(FF-CC)}}$.

Existente igitur X functione quadam abscissarum x, & designante z arcum hujus curvæ: si fuerit $\frac{dZ}{dz} = X$; & quæratur curva, in qua Zz sit omnium minima: est $\frac{dy}{dz} = \text{cos}.TMP = \frac{C}{X}$; unde $\frac{dy}{dx} = \frac{C}{\sqrt{(XX-CC)}}$.

Exemplum primum. Sit $X = \sqrt{x}$: erit $\frac{dy}{dx} = \frac{\sqrt{c}}{\sqrt{(x-c)}}$; $y = c' + \sqrt{(c(x-c))}$. Sit $y = 0$, quando $x = c$: erit $c' = 0$, $y = \sqrt{(c(x-c))}$. Æquatio hæc est ad parabo-

rabolam; & minimum hoc pertinet ad motum projectilium terrestrium in spatio libero.

Exemplum secundum. Sit $X=\frac{1}{\surd x}$: fit $-\frac{dy}{dx}=\surd\frac{x}{c-x}$,

$y=c'+\surd(cx-xx)-\frac{1}{2}c\,\text{arc.sin.v.}\frac{x}{\frac{1}{2}c}$; æquationes cycloidis vulgaris. Sit $x=0$: tum $\frac{dy}{dx}=0$; seu tangens curvæ fit axi parallela.

Sit $y=0$, quando $x=c$: tum $c'=0$, & $\frac{dy}{dx}=\frac{1}{0}$; seu tangens fit axi perpendicularis. Minimum hoc refertur ad descensum omnium celerrimum.

§. 203. In §. 201. loco summæ $Fz+F'z'$ proponatur summa $Fz^n+F'z'^n$, quæ debeat esse omnium minima.

Erit eodem modo $nFz^{n-1}\frac{dz}{dv}+nF'z'^{n-1}\frac{dz'}{dv}=0$

$$\text{seu}\quad Fz^{n-1}\frac{dz}{dv}+F'z'^{n-1}\frac{dz'}{dv}=0:$$

$$\text{unde}\quad Fz^{n-1}\,\text{cos}.MAB=F'z'^{n-1}\,\text{cos}.A'MP.$$

Fiat eadem constructio, quæ §. 202. In polygono $AMM'M''M'''M''''\ldots$, in quo summa $Fz^n+F'z'^n+F''z''^n+F'''z'''^n+\ldots$ est omnium minima, est

$$\begin{aligned} Fz^{n-1}\,\text{cos}.MAB &= F'z'^{n-1}\,\text{cos}.M'MP\\ &= F''z''^{n-1}\,\text{cos}.M''M'P'\\ &= F'''z'''^{n-1}\,\text{cos}.M'''M''P''\\ &= F^{\text{iv}}z^{\text{iv}\,n-1}\,\text{cos}.M^{\text{iv}}M'''P'''\\ &= \text{- - - - -} \end{aligned}$$

Quare pro vertice quolibet M, inter duos vertices $'M$, M' posito, est $Fz^{n-1}\,\text{cos}.M'MP$ quantitas constans. Sit pars quælibet axis $PP'=\Delta x$: erit etiam $F\frac{z^{n-1}}{\Delta x^{n-1}}\text{cos}.M'MP$ quantitas constans. Unde, aucto partium numero, in curva, quæ limes est figurarum rectilinearum, quantitatis $F\frac{z^{n-1}}{\Delta x^{n-1}}\text{cos}.M'MP$ limes erit quantitas constans; seu $F\left(\frac{dz}{dx}\right)^{n-1}\frac{dy}{dz}$ est quantitas constans.

Exemplum primum. Sit F quantitas constans, & $n=2$: erit $\frac{dz}{dx}.\frac{dy}{dz}=c$; proinde $\frac{dy}{dx}=c$; cui æquationi satisfacit linea recta. Generatim, posito F constante, est $\left(\frac{dz}{dx}\right)^{n-1}.\frac{dy}{dz}=c$; æquatio ad lineam rectam.

E.em.

Exemplum secundum. Sit F ipsi x proportionalis. Erit $x\left(\frac{dz}{dx}\right)^{n-1}\cdot\frac{dy}{dz}=c$.

Casu particulari, quo $n=-2$, fit $x\left(\frac{dx}{dz}\right)^{3}\cdot\frac{dy}{dz}=c$; quæ est æquatio curvæ genitricis solidi, quod resistentiam omnium minimam patitur a fluido, in quo juxta directionem axi rotationis parallelam movetur.

§. 204. Loco summæ $Fz^n+F'z'^n$ proponatur summa $F\varphi z+F'\varphi z'$, quæ omnium minima esse debeat: positis φz, $\varphi z'$ similibus ipsarum z, z' functionibus, talibus, ut $\frac{d\varphi z}{dv}=\pi z\frac{dz}{dv}$, ideoque $\frac{d\varphi z'}{dv}=\pi z'\frac{dz'}{dv}$; in quibus πz & $\pi z'$ sunt etiam functiones ipsarum z & z' similes.

Erit eodem modo $F\pi z\frac{dz}{dv}+F'\pi z'\frac{dz'}{dv}=0$.

Igitur $F\pi z\cos.MAB=F'\pi z'\cos.A'MP$. Unde eadem facta constructione, quæ §. 202. deducitur, quantitatem $F\pi z\cos.M'MP$ esse constantem; proinde in curva, quæ limes est figuræ rectilineæ $AMM'M''M'''M^{IV}\ldots$, est $\lim.F\pi\frac{\Delta z}{\Delta x}\cos.M'MP$, seu $F.\pi\left(\frac{dz}{dx}\right)\frac{dy}{dz}$ quantitas constans. Sit nempe $\frac{dZ}{dx}=X\varphi\frac{dz}{dx}$: erit Z omnium minimum, quando $X\pi\left(\frac{dz}{dx}\right)\cdot\frac{dy}{dz}$ est quantitas constans.

§. 205. Investigatio curvæ maximi minimive proprietate præditæ paulo aliter institui potuisset modo sequente; illi admodum analogo, qui §. 185. 186. traditur.

Sit BB' axis magnitudine datus, in partes quotcunque æquales in P, P', P'', P''', $P^{IV}\ldots$ punctis divisus.

Agantur perpendicula	BA,	MP,	$M'P'$,	$M''P''$,	$M'''P'''$,	$M^{IV}P^{IV}\ldots$
quæ dicantur	a,	y,	y',	y'',	y''',	$y^{IV}\ldots$
& rectæ		AM,	MM',	$M'M''$,	$M''M'''$,	$M'''M^{IV}\ldots$
quæ dicantur		z,	z',	z'',	z''',	$z^{IV}\ldots$
Quæritur summa		$F\varphi z+$	$F'\varphi z'+$	$F''\varphi z''+$	$F'''\varphi z'''+$	$F^{IV}\varphi z^{IV}+\ldots$
quæ sit omnium minima; positis		φz,	$\varphi z'$,	$\varphi z''$,	$\varphi z'''$,	$\varphi z^{IV}\ldots$
functionibus similibus ipsius z talibus, ut exponentes differentiales		$\frac{d\varphi z}{dv}$,	$\frac{d\varphi z'}{dv}$,	$\frac{d\varphi z''}{dv}$,	$\frac{d\varphi z'''}{dv}$,	$\frac{d\varphi z^{IV}}{dv}\ldots$
sint respective		$\pi z\frac{dz}{dv}$,	$\pi z'\frac{dz'}{dv}$,	$\pi z''\frac{dz''}{dv}$,	$\pi z'''\frac{dz'''}{dv}$,	$\pi z^{IV}\frac{dz^{IV}}{dv}\ldots$

Erit

Erit ideo $F\pi z\frac{dz}{dv}+F'\pi z'\frac{dz'}{dv}+F''\pi z''\frac{dz''}{dv}+F'''\pi z'''\frac{dz'''}{dv}+F^{iv}\pi z^{iv}\frac{dz^{iv}}{dv}+\ldots=0.$

Hinc
$$\left.\begin{array}{l}\frac{dy}{dv}(F'\pi z'\,\text{cof.}\,M'MP - F\pi z\,\text{cof.}\,MAB)\\ +\frac{dy'}{dv}(F''\pi z''\,\text{cof.}\,M''M'P' - F'\pi z'\,\text{cof.}\,M'MP)\\ +\frac{dy''}{dv}(F'''\pi z'''\,\text{cof.}\,M'''M''P'' - F''\pi z''\,\text{cof.}\,M''M'P')\\ +\frac{dy'''}{dv}(F^{iv}\pi z^{iv}\,\text{cof.}\,M^{iv}M'''P''' - F'''\pi z'''\,\text{cof.}\,M'''M''P'')\\ + - - - - - - -\\ + - - - - - - -\end{array}\right\} = 0.$$

Omnes quantitates mutabiles $y, y', y'', y''', y^{iv}\ldots$ fiant fimul conftantes, una excepta; ac proinde omnes exponentes differentiales $\frac{dy}{dv}, \frac{dy'}{dv}, \frac{dy''}{dv}, \frac{dy'''}{dv}\ldots$ fimul evanefcant, uno excepto. Erit, ut prius,
$$\begin{aligned}F\pi z\,\text{cof.}\,MAB &= F'\pi z'\,\text{cof.}\,M'MP\\ &= F''\pi z''\,\text{cof.}\,M''M'P'\\ &= F'''\pi z'''\,\text{cof.}\,M'''M''P''\\ &= F^{iv}\pi z^{iv}\,\text{cof.}\,M^{iv}M'''P'''\\ &= - - - -\\ &= - - - -\end{aligned}$$

Unde eædem, quæ prius, fluunt confequentiæ.

Occurrunt quandoque cafus, quibus fatius effe videtur inveftigationem propofitam pofteriori modo aggredi; uti uno alterove exemplo oftendam.

§. 206. *Problema.* Omnibus uti in §. 202. pofitis, quæritur figura $BAMM'M''\ldots A'B'$, cujus area per perimetrum $AMM'M''M'''\ldots A'$ divifa præbeat quotum omnium maximum.

Pars quælibet axis PP' dicatur b.

Erit ideo $\frac{b(a+2y+2y'+2y''+2y'''+2y^{iv}+\ldots+a')}{z+z+z'+z''+z'''+z^{iv}+\ldots} =$ maximo.

Hinc

Hinc $2(z+z'+z''+z'''+z^{\text{iv}}+\ldots)\left(\frac{dy}{dv}+\frac{dy'}{dv}+\frac{dy''}{dv}+\frac{dy'''}{dv}+\frac{dy^{\text{iv}}}{dv}+\ldots\right)$

$$= (a+2y+2y'+2y''+2y'''+2y^{\text{iv}}+\ldots)\left(\frac{dz}{dv}+\frac{dz'}{dv}+\frac{dz''}{dv}+\frac{dz'''}{dv}+\frac{dz^{\text{iv}}}{dv}+\ldots\right)$$

$$\begin{aligned} = (a+2y+2y'+2y''+2y'''+2y^{\text{iv}}+\ldots)\Big\{ & \frac{dy}{dv}(\text{cos.}M'MP-\text{cos.}MAB) \\ & +\frac{dy'}{dv}(\text{cos.}M''M'P'-\text{cos.}M'MP) \\ & +\frac{dy''}{dv}(\text{cos.}M'''M''P''-\text{cos.}M''M'P') \\ & +\frac{dy'''}{dv}(\text{cos.}M^{\text{iv}}M'''P'''-\text{cos.}M'''M''P'') \\ & +\frac{dy^{\text{iv}}}{dv}(\text{cos.}M^{\text{v}}M^{\text{iv}}P^{\text{iv}}-\text{cos.}M^{\text{iv}}M'''P''') \\ & + \text{ - - - - - } \\ & + \text{ - - - - - }\Big\} \end{aligned}$$

Omnes quantitates mutabiles fiant simul constantes, una excepta: erit

$$\begin{aligned} \frac{z+z'+z''+z'''+z^{\text{iv}}+\ldots}{a+2y+2y'+2y''+2y'''+2y^{\text{iv}}+\ldots} & = \text{cos.}M'MP-\text{cos.}MAB \\ & = \text{cos.}M''M'P'-\text{cos.}M'MP \\ & = \text{cos.}M'''M''P''-\text{cos.}M''M'P' \\ & = \text{cos.}M^{\text{iv}}M'''P'''-\text{cos.}M'''M''P'' \\ & = \text{ - - - - - } \\ & = \text{ - - - - - } \end{aligned}$$

Quare pro vertice quolibet M duobus verticibus $'M$, M' interjacente est

$$\text{cos.}M'MP-\text{cos.}M'M'P = \frac{z+z'+z''+z'''+z^{\text{iv}}+\ldots}{a+2y+2y'+2y''+2y'''+2y^{\text{iv}}+\ldots}$$

Æquatio hæc locum habet, quicunque sit numerus laterum figuræ. Axis igitur BB' dividatur in partes quotcunque æquales, quarum quælibet sit Δx: erit etiam $\frac{\text{cos.}M'MP-\text{cos.}M'M'P}{\Delta x} = \frac{z+z'+z''+z'''+z^{\text{iv}}+\ldots}{\Delta x(a+2y+2y'+2y''+2y'''+2y^{\text{iv}}+\ldots)}$. Quare & prioris quantitatis limes æqualis est limiti posterioris: unde in curva, proposita

 maxi-

maximi proprietate prædita, eſt (§. 192. Obſ. 3.) $\frac{-\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = \frac{P}{S}$ (denotantibus P & S perimetrum & arcum figuræ). Proinde in curva propoſita radius curvaturæ eſt conſtans; ideoque hæc curva eſt circumferentia circuli.

Alterum exemplum. Sit S punctum aliquod in A axe BB' producto, per quod ducatur recta ipſi BB' perpendicularis; requiritur, ut centrum gravitatis perimetri $AMM'M''M'''M^{iv}$.... ſit reſpectu hujus perpendicularis omnium maxime depreſſum.

Diſtantiæ SB, SP, SP', SP'', SP''', SP^{iv}, ... vocentur reſpective b, c, c', c'', c''', c^{iv}, ...

Erit ideo $\frac{z(b+c)+z'(c+c')+z''(c'+c'')+z'''(c''+c''')+z^{iv}(c'''+c^{iv})+\dots}{z+z'+z''+z'''+z^{iv}+\dots} =$ maximo.

Compendii cauſa numerator & denominator hujus fractionis vocentur M & P reſpective.

Erit $P\left((b+c)\frac{dz}{dv}+(c+c')\frac{dz'}{dv}+(c'+c'')\frac{dz''}{dv}+(c''+c''')\frac{dz'''}{dv}+(c'''+c^{iv})\frac{dz^{iv}}{dv}+\dots\right)$

$$= M\left(\frac{dz}{dv}+\frac{dz'}{dv}+\frac{dz''}{dv}+\frac{dz'''}{dv}+\frac{dz^{iv}}{dv}+\dots\right)$$

$$\begin{array}{lcl}
\text{ſeu } P\Big(\frac{dy}{dv}((b+c)\cos PMA-(c+c')\cos P'M'M) & & M\Big(\frac{dy}{dv}(\cos PMA-\cos P'M'M) \\
+\frac{dy'}{dv}((c+c')\cos P'M'M-(c'+c'')\cos P''M''M') & & +\frac{dy'}{dv}(\cos P'M'M-\cos P''M''M') \\
+\frac{dy''}{dv}((c'+c'')\cos P''M''M'-(c''+c''')\cos P'''M'''M'') & = & +\frac{dy''}{dv}(\cos P''M''M'-\cos P'''M'''M'') \\
+\frac{dy'''}{dv}((c''+c''')\cos P'''M'''M''-(c'''+c^{iv})\cos P^{iv}M^{iv}M''') & & +\frac{dy'''}{dv}(\cos P'''M'''M''-\cos P^{iv}M^{iv}M''') \\
+\ -\ -\ -\ -\ -\ -\ -\ - & & +\ -\ -\ -\ -\ - \\
+\ -\ -\ -\ -\ -\ -\ -\ - & & +\ -\ -\ -\ -\ -
\end{array}$$

Omnes quantitates mutabiles y, y', y'', y''', y^{iv}... fiant ſimul conſtantes præter unam: erit pro vertice quolibet M, cui reſpondet abſciſſa x, & qui vertices inter $'M$, M' jacet,

$P((2x$

$$P((2x-\Delta x)\text{cof.}PM'M-(2x+\Delta x)\text{cof.}P'M'M) = M(\text{cof.}PM'M-\text{cof.}P'M'M);$$

ideoque

$$P(2x\frac{\text{cof.}PM'M-\text{cof.}P'M'M}{\Delta x}-(\text{cof.}PM'M+\text{cof.}P'M'M))=M\frac{\text{cof.}PM'M-\text{cof.}P'M'M}{\Delta x};$$

unde & limites ſunt inter ſe æquales, nempe

$$2P\left(x\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}+\frac{dy}{dz}\right) = M\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}$$

$$\text{Igitur}\quad 2Px\frac{dy}{dz} = M\left(C+\frac{dy}{dz}\right)$$

$$\frac{dy}{dz} = \frac{MC}{2Px-M}$$

$$\text{unde}\quad \frac{dy}{dz} = \frac{MC}{\sqrt{((2Px-M)^2-MMCC)}}.$$

Obſervatio. In duobus his exemplis maximi minimive proprietas intra limites datos continetur, atque ad partem tantum curvæ inter puncta data A & A' comprehenſam pertinet. Quoti ſcilicet $\frac{S}{P}$, $\frac{M}{P}$, quatenus curva intra hos limites continetur, tanquam immmutati ſpectantur.

§. 207. Figura propoſita, ſive rectilinea, ſive curvilinea, referatur ad punctum aliquod tanquam focum. Inveſtigatio maximi minimive eodem fere modo inſtituitur.

Problema. Sit S punctum poſitione datum; poſitione pariter dentur duo puncta A & A'. Centro S radiis SA, SA' deſcribantur circuli, qui rectæ cuicunque per S ductæ in B & B' occurrant. Centro S, radio quolibet SP inter SB, SB' medio, deſcribatur circulus. Sint F, F' coëfficientes magnitudine dati. In circumferentia radii SP aſſignandum ſit M punctum ſic, ut ſumma $F\times AM+F'\times A'M$ ſit omnium minima. Fig. 62.

Sint $SA=a$, $SA'=a'$, $SM=b$, $ASM=y$, $A'SM'=y'$, $AM=z$, $A'M=z'$.

Erit ideo $F\times z + F'\times z' =$ minimo;

hinc $F\times\frac{dz}{du}+F'\times\frac{dz'}{dv} = 0.$

Atqui

Atqui $\begin{matrix} zz = aa - 2ab\cos.y + bb, \\ z'z' = a'a' - 2a'b\cos.y' + bb' \end{matrix}$

ideoque $\begin{matrix} \dfrac{dz}{dv} = \dfrac{ab\sin.y\dfrac{dy}{dv}}{z} = a\sin.SAM\dfrac{dy}{dv} \\ \dfrac{dz'}{dv} = \dfrac{a'b\sin.y'\dfrac{dy'}{dv}}{z'} = b\sin.SMA'\dfrac{dy'}{dv}. \end{matrix}$

Quare $F\times a\sin.SAM\times\dfrac{dy}{dv} + F'\times b\sin.SMA'\times\dfrac{dy'}{dv} = 0.$

Sed $y + y' = ASA' =$ dato; ergo $\dfrac{dy}{dv} + \dfrac{dy'}{dv} = 0$. Proinde $F\times a\sin.SAM = F'\times b\sin.SMA'$.

Fig. 63. Recta BB' dividatur in partes quotcunque æquales in punctis P, P', P'', P''', P^{v}... Centro S, radiis SP, SP', SP'', SP''', SP^{iv}... describantur circuli; sint F, F', F'', F''', F^{iv}... totidem coëfficientes dati. Ut summa $F\times AM + F'\times MM' + F''\times M'M'' + F'''\times M''M''' + F^{\text{v}}\times M'''M^{\text{iv}} + \ldots$ sit omnium minima, debet esse

$$\begin{aligned} F\times SA\sin.SAM &= F'\times SM\sin.SMM' \\ &= F''\times SM'\sin.SM'M'' \\ &= F'''\times SM''\sin.SM''M''' \\ &= F^{\text{iv}}\times SM'''\sin.SM'''M^{\text{iv}} \\ &= - \quad - \quad - \quad - \end{aligned}$$

Hinc pro vertice quolibet M, duos inter $'M$, M' posito, est $F\times SM\times\sin.SMM'$ quantitas constans.

Ab foco S concipiatur acta SQ ipsi MM perpendicularis: erit $F\times SQ$ quantitas constans.

Quoniam $F\times SM\times\sin.SMM'$ est quantitas constans; limes etiam quantitatis hujus est quantitas constans: proinde, aucto partium numero, in curva, quæ limes est figuræ rectilineæ $AMM'M''M'''$... per M ducta tangente MT, in quam demittatur perpendiculum SQ, est $F\times SM\sin.SMT = F\times SQ$ quantitas constans.

Posito igitur $\dfrac{dZ}{dx} = \varphi x.\dfrac{dz}{dx}$, denotante φx functionem aliquam distantiæ FM:

curva,

curva, in qua Z eſt minima, ea eſt, ad quam $\varphi x \times SM \,\text{ſin.}\, SMT$, ſeu $\varphi x \times SQ$ eſt quantitas conſtans.

Scholium. Exemplum hoc refertur ad minimum quoddam, quod in motu projectilium circa centrum aliquod revolutorum locum habet. Nempe quotiescunque velocitas horum projectilium eſt functio aliqua diſtantiæ eorum a centro revolutionis, quæ dicatur V; facto $\frac{dZ}{dx} = V\frac{dz}{dx}$, eſt Z in curva deſcripta omnium minimum. Etenim $V \times SQ$ eſt quantitas conſtans per legem arearum a KEPLERO detectam, & a NEWTONO demonſtratam. (Vid. EULERI *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes. Additamentum 2.*)

§. 208. In exemplo præcedente ponebam, radios vectores æqualibus differentiis creſcere. Inveſtigatio $\begin{matrix}\text{maximi}\\ \text{minimi}\end{matrix}$ fit eodem modo, poſito angulos ad focum æqualibus differentiis variare.

Angulus conſtans ASM dicatur α; & debeat eſſe $Fz + F'z' + F''z'' + F'''z''' + F^{\text{iv}}z^{\text{iv}} + \ldots$ omnium $\begin{matrix}\text{maxima}\\ \text{minima}\end{matrix}$. Fig. 64.

Hinc $F\frac{dz}{dv} + F'\frac{dz'}{dv} + F''\frac{dz''}{dv} + F'''\frac{dz'''}{dv} + F^{\text{iv}}\frac{dz^{\text{iv}}}{dv} + \ldots = 0.$

Radii SA, SM, SM', SM'', SM''', $SM^{\text{iv}} \ldots$

dicantur a, y, y', y'', y'''; $y^{\text{iv}} \ldots$ reſpective.

Quoniam $zz = aa\,\text{ſin.}^2\alpha + (y - a\,\text{coſ.}\alpha)^2$ fit $\frac{dz}{dv} = \frac{dy}{dv}\text{coſ.}SMA$

$$z'z' = yy\,\text{ſin.}^2\alpha + (y' - y\,\text{coſ.}\alpha)^2 \qquad \frac{dz'}{dv} = \frac{dy'}{dv}\text{coſ.}SM'M - \frac{dy}{dv}\text{coſ.}(\alpha + SM'M)$$

$$z''z'' = y'y'\,\text{ſin.}^2\alpha + (y'' - y'\,\text{coſ.}\alpha)^2 \qquad \frac{dz''}{dv} = \frac{dy''}{dv}\text{coſ.}SM''M' - \frac{dy'}{dv}\text{coſ.}(\alpha + SM''M')$$

$$z'''z''' = y''y''\,\text{ſin.}^2\alpha + (y''' - y''\,\text{coſ.}\alpha)^2 \qquad \frac{dz'''}{dv} = \frac{dy'''}{dv}\text{coſ.}SM'''M'' - \frac{dy''}{dv}\text{coſ.}(\alpha + SM'''M'')$$

$$z^{\text{iv}}z^{\text{iv}} = y'''y'''\,\text{ſin.}^2\alpha + (y^{\text{iv}} - y'''\,\text{coſ.}\alpha)^2 \qquad \frac{dz^{\text{iv}}}{dv} = \frac{dy^{\text{iv}}}{dv}\text{coſ.}SM^{\text{iv}}M''' - \frac{dy'''}{dv}\text{coſ.}(\alpha + SM^{\text{iv}}M''')$$

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unde

$$\left.\begin{array}{l}\text{unde } \frac{dy}{dv}(F\text{cof.}SMA - F'\text{cof.}(\alpha+SM'M)) \\ + \frac{dy'}{dv}(F'\text{cof.}SM'M - F''\text{cof.}(\alpha+SM''M')) \\ + \frac{dy''}{dv}(F''\text{cof.}SM''M' - F'''\text{cof.}(\alpha+SM'''M'')) \\ + \frac{dy'''}{dv}(F'''\text{cof.}SM'''M'' - F^{\text{iv}}\text{cof.}(\alpha+SM^{\text{iv}}M''')) \\ + \text{ - - - - - - - -} \\ + \text{ - - - - - - - -}\end{array}\right\} = 0.$$

Omnes quantitates mutabiles y, y', y'', y''', y^{iv} . . . fiant fimul conftantes, una excepta: erit pro vertice quolibet M, duobus $'M$, M' interjacente, $F\text{cof.SM'M} = F'\text{cof.}(\alpha+\text{SM'M}) = -F'\text{cof.SMM'}$.

Exemplum. Sit $F = F' = F'' = F''' = F^{\text{iv}}$...: erit $F\text{cof.SM'M} = -F\text{cof.SMM'}$; ideoque cof.SM'M $=$ — cof.SMM'; SMM' $= 180° -$ SM'M: proinde puncta 'M, M, M' jacent in directum; & tota via $AMM'M''M'''$ . . . eft linea recta.

Dividatur angulus AFA' in partes quotcunque invicem æquales, quarum quævis dicatur Δx; & fit F' quantitas mutabilis ab x dependens, ita ut fit $F' = F + \frac{\Delta x}{1}\cdot\frac{dF}{dx} + \frac{\Delta x^2}{1.2}\cdot\frac{ddF}{dx^2} + \ldots$

Erit $F\text{cof.SM'M} = (F + \frac{\Delta x}{1}\cdot\frac{dF}{dx} + \frac{\Delta x^2}{1.2}\cdot\frac{ddF}{dx^2} + \ldots)\,\text{cof.}(\alpha+\text{SM'M})$;
unde

$$F(\text{cof.SM'M} - \text{cof.}(\alpha+\text{SM'M})) = \left(\frac{\Delta x}{1}\cdot\frac{dF}{dx} + \frac{\Delta x^2}{1.2}\cdot\frac{ddF}{dx^2} + \ldots\right)\text{cof.}(\alpha+\text{SM'M})$$

$$F\frac{\text{cof.SM'M} - \text{cof.}(\alpha+\text{SM'M})}{\Delta x} = \left(\frac{dF}{dx} + \frac{\Delta x}{1.2}\cdot\frac{ddF}{dx^2} + \frac{\Delta x^2}{1.2.3}\cdot\frac{d^3F}{dx^3} + \ldots\right)\text{cof.}(\alpha+\text{SM'M}).$$

Quare & membrorum æquationis hujus limites funt invicem æquales; nempe

$$-F\frac{d.\frac{dy}{dz}}{dx} = \frac{dF}{dx}\cdot\frac{dy}{dz}:$$

$$\text{unde } \frac{dF}{dx}\cdot\frac{dy}{dz} + F\frac{d.\frac{dy}{dz}}{dx} = 0$$

$$F\frac{dy}{dz} = C.$$ Itaque res ad calculum integralem femper reducitur.

Ex æquatione $\frac{dy}{dz} = \frac{C}{F}$ deducitur $\frac{dx}{dy} = \frac{C}{\sqrt{(FF - CCyy)}}$.

§. 209.

§. 209. Transeo ad exempla, quibus factor F est functio aliqua ordinatarum.

Problema. Omnibus uti §. 202. positis, quæritur summa

$$z(a+y)+z'(y+y')+z''(y'+y'')+z'''(y''+y''')+\ldots = \begin{matrix}\text{maximo}\\ \text{minimo}\end{matrix}.$$

Fig. 61.

Erit ideo $\frac{dy}{dv}(z+z'+(a+y)\operatorname{cof.}PMA-(y+y')\operatorname{cof.}P'M'M)$

$$+\frac{dy'}{dv}(z'+z''+(y+y')\operatorname{cof.}P'M'M-(y'+y'')\operatorname{cof.}P''M''M')$$

$$+\frac{dy''}{dv}(z''+z'''+(y'+y'')\operatorname{cof.}P''M''M'-(y''+y''')\operatorname{cof.}P'''M'''M'')$$

$$+\frac{dy'''}{dv}(z'''+z^{\text{iv}}+(y''+y''')\operatorname{cof.}P'''M'''M''-(y'''+y^{\text{iv}})\operatorname{cof.}P^{\text{iv}}M^{\text{iv}}M''')$$

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Omnes quantitates mutabiles y, y', y'', y''', y^{iv}... fiant simul constantes, una excepta: erit pro vertice quolibet M, inter duos vertices $'M$, M' comprehenso, $\frac{dy}{dv}(z+z'+('y+y)\operatorname{cof.}PM'M-(y+y')\operatorname{cof.}P'M'M) = 0$.

Atqui $'y = y-\frac{\Delta x}{1}\cdot\frac{dy}{dx}+\frac{\Delta x^2}{1.2}\cdot\frac{ddy}{dx^2}-\frac{\Delta x^3}{1.2.3}\cdot\frac{d^3y}{dx^3}+\ldots$

$$y' = y+\frac{\Delta x}{1}\cdot\frac{dy}{dx}+\frac{\Delta x^2}{1.2}\cdot\frac{ddy}{dx^2}+\frac{\Delta x^3}{1.2.3}\cdot\frac{d^3y}{dx^3}+\ldots$$

Igitur $z+z'+2y(\operatorname{cof.}PM'M-\operatorname{cof.}P'M'M)-\frac{\Delta x}{1}\cdot\frac{dy}{dx}(\operatorname{cof.}PM'M+\operatorname{cof.}P'M'M)$

$$+\frac{\Delta x^2}{1.2}\cdot\frac{ddy}{dx^2}(\operatorname{cof.}PM'M-\operatorname{cof.}P'M'M) = 0.$$

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Proinde & limes semissis hujus quantitatis est zero; nempe

$$\frac{dz}{dx}-y\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}-\frac{dy}{dx}\cdot\frac{dy}{dz}=0, \text{ seu } \frac{1}{\frac{dz}{dx}}-y\cdot\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}=0: \text{ ideoque}$$

$$\frac{\frac{dy}{dx}}{\frac{dz}{dx}}-y\times\frac{\frac{dy}{dx}\cdot\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}=0;\ \frac{y}{\frac{dz}{dx}}=C;\ \frac{dx}{dz}=\frac{C}{y}.$$ Unde res ad calculum integralem revocatur.

Sit

Sit nempe $\frac{dZ}{dx} = y\frac{dz}{dx}$: functio Z fit omnium maxima, quando $\frac{dx}{dz} = \frac{C}{y}$.

Generatim. Sit $z(\varphi a+\varphi y)+z'(\varphi y+\varphi y')+z''(\varphi y'+\varphi y'')+z'''(\varphi y''+\varphi y''')+\ldots=$maximo.

$$\left.\begin{aligned}
&\text{Erit } \frac{dy}{dv}(\pi y(z+z')+(\varphi a+\varphi y)\,\text{cof.}\,PM'M-(\varphi y+\varphi y')\,\text{cof.}\,P'M'M)\\
&+\frac{dy'}{dv}(\pi y'(z'+z'')+(\varphi y+\varphi y')\,\text{cof.}\,P'M'M-(\varphi y'+\varphi y'')\,\text{cof.}\,P''M''M')\\
&+\frac{dy''}{dv}(\pi y''(z''+z''')+(\varphi y'+\varphi y'')\,\text{cof.}\,P''M''M'-(\varphi y''+\varphi y''')\,\text{cof.}\,P'''M'''M'')\\
&+\frac{dy'''}{dv}(\pi y'''(z'''+z^{\text{iv}})+(\varphi y''+\varphi y''')\,\text{cof.}\,P'''M'''M''-(\varphi y'''+\varphi y^{\text{iv}})\,\text{cof.}\,P^{\text{iv}}M^{\text{iv}}M''')\\
&+\;\;-\;\;-\;\;-\;\;-\;\;-\;\;-\\
&+\;\;-\;\;-\;\;-\;\;-\;\;-\;\;-
\end{aligned}\right\} = 0.$$

Hinc pofitis omnibus quantitatibus mutabilibus $y, y', y'', y''', y'''', y^{\text{v}}\ldots$ fimul conftantibus, præter unam, fit pro vertice quolibet M, duobus 'M, M' interjacente, $\pi y(z+z')+(\varphi' y+\varphi y)\,\text{cof.PM'M}-(\varphi y+\varphi y')\,\text{cof.P'M'M} = 0$.

$$\text{Atqui } \varphi' y = \varphi y - \frac{\Delta x}{1}\cdot\pi y\frac{dy}{dx} + \frac{\Delta x^2}{1.2}\frac{d^2\varphi y}{dx^2} + \ldots.$$

$$\varphi y' = \varphi y + \frac{\Delta x}{1}\cdot\pi y\frac{dy}{dx} + \frac{\Delta x^2}{1.2}\frac{d^2\varphi y}{dx^2} + \ldots$$

Ergo

$$\left.\begin{aligned}
&\pi y\left(\frac{z+z'}{\Delta x}\right)+\varphi y\frac{\text{cof.PM'M}-\text{cof.P'M'M}}{\Delta x}-\pi y\frac{dy}{\Delta x}(\text{cof.PM'M}+\text{cof.P'M'M})\\
&\qquad+\frac{\Delta x}{1.2}\frac{d^2\varphi y}{dx^2}(\text{cof.PM'M}-\text{cof.P'M'M})\\
&\qquad-\;\;-\;\;-\;\;-\;\;-
\end{aligned}\right\} = 0.$$

Unde & limes dimidiæ hujus quantitatis $= 0$; nempe

$$\pi y\frac{dz}{dx} - \varphi y\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} - \pi y\frac{dy}{dx} - \frac{dz}{dx} = 0$$

$$\pi y\frac{1}{\frac{dz}{dx}} - \varphi y\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = 0$$

πy

$$\pi y \frac{\frac{dy}{dx}}{\frac{dz}{dx}} - \varphi y \frac{\frac{dy}{dx} \cdot \frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = 0$$

tandemque $\varphi y \frac{1}{\frac{dz}{dx}} = C$, & $\frac{dx}{dz} = \frac{C}{\varphi y}$.

In curva igitur, ad quam est $\frac{dZ}{dx} = \frac{dz}{dx}\varphi y$, Z est omnium maxima, quando $\frac{dx}{dz} = \frac{C}{\varphi y}$.

§. 210. Investigatio eodem modo instituitur in curvis ad focum aliquem relatis.

Sint nempe y, y', y'', y''', y^{iv} . . . radii vectores, quos inter angulus constans α comprehenditur; & debeat esse

$$z(\varphi a+\varphi y)+z'(\varphi y+\varphi y')+z''(\varphi y'+\varphi y'')+z'''(\varphi y''+\varphi y''')+z^{\text{iv}}(\varphi y'''+\varphi y^{\text{iv}})+\ldots = \text{max.}$$

Erit
$$\left.\begin{array}{l}
\frac{dy}{dv}(\pi y(z+z')+(\varphi a+\varphi y)\,\text{cos}.SMA-(\varphi y+\varphi y')\,\text{cos}.(\alpha+SM'M)) \\
+\frac{dy'}{dv}(\pi y'(z'+z'')+(\varphi y+\varphi y')\,\text{cos}.SM'M-(\varphi y'+\varphi y'')\,\text{cos}.(\alpha+SM''M')) \\
+\frac{dy''}{dv}(\pi y''(z''+z''')+(\varphi y'+\varphi y'')\,\text{cos}.SM''M'-(\varphi y''+\varphi y''')\,\text{cos}.(\alpha+SM'''M'')) \\
+\frac{dy'''}{dv}(\pi y'''(z'''+z^{\text{iv}})+(\varphi y''+\varphi y''')\,\text{cos}.SM'''M''-(\varphi y'''+\varphi y^{\text{iv}})\,\text{cos}.(\alpha+SM^{\text{iv}}M''')) \\
+ \quad - \quad - \quad - \quad - \quad - \quad - \\
+ \quad - \quad - \quad - \quad - \quad - \quad -
\end{array}\right\} = 0.$$

Unde pro quolibet vertice M, duos inter $'M$, M' comprehenso, est

$$\pi y(z+z')+(\varphi' y+\varphi y)\,\text{cos}.SM'M-(\varphi y+\varphi y')\,\text{cos}.SM'M) = 0.$$

Hinc

$$\left.\begin{array}{l}
\pi y(z+z')+2\varphi y(\text{cos}.SM'M-\text{cos}.(\alpha+SM'M))-\frac{\Delta x}{1}.\pi y\frac{dy}{dx}(\text{cos}.SM'M+\text{cos}.(\alpha+SM'M)) \\
\qquad +\frac{\Delta x^2}{1.2}\frac{d^2\varphi y}{dx^2}(\ldots\ldots\ldots) \\
\qquad + \quad - \quad - \quad - \quad - \quad - \quad -
\end{array}\right\} = 0.$$

Ideo-

Ideoque etiam

$$\pi y\frac{z+z'}{\Delta x}+2\varphi y\frac{\text{cos.}SM'M-\text{cos.}(\alpha+SM'M)}{\Delta x}-\pi y\frac{dy}{dx}(\text{cos.}SM'M+\text{cos.}\alpha+SM'M)$$
$$+\frac{\Delta x}{1.2}\frac{d^2\varphi y}{dx^2}(\ldots\ldots\ldots)=0.$$

Ideo & limes quantitatis hujus eſt zero.

Atqui $\lim.\frac{\text{cos.}SM'M\text{-cos.}(\alpha+SM'M)}{\Delta x}=\frac{-d.\frac{dy}{dz}}{dx}=\frac{y\left(\frac{dy}{dx}\right)^2-yy\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}.$

Ergo $\pi y\left(\frac{dz}{dx}-\frac{dy}{dx}.\frac{dy}{dz}\right)+\varphi y\frac{y\left(\frac{dy}{dx}\right)^2-yy\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}=0$

ſeu $\pi y\frac{\frac{yy}{dz}}{dx}+\varphi y\frac{y\left(\frac{dy}{dx}\right)^2-yy\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}=0$

$$\pi y\frac{y\frac{dy}{dx}}{\frac{dz}{dx}}+\varphi y\frac{\left(\frac{dy}{dx}\right)^3-y\frac{dy}{dx}.\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}=0$$

Unde $\varphi y\frac{y}{\frac{dz}{dx}}=C$, & $\frac{C}{\varphi y}=y\frac{dx}{dz}=$ ſin. FMT.

Hactenus dicta pertinent ad methodum $\frac{\text{maximorum}}{\text{minimorum}}$ abſolutam; ſeu eam, in qua curvis aut figuris propoſitis nulla proprietas communis tribuitur. Transeo ad exempla, quibus non inter omnes abſolute figuras vel curvas, ſed eas inter, quæ data quadam proprietate communi gaudeant, quæritur figura vel curva, $\frac{\text{maximi}}{\text{minimi}}$ proprietate aliqua prædita.

Fig. 65. §. 211. *Problema.* Sint A, A' duo puncta poſitione data, relata ad rectam BB' per demiſſa in eam perpendicula AB, $A'B'$. Axi BB', in tres partes æquales diviſo in punctis P, P', perpendiculares in his punctis conſtituantur rectæ PM, $P'M'$. Quæruntur earum puncta M & M': ut ſumma rectarum AM, MM', $M'A'$ detur magnitudine; & ſpatium $BAMM'B'A'$ ſit omnium maximum.

Sint

Sint $AB = a$, $A'B' = a'$, $BP = b$, $MP = y$, $M'P' = y'$, $AM = z$, $MM' = z'$, $M'A' = z''$.

$$\text{Erit ideo} \quad \begin{aligned} z + z' + z'' &= \text{dato} \\ b(a + 2y + 2y' + a') &= \text{maximo} \end{aligned} \quad \text{seu} \quad \begin{aligned} z + z' + z'' &= \text{dato} \\ y + y' &= \text{maximo.} \end{aligned}$$

$$\text{Hinc} \quad \frac{dz}{dv} + \frac{dz'}{dv} + \frac{dz''}{dv} = 0$$

$$\frac{dy}{dv} + \frac{dy'}{dv} = 0.$$

$$\text{Atqui} \quad \frac{dz}{dv} = \frac{dy}{dv} \operatorname{cos}. MAB$$

$$\frac{dz'}{dv} = \left(\frac{dy'}{dv} - \frac{dy}{dv}\right) \operatorname{cos}. M'MP$$

$$\frac{dz''}{dv} = -\frac{dy'}{dv} \operatorname{cos}. A'M'P'.$$

$$\text{Proinde} \quad \frac{dy}{dv}(\operatorname{cos}. MAB - \operatorname{cos}. M'MP) + \frac{dy'}{dv}(\operatorname{cos}. M'MP - \operatorname{cos}. A'M'P') = 0$$

$$\frac{dy}{dv} + \frac{dy'}{dv} = 0.$$

Hinc $\operatorname{cos}. MAB - \operatorname{cos}. M'MP = \operatorname{cos}. M'MP - \operatorname{cos}. A'M'P'$.

Dividatur axis BB' in partes quotcunque æquales, quarum quævis dicatur Δx, in P, P', P'', P''', P^{IV} ... punctis; ex quibus agantur rectæ ipsi BB' perpendiculares. Summa $z + z' + z'' + z''' + z^{\text{IV}} + \ldots$ magnitudine data; maxima est summa $\Delta x(a + 2y + 2y' + 2y'' + 2y''' + 2y^{\text{IV}} + \ldots)$, quando Fig. 61.

$$\begin{aligned} \operatorname{cos}. MAB - \operatorname{cos}. M'MP &= \operatorname{cos}. M'MP - \operatorname{cos}. M''M'P' \\ &= \operatorname{cos}. M''M'P' - M'''M''P'' \\ &= \operatorname{cos}. M'''M''P'' - \operatorname{cos}. M^{\text{IV}}M'''P''' \\ &= - \quad - \quad - \quad - \quad - \quad - \\ &= - \quad - \quad - \quad - \quad - \quad - \end{aligned}$$

Pro quolibet igitur vertice M, duos inter 'M, M' posito, est $\operatorname{cos}. M'MP - \operatorname{cos}. M'MP$ quantitas constans, manente Δx eadem; proinde & $\operatorname{cos}. \dfrac{M'M'P - M'MP}{\Delta x}$, pariterque limes ejus, hoc est (§. 192. *Obs.* 3.), $\dfrac{-\dfrac{ddy}{dx^2}}{\left(\dfrac{dz}{dx}\right)^3}$ est quantitas constans. Quare

 inter

inter curvas, data perimetro terminatas, ea maximam comprehendit aream, cujus radius curvaturæ eſt datæ magnitudinis; ac proinde curva hæc eſt circumferentia circuli, uti methodo mere elementari demonſtrari poteſt.

Quoniam $\dfrac{-\dfrac{ddy}{dx^2}}{\left(\dfrac{dz}{dx}\right)^3}=\dfrac{1}{C}$: eſt $\dfrac{-\dfrac{dy}{dx}\cdot\dfrac{ddy}{dx^2}}{\left(\dfrac{dz}{dx}\right)^3}=\dfrac{dy}{dx}\cdot\dfrac{1}{C}$; hinc $\dfrac{1}{\dfrac{dz}{dx}}=\dfrac{C'+y}{C}$,

$\dfrac{dx}{dz}=\dfrac{C'+y}{C}=$ ſin.PMT.

Hinc deducitur $\dfrac{dx}{dy}=\dfrac{C'+y}{\sqrt{(CC-(C'+y)^2)}}$; $x=C''-\sqrt{(CC-(C'+y)^2)}$: unde $CC=(C''-x)^2+(C'-y)^2$, quæ eſt etiam æquatio circumferentiæ circuli.

§. 212. *Problema.* Perimetro $AMM'A'$ magnitudine data; requiritur, ut ſolidum gyratione figuræ $BAMM'A'B'$ circa axem BB' genitum ſit omnium maximum.

Erit ideo $z+z'+z''=$ dato

$$b((aa+ay+yy)+(yy+yy'+y'y')+(y'y'+y'a'+a'a'))=\text{maximo}$$

ſeu $z+z'+z''=$ dato

$$ay+2yy+yy'+2y'y'+y'a'=\text{maximo.}$$

Hinc $\dfrac{dz}{dv}+\dfrac{dz'}{dv}+\dfrac{dz''}{dv}=0$

ſeu $\dfrac{dy}{dv}(\text{coſ.}MAB-\text{coſ.}M'MP)+\dfrac{dy'}{dv}(\text{coſ.}M'MP-\text{coſ.}A'M'P')=0$

& $\dfrac{dy}{dv}(a+4y+y')+\dfrac{dy'}{dv}(y+4y'+a')=0.$

Quare $\dfrac{\text{coſ.}MAB-\text{coſ.}M'MP}{a+4y+y'}=\dfrac{\text{coſ.}M'MP-\text{coſ.}A'M'P'}{y+4y'+a'}$.

Dividatur axis BB' in partes quotcunque æquales in P, P', P'', P''', P''''... punctis: erit pro dato partium numero, & pro vertice quolibet M, duos inter 'M, M' comprehenſo, $\dfrac{\text{coſ.}M'M'P-\text{coſ.}M'MP}{'y+4y+y'}$ quantitas conſtans; proinde &

quan-

quantitas $\frac{\text{cof.}M'M'P - \text{cof.}M'MP}{\Delta x('y+4y+y')}$, pariterque ejus limes eſt quantitas conſtans. Atqui limes hic eſt $\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} \times \frac{1}{6y}$. In curva igitur longitudine data, quæ circa axem BB' rotata maximum generat ſolidum, eſt $\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} \times \frac{1}{y}$ quædam quantitas conſtans; ſeu rectangulum ſub recta axi huic ordinatim applicata & ſub radio curvaturæ ſemper eſt idem.

Sit $\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} \times \frac{1}{y} = \frac{2}{CC}$: erit $\frac{-\frac{dy}{dx} \cdot \frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = -\frac{2y\frac{dy}{dx}}{CC}$.

$$\text{hinc } \frac{1}{\frac{dz}{dx}} = \frac{C'C' - yy}{CC} = \frac{dx}{dz} = \text{fin.}PMT.$$

Unde res ad calculum integralem reducitur.

Scholium. Curva problemate hoc determinata dicitur *elaſtica.*

§. 213. *Generatim.* Sit $z + z' + z'' + z''' + z'''' + \ldots$ magnitudine data; & ſumma $(\varphi a + \varphi y) + (\varphi y + \varphi y') + (\varphi y' + \varphi y'') + (\varphi y'' + \varphi y''') + \ldots$ ſeu ſumma $\varphi y + \varphi y' + \varphi y'' + \varphi y''' + \varphi y^{\text{IV}} + \ldots$ debeat eſſe omnium maxima.

Erit $\frac{dz}{dv} + \frac{dz'}{dv} + \frac{dz''}{dv} + \frac{dz'''}{dv} + \frac{dz^{\text{IV}}}{dv} + \ldots = 0$

$$\pi y \frac{dy}{dz} + \pi y' \frac{dy'}{dv} + \pi y'' \frac{dy''}{dv} + \pi y''' \frac{dy'''}{dv} + \pi y^{\text{IV}} \frac{dy^{\text{IV}}}{dv} + \ldots = 0.$$

$$\begin{aligned}\text{Unde } &\frac{dy}{dv}(\text{cof.}PMA - \text{cof.}P'M'M) \\ &+ \frac{dy'}{dv}(\text{cof.}P'M'M - \text{cof.}P''M''M') \\ &+ \frac{dy''}{dv}(\text{cof.}P''M''M' - \text{cof.}P'''M'''M'') \qquad = 0. \\ &+ \frac{dy'''}{dv}(\text{cof.}P'''M'''M'' - \text{cof.}P^{\text{IV}}M^{\text{IV}}M''') \\ &+ \text{- - - - - -}\end{aligned}$$

Omnes

Omnes quantitates mutabiles y, y', y'', y''', y^{iv} ... fiant simul constantes, præter duas. Erit $\frac{\text{cos.}PMA - \text{cos.}P'M'M}{\pi y} = \frac{\text{cos.}P'M'M - \text{cos.}P''M''M'}{\pi y'}$

$$= \frac{\text{cos.}P''M''M' - \text{cos.}P'''M'''M''}{\pi y''}$$

$$= \frac{\text{cos.}P'''M'''M'' - \text{cos.}P^{\text{iv}}M^{\text{iv}}M'''}{\pi y'''}$$

$$= - \quad - \quad - \quad - \quad -$$

Hinc pro vertice quolibet M, duobus 'M, M' interjacente, est $\frac{\text{cos.}PM'M - \text{cos.}P'M'M}{\pi y}$ quantitas constans, a quantitate Δx pendens; proinde & quantitas $\frac{\text{cos.}PM'M - \text{cos.}P'M'M}{\Delta x . \pi y}$, pariterque limes ejus, nempe $\frac{-\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3 \pi y}$ est quantitas constans C.

Hinc $\frac{-\frac{dy}{dx} \cdot \frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = C\pi y \frac{dy}{dx}$

ideoque $\frac{1}{\frac{dz}{dx}} = C' + C\varphi y$; seu $\frac{dx}{dz} = C' + C\varphi y = \text{sin.}PMT$. Unde res ad calculum integralem reducitur.

In curva igitur, cujus perimeter magnitudine datur, si fuerit $\frac{dZ}{dx} = \varphi y$; functio Z sit omnium maxima, quando radius curvaturæ est in ratione inversa quantitatis πy seu $\frac{d\varphi y}{dx}$, seu quando $\text{sin.}PMT = \frac{dx}{dz} = C' + C\varphi y$.

§. 214. Omnibus uti §. 211. positis, requiritur, ut momentum figuræ $BAMM'A'B'$ respectu lineæ DD' ipsi BB' perpendicularis sit omnium maximum.

Fig. 66. Sint AD, $A'D'$, MQ, $M'Q$ ipsi DD' perpendiculares.

Lemma. Momentum trapezii $BAMP$ respectu rectæ DD', lateribus ejus AB, MP parallelæ, proportionale est summæ $(AB(2AD+MQ)+MP(2MQ+AD))\,BP$.

Sint $AD = b$, $A'D' = b'$, $MQ = c$, $M'Q' = c'$.

Igitur $\quad z \; + \; z' \; + \; z'' \quad = \text{dato}$

$a(2b+c)+y(2c+b)+y(2c+c')+y'(2c'+c)+y'(2c'+b')+a'(2b'+c) = \text{maximo},$

seu $\quad y(b+4c+c')+y'(c+4c'+b') = \text{max.}$

Hinc

Hinc $\frac{dy}{dv}(\text{cof.PMA}-\text{cof.P'M'M}) + \frac{dy'}{dv}(\text{cof.P'M'M}-\text{cof.BA'M'}) = 0$

$$\frac{dy}{dv}(b+4c+c') \qquad + \frac{dy'}{dv}(c+4c'+b') \qquad = 0.$$

Ideoque $\frac{\text{cof.PMA}-\text{cof.P'M'M}}{b+4c+c'} = \frac{\text{cof.P'M'M}-\text{cof.B'A'M'}}{c+4c'+b'}$.

Hinc axe BB' divifo in partes quotcunque æquales, quarum quælibet fit Δx; & pofita $SP=x$: erit pro vertice quolibet M, duos inter 'M, M' fito, $\frac{\text{cof.PM'M}-\text{cof.P'M'M}}{6x\Delta x}$ quantitas conftans; quare etiam $\frac{\text{cof.PM'M}-\text{cof.P'M'M}}{x\Delta x}$, atque ejus limes eft quædam conftans, nempe $\frac{-\frac{ddy}{dx^2}}{x\left(\frac{dz}{dx}\right)^3} = C$: ac proinde difficultas curvam propofitam determinandi pendet ab imperfectione calculi integralis. Fig. 61.

Eadem difficultas fe offert: fi, data perimetro, fuerit $\frac{dZ}{dx} = Fx \times \varphi y$; & debeat effe Z omnium maximæ. Erit enim $\frac{-\frac{ddy}{dx^2}}{Fx \cdot \left(\frac{dz}{dx}\right)^3 \pi y} = C$, feu

$\frac{-\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = CF \times \pi y$, cujus integratio in terminis finitis defideratur.

§. 215. Quodfi figuræ, five rectilineæ, five curvilineæ, ad punctum aliquod referuntur; modus procedendi non erit multum diverfus.

Sit S focus, ad quem figura refertur per radios vectores SA, SM, SM', SM'', SM'''... Data perimetro $AMM'M''M'''$ ... figura $SAMM'M''M'''$ ... debeat effe omnium maxima. Sint anguli ASM, MSM', $M'SM''$, $M''SM'''$... invicem æquales, qui dicantur α. Fig. 64.

Erit $z + z' + z'' + z''' + z'''' + \ldots = \text{dato}$

$\text{fin.}\alpha(ay + yy' + y'y'' + y''y''' + y'''y^{\text{iv}} + \ldots = \text{maximo}.$

Hinc fit $\frac{dz}{dv} + \frac{dz'}{dv} + \frac{dz''}{dv} + \frac{dz'''}{dv} + \frac{dz^{\text{iv}}}{dv} + \ldots = 0$

 feu

$$\text{seu}\quad \begin{aligned} &\frac{dy}{dv}(\text{cof.}SMA - \text{cof.}(\alpha + SM'M)) \\ + &\frac{dy'}{dv}(\text{cof.}SM'M - \text{cof.}(\alpha + SM''M')) \\ + &\frac{dy''}{dv}(\text{cof.}SM''M' - \text{cof.}(\alpha + SM'''M'')) \\ + &\frac{dy'''}{dv}(\text{cof.}M'''M'' - \text{cof.}(\alpha + SM^{\text{IV}}M''')) \\ + &\; - \; - \; - \; - \; - \; - \end{aligned} \quad = 0$$

$$\text{Item}\quad \frac{dy}{dv}(a+y') + \frac{dy'}{dv}(y+y') + \frac{dy''}{dv}(y'+y'') + \frac{dy'''}{dv}(y''+y''') + \ldots = 0$$

Omnibus igitur quantitatibus mutabilibus y, y', y'', y''', y^{IV}... præter duas, positis conſtantibus, fit

$$\frac{\text{cof.}SMA - \text{cof.}(\alpha + SM'M)}{a+y'} = \frac{\text{cof.}SM'M - \text{cof.}(\alpha + SM''M')}{y+y'}$$
$$= \frac{\text{cof.}SM''M' - \text{cof.}(\alpha + SM'''M'')}{y'+y''}$$
$$= \frac{\text{cof.}SM'''M'' - \text{cof.}(\alpha + SM^{\text{IV}}M''')}{y''+y'''}$$
$$= - \; - \; - \; - \; -$$

Quare pro vertice quovis M, duos inter $'M$, M' jacente, fit $\frac{\text{cof.}SMM' - \text{cof.}(\Delta x + SM'M)}{\Delta x('y+y')}$ quantitas conſtans: proinde & limes quantitatis hujus duplæ conſtans eſt, nempe $\frac{yy + 2\left(\frac{dy}{dx}\right)^2 - y\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = \frac{1}{C}$; cui æquationi ſatisfacit circumferentia circuli (§. 196.).

§. 216. *Generatim* ſit

$$z + z' + z'' + z''' + z^{\text{IV}} + \ldots = \text{dato}$$
$$\varphi y + \varphi y' + \varphi y'' + \varphi y''' + \varphi y^{\text{IV}} + \ldots = \text{maximo.}$$

Fit $\frac{\text{cof.}SM'M - \text{cof.}(\alpha + S'M'M)}{\Delta x \cdot \pi y} = C$; & limes hujus quantitatis, nempe

$$\frac{y^3 + 2y\left(\frac{dy}{dx}\right)^2 - yy\frac{ddy}{dx^2}}{\pi y\left(\frac{dz}{dx}\right)^3} = C; \;\; \& \;\; \frac{y\left(yy + 2\left(\frac{dy}{dx}\right)^2 - y\frac{ddy}{dx^2}\right)}{\left(\frac{dz}{dx}\right)^3} = C\pi y.$$

Proinde radius curvaturæ eſt in ratione directa radii vectoris, & inverſa functionis πy.

§. 217.

§. 217. Curva referatur ad aliquem axem per coordinatas rectilineas; & proponantur duæ æquationes

$$zFy + z'Fy' + z''Fy'' + z'''Fy''' + z^{iv}Fy^{iv} + \ldots = \text{dato}$$
$$\varphi y + \varphi y' + \varphi y'' + \varphi y''' + \varphi y^{iv} + \ldots = \text{maximo.}$$

Posito $\frac{dFy}{dx} = fy\frac{dy}{dv}$,

erit $\frac{dy}{dv}(zfy + Fy\,\text{cof.}PM'M - Fy'\,\text{cof.}P'M'M)$ Fig. 61.

$$\left.\begin{aligned}&+\frac{dy'}{dv}(z'fy' + Fy'\,\text{cof.}P'M'M - Fy''\,\text{cof.}P''M''M')\\&+\frac{dy''}{dv}(z''fy'' + Fy''\,\text{cof.}P''M''M' - Fy'''\,\text{cof.}P'''M'''M'')\\&+\frac{dy'''}{dv}(z'''fy''' + Fy'''\,\text{cof.}P'''M'''M'' - Fy^{iv}\,\text{cof.}P^{iv}M^{iv}M''')\\&+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\end{aligned}\right\} = 0$$

Et fimul $\frac{dy}{dv}.\pi y + \frac{dy'}{dv}.\pi y' + \frac{dy''}{dv}.\pi y'' + \frac{dy'''}{dv}.\pi y''' + \ldots = 0$

Hinc fit $\frac{zfy + Fy\,\text{cof.}PM'M - Fy'P'M'M}{\pi y} = C.$

Atqui $Fy' = Fy + \frac{\Delta x}{1}fy + \frac{\Delta x^2}{1.2}\frac{d^2Fy}{dx^2} + \frac{\Delta x^3}{1.2.3}\frac{d^3Fy}{dx^3} + \ldots.$

Quare $\frac{\frac{z}{\Delta x}fy + Fy(\text{cof.}PM'M - \text{cof.}P'M'M) - fy\frac{dy}{dx}\text{cof.}P'M'M - \frac{\Delta x}{1.2}\frac{d^2Fy}{dx^2}\text{cof.}P'M'M - \ldots}{\pi y} = C.$

Quare & limes hujus quantitatis eft quantitas conftans: unde fit

$$fy.\frac{1}{\frac{dz}{dx}} - Fy\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = C\pi y$$

$$fy.\frac{\frac{dy}{dx}}{\frac{dz}{dx}} - Fy\frac{\frac{dy}{dx}.\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = C\pi y.\frac{dy}{dx}$$

$$Fy:\frac{1}{\frac{dz}{dx}} = C' + C\varphi y$$

$$\frac{dx}{dz} = \frac{C' + C\varphi y}{Fy} = \text{fin.}PMT.$$ Unde res ad calculum integralem redit.

 Sit

Sit nempe Z functio aliqua communis ejusmodi, ut $\frac{dZ}{dx} = \frac{dz}{dx} Fy$;

et sit Z' alia quaedam functio talis, ut $\frac{dZ'}{dx} = \varphi y$:

Functio Z' fit omnium maxima, quando $\frac{dx}{dz} = \frac{C' + C\varphi y}{Fy}$.

Exemplum primum. Superficies solidi rotatione curvæ alicujus circa axem BB' geniti detur magnitudine; & area curvæ genitricis debeat esse omnium maxima.

Erit $\left.\begin{array}{l} \frac{dZ}{dx} = y\frac{dz}{dx} \\ \frac{dZ'}{dx} = y \end{array}\right\}$: hinc $\frac{dx}{dz} = \frac{Cy \pm C'}{y}$; quæ est æquatio ad lineam rectam casu, quo $C' = 0$.

Exemplum secundum. Superficies solidi rotatione curvæ alicujus circa axem BB' geniti detur magnitudine; & capacitas solidi debeat esse omnium maxima.

Erit $\left.\begin{array}{l} \frac{dZ}{dx} = y\frac{dz}{dx} \\ \frac{dZ'}{dx} = yy \end{array}\right\}$: $\frac{dx}{dz} = \frac{Cyy \pm C'}{y}$; quæ est æquatio ad circumferentiam circuli casu, quo $C' = 0$.

§. 218. *Problema.* Inter figuras ejusdem perimetri eam determinare, cujus centrum gravitatis sit omnium maxime depressum respectu alicujus rectæ axi ejus perpendicularis.

Itaque est $z + z' + z'' + z''' + z^{\text{IV}} + \ldots = \text{dato}$

$$\frac{3M}{S} = \frac{a(2b+c)+y(b+4c+c')+y'(c+4c'+c'')+y''(c'+4c''+c''')+y'''(c''+4c'''+c^{\text{IV}})+\ldots}{a + 2y + 2y' + 2y'' + 2y''' + 2y^{\text{IV}} + \ldots} = \text{max.}$$

Unde $\frac{dz}{dv} + \frac{dz'}{dv} + \frac{dz''}{dv} + \frac{dz'''}{dv} + \frac{dz^{\text{IV}}}{dv} + \ldots = 0$

$$S\left(\frac{dy}{dv}(b+4c+c') + \frac{dy'}{dv}(c+4c'+c'') + \frac{dy''}{dv}(c'+4c''+c''') + \frac{dy'''}{dv}(c''+4c'''+c^{\text{IV}}) + \ldots\right.$$

$$= 6M\left(\frac{dy}{dv} + \frac{dy'}{dv} + \frac{dy''}{dv} + \frac{dy'''}{dv} + \ldots\right)$$

Hinc

Hinc $\frac{dy}{dv}(\text{cof.}PMA - \text{cof.}P'M'M)$

$+ \frac{dy'}{dv}(\text{cof.}P'M'M - \text{cof.}P''M''M')$

$+ \frac{dy''}{dv}(\text{cof.}P''M''M' - \text{cof.}P'''M'''M'') \qquad = 0$

$+ \frac{dy'''}{dv}(\text{cof.}P'''M'''M'' - \text{cof.}P^{\text{IV}}M^{\text{IV}}M''')$

$+ \quad - \quad - \quad - \quad - \quad - \quad -$

Igitur $\frac{\text{cof.}PM'M - \text{cof.}P'M'M}{S('c+4c+c') - 6M} = C$

pariterque lim. $\frac{\frac{\text{cof.}PM'M - \text{cof.}P'M'M}{\Delta x}}{S('x+4x+x') - 6M} = C$; seu $\frac{-\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = C(Sx - M)$:

unde $\frac{dy}{dz} = C(\frac{1}{2}Sxx - Mx - C')$.

Hoc exemplo ad curvam translato, fit $\frac{dZ}{dx} = \frac{dz}{dx}$, $\frac{dM}{dx} = xy$, $\frac{dS}{dx} = y$: & posita Z magnitudine data, $\frac{M}{S}$ fit omnium maxima, quando $\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3(Sx-M)} = C$.

Eodem modo tractari possunt aliæ formulæ, quibus fit $\frac{M}{S}$ = maxima, positis $\begin{matrix} \frac{dM}{dx} = Fx.\varphi y \\ \frac{dS}{dx} = \varphi' y \end{matrix}$. Sed imperfecta calculi integralis conditio obstat, quominus inde possit determinatio curvæ deduci.

§. 219. *Problema.* Magnitudine detur perimeter $AMM'M''M'''$ ....; sed centrum gravitatis ejus respectu alicujus rectæ axi BB' perpendicularis debeat esse omnium maxime depressum. Fig. 61.

Est ideo $z + z' + z'' + z''' + z^{\text{IV}} + \ldots$ = dato

$z(b+c) + z'(c+c') + z''(c'+c'') + z'''(c''+c''') + \ldots$ = maximo.

Hinc $\frac{dz}{dv} + \frac{dz'}{dv} + \frac{dz''}{dv} + \frac{dz'''}{dv} + \frac{dz^{\text{IV}}}{dv} + \ldots = 0$

$(b+c)\frac{dz}{dv} + (c+c')\frac{dz'}{dv} + (c'+c'')\frac{dz''}{dv} + (c''+c''')\frac{dz'''}{dv} + \ldots = 0$

 dy

$$\frac{dy}{dv}(\cos. PMA - \cos. P'M'M) + \frac{dy'}{dv}(\cos. P'M'M - \cos. P''M''M') + \frac{dy''}{dv}(\cos. P''M''M' - \cos. P'''M'''M'') + \frac{dy'''}{dv}(\cos. P'''M'''M'' - \cos. P^{iv}M^{iv}M''') + \ldots = 0;$$

$$\frac{dy}{dv}((b+c)\cos. PMA - (c+c')\cos. P'M'M) + \frac{dy'}{dv}((c+c')\cos. P'M'M - (c'+c'')\cos. P''M''M') + \frac{dy''}{dv}((c'+c'')\cos. P''M''M' - (c''+c''')\cos. P'''M'''M'') + \frac{dy'''}{dv}((c''+c''')\cos. P'''M'''M'' - (c'''+c^{iv})\cos. P^{iv}M^{iv}M''') + \ldots = 0.$$

Omnes quantitates mutabiles $y, y', y'', y''', y^{iv} \ldots$ fiant simul constantes, duabus exceptis: erit pro vertice quolibet M, duos inter 'M, M' sito, $\frac{\cos. PM'M - \cos. P'M'M}{('x+x)\cos. PM'M - (x+x')\cos. P'M'M}$ quantitas constans ab Δx pendens; seu $\frac{\cos. PM'M - \cos. P'M'M}{2x(\cos. PM'M - \cos. P'M'M) - \Delta x(\cos. PM'M + \cos. P'M'M)}$ est quantitas constans: proinde

$$\frac{\frac{\cos. PM'M - \cos. P'M'M}{\Delta x}}{2x\frac{\cos. PM'M - \cos. P'M'M}{\Delta x} - (\cos. PM'M + \cos. P'M'M)} = \frac{1}{2C}$$

unde & limes hujus quantitatis, nempe

$$\frac{\frac{-\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}}{x\frac{-\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} - \frac{dy}{dz}} = \frac{1}{C}.$$ Hinc $$\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = \left(x\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} + \frac{dy}{dz}\right)\frac{1}{C}.$$

$$\frac{dy}{dz} = \frac{x\frac{dy}{dz} - C'}{C}$$

$$\frac{dy}{dz} = \frac{C'}{x - C} = \cos. PMT$$

$$\frac{dy}{dx} = \frac{C'}{\sqrt{((x-C)^2 - CC')}}.$$

Hoc

Hoc exemplo ad curvam translato, est $\begin{aligned}\frac{dZ}{dx} &= \frac{dz}{dx}\\ \frac{dZ'}{dx} &= x\frac{dz}{dx}:\end{aligned}$

et posita Z magnitudine data, Z' fit omnium maxima, quando $\frac{dy}{dz} = \frac{C'}{x-C}$.

Exemplum hoc refertur ad curvam *catenariam* dictam.

§. 220. Exempla hactenus proposita sufficiunt, ut tirones videant, quomodo quæstiones illis similes, quæ evolutæ fuerunt, ad principia limitum possint reduci; & ut intelligant, methodos a mathematicis huic quæstionum generi solvendo applicatas spectari debere tanquam compendia, solutioni brevius consequendæ apta. Quoniam autem methodi hæ admodum sunt generales, & iis quoque functionibus applicantur, quæ exponentes differentiales altiorum ordinum involvunt; quædam adhuc exempla facillima explicabo, quæ secundi altiorumve ordinum exponentes attingunt.

Sit $\frac{dZ}{dx} = \left(\frac{ddy}{dx^2}\right)^n$; & quæratur functio Z talis, ut sit omnium maxima.

Quoniam $\frac{ddy}{dx^2} = \lim.\frac{\Delta^2 y}{\Delta x^2}$;

$$\begin{array}{ll}
\text{fiat } Z' = ({}'y - 2y + y')^n & \text{Erit } \frac{dZ'}{dv} = n({}'y - 2y + y')\left(\frac{d'y}{dv} - 2\frac{dy}{dv} + \frac{dy'}{dv}\right) \\
\quad + (y - 2y' + y'')^n & \quad + n(y - 2y' + y'')\left(\frac{dy}{dv} - 2\frac{dy'}{dv} + \frac{dy''}{dv}\right) \\
\quad + (y' - 2y'' + y''')^n & \quad + n(y' - 2y'' + y''')\left(\frac{dy'}{dv} - 2\frac{dy''}{dv} + \frac{dy'''}{dv}\right) \quad = 0 \\
\quad + (y'' - 2y''' + y^{iv})^n & \quad + n(y'' - 2y''' + y^{iv})\left(\frac{dy''}{dv} - 2\frac{dy'''}{dv} + \frac{dy^{iv}}{dv}\right) \\
\quad + \;-\;-\;- & \quad + \;-\;-\;-\;-\;-\;- \\
\quad + \;-\;-\;- & \quad + \;-\;-\;-\;-\;-\;-
\end{array}$$

Hinc omnibus quantitatibus mutabilibus ${}'y$, y, y', y'', y''', y^{iv} . . . factis simul constantibus, una excepta; fit pro quantitate quacunque mutabili y, inter quantitates ${}'y$, y' media, $({}'y - 2y + y')^{n-1} - 2(y - 2y' + y'')^{n-1} + (y' - 2y'' + y''')^{n-1} = 0$.

Fiant

Fiant $('y - 2y + y')^{n-1} = 'Q$

$$(y - 2y' + y'')^{n-1} = Q: \text{ erit } \Delta''Q = 0; \text{ unde } \frac{\Delta''Q}{\Delta x^2} = 0, \text{ \& quantitatis hujus}$$

$$(y' - 2y'' + y''')^{n-1} = Q'$$

limes, nempe $\frac{ddQ}{dx^2} = 0$: unde $\frac{dQ}{dx} = C^{N-1}$; $Q = C'^{N-1} + C^{N-1}x = \left(\frac{ddy}{dx^2}\right)^{n-1}$, &

$\frac{ddy}{dx^2} = (C'^N + C^{N-1}x)^{\frac{1}{n-1}}$; $\frac{dy}{dx} = c + \frac{n-1}{n}(C'^N + C^{N-1}x)^{\frac{n}{n-1}}$

$$y = c' + cx + \frac{n-1}{n} \cdot \frac{n-1}{2n-1}(C'^N + C^{N-1}x)^{\frac{2n-1}{n-1}}.$$

Proinde posito $\frac{dZ}{dx} = \left(\frac{ddy}{dx^2}\right)^n$, fit Z omnium maximum, si fuerit

$$y = c' + cx + \frac{n-1}{n} \cdot \frac{n-1}{2n-1}(C'^N + {}^{N-1}Cx)^{\frac{2n-1}{n-1}}.$$

Eodem modo, si sit $\frac{dZ}{dx} = \left(\frac{d^3y}{dx^3}\right)^n$: erit

$$y = c'' + c'x + \tfrac{1}{2}cxx + \frac{n-1}{n} \cdot \frac{n-1}{2n-1} \cdot \frac{n-1}{3n-1}(C''^N + C'^{N-1}x + \tfrac{1}{2}C''^{N-1}x^2)^{\frac{3n-1}{n-1}}.$$

Unde lex manifesta est pro quolibet exponente differentiali.

Caput vicesimum primum.

Delineatio succincta applicationis calculi differentialis et integralis ad physicam.

§. 221.

Hucusque nonnisi objecta ad mathesin puram spectantia tractavi. Quare ab rigore demonstrandi, quem ea admittit atque exigit, deflectere non debui. Speciatim non licuit figuras curvilineas considerare tanquam rectilineas, quicunque harum sit numerus laterum; nec solida superficiebus curvis terminata tanquam polyhedra, utcunque magno hedrarum numero contineantur. Universim ab magnitudinibus limitum capacibus ad ipsos hos limites transire non licuit, nisi quoad transitus hic propositionibus eum firmantibus muniebatur; cujusmodi tum passim, tum Capite præsertim primo stabilire annixus sum.

§. 222.

§. 222. Rigorem hunc in mathesi pura omnino necessarium, atque unum ex propriis ipsius characteribus constituentem, applicationes ejus ad objecta, quorum cognitio ab sensuum testimonio pendet (sive ad artes pertineant, sive ad scientias), non solum non exigunt; sed & affectationem illius ut supervacaneam, quin sæpe noxiam respuunt. (a) Certitudo mathematica ab constitutione organorum nostrorum independens debet esse absoluta: cognitioni nostræ physicæ, ab indole sensuum nostrorum pendenti, semper aliquid labis adhærebit ab imperfectione organorum nostrorum, utcunque exercitatorum, & quacunque arte adjutorum. Translationem abstractionum mathematicarum ad scientias physicas harum progressui officere posse experientia testatur: quod comprobantium exemplorum pauca exstantiora recensebo.

Quoad objectorum tantum terrestrium & lunæ distantiæ ad indagandam velocitatem lucis suppetebant; determinari haud potuit tempus, quo lux datum spatium emetitur. Unde judicio præcipiti illatum fuit, lucem spatio cuicunque emetiendo nullum impendere tempus; velocitas ejus pronunciata fuit infinita: explicationes facti hujus nullo vellicati dubio traditæ fuerunt; quibus etiamnum adstipulaturi essemus, nisi observationes in corporibus longe remotioribus institutæ falsum esse, quod explicandum sumebatur, docuissent.

Ex apparente directionum gravium parum invicem distantium parallelismo superficies telluris diu plana fuit judicata: ad rotunditatem terræ stabiliendam alius generis observationibus opus fuit, quæ philosophorum oculos serius demum perculerunt.

Ex sensibili ponderum corporum ad distantias parum diversas a superficie telluris æqualitate diu inferri consuevit æqualitas actionis causæ gravitatis: donec physica exactior testimonii sensuum æstimatione accuratiori & discussione phænomenorum immediate illis haud obviorum opinionem hanc delevit.

Alia

(a) Animadversiones in applicationem matheseos ad physicam Capite hoc traditæ, pariter ac deductiones expositæ Capite XII. Dissertationis meæ: *Exposition des principes des calculs superieurs*, conformes sunt doctrinis, quas ex ore Dni. Le Sage hausi, quo tempore studia mea philosophica & mathematica affectu paterno regebat. Vid. Diss. cit. Nota p. 210.

Alia afferre possem exempla errorum, in quos præcipites ab phænomenis ad causas ipsorum conclusiones induxerunt; & qui ab simili judicii præcipitantia cavere monent.

§. 223. Per totum hunc librum ostensa infinite parvorum, quæ vocantur, mathematicorum inutilitate; & quibusdam locis, repugnantia conceptus ipsorum: diversam ab ea, quod ad objecta physica, sententiam cum Dno. Le Sage profiteor. In scientiis ad hæc applicatis pro nihilo habemus, quod sensus nostros quacunque adjutos arte effugit: & explicatio phænomenorum naturæ, conformitatem eorum non rigorosam quidem cum legibus arbitrariis inde deductis, sed eousque accuratam, quousque observationum nostrarum *ακριβεια* pertingit, ostendens, hoc respectu satisfacere proposito censenda est. (*b*)

§. 224. Principio hoc admisso, Dn. Le Sage omnino convellit objectiones mechanicæ phænomeni naturæ latissime patentis, gravitatis universalis, explicationi adversas; adhibitis (juxta analysin æque ingeniosam ac solidam) *corpuscu-*

(*b*) Quo tempore infinite parva tanquam basis necessaria calculorum superiorum spectabantur: Dn. Le Sage usum eorum in mathesi pura ab contradictionibus, quas implicat, tueri satagebat conceptu sequenti: quem deinde ad eam nonnisi sensu indirecto vel devio & explicatu prolixo applicari posse agnovit; ac quem ipsiusmet verbis, 14 Julii 1786. ad me perscriptis, uti in Dissertatione mea priore pag. 211. exponam:

„Mon but dans cette ancienne recherche etoit d'assigner une bonne fois aux infi„niment petits de tout genre & de tout ordre une constitution, qui me permit en„suite & toujours de les traiter hardiment comme des quantités determinées, finies, „mais negligibles; & qui fut applicable aux etres réels, comme aux etres hypothe„tiques.

„Pour cet effet je substituai aux quantités moindres qu'aucune assignée, & à cel„les qu'on pourroit pretendre etre moindres qu'aucune assignable, des quantités moin„dres qu'aucune qu'on assignera réellement jamais; ou plutot des parcelles, qu'on „peut negliger relativement à leur tout, sans aucune erreur qui devienne jamais réel„lement sensible.

„Effectivement de telles quantités sont vraiment *determinées*; puisque toute ap„plication, qu'on fera réellement jamais, de quelque proposition que ce soit, est de„terminée en soi, quoique actuellement inconnue: vu que tous les futurs contingens „sont determinés, n'y eut-il meme aucun etre qui les previt, ni aucun *nexus* entre „ces futurs-là & le present.

pusculis, quæ intervallis relate ad dimensiones ipsorum admodum magnis invicem distent, & quaquaversus (c) secundum lineas rectas celeritate prægrandi moveantur.

Dn. Le Sage actionis fluidi hujus consequentias mathematicas accurate evolvit, & consensum earum cum phænomenis gravitatis ostendit. Meditationes ipsius de arduo hoc physicæ generalis objecto minus adhuc notæ sunt. Studio, qua possunt, perfectione redactas publico exponendi, scripta sua divulgare non festinavit: nec nisi primas systematis sui lineas in Dissertatione inscripta *Essai de chymie mecanique*, quam Academia Rhotomagensis anno 1758. præmio ornavit, ac nuper in *Lucretio Newtoniano*, Commentariis Academiæ Berolinensis anni 1782. inserto, exposuit. Ambitus ejus opera nonnullorum ipsius amicorum quodammodo innotuit: quos inter Dni. De Luc in variis suis scriptis, iis nominatim, quæ sub titulo: *Idées sur la Meteorologie*, *Lettres à la Reine d'Angleterre*, *Lettres à Mr. de la Metherie*, edidit; & Dni. Prevost in libro inscripto *Origine des forces magnetiques*. (d) Grato in agnatum hunc pie devenerandum, cujus curæ quicquid in me est doctrinæ debeo, affectu motus, cum septennio abhinc ex Polonia reverterer, summopere optavi exiguum ejus documentum ipsi exhibere opera mea qualicunque ad manuscriptorum ipsius multiplicium editionem promovendam. Quo jucundior mihi fuit, quam diu animo fovi, spes hoc voto potiundi, eo magis irritam ab me conceptam fuisse dolui.

 §. 225.

(c) Hæc quoque expressio physice est intelligenda. Quod innuo difficultatis solvendæ gratia, nuper ab illustri Mathematico motæ, & uberius ab Dno. Wilckens expositæ, contra pag. 71 sq. Dissertationis: *Essai de chymie mecanique*; ubi dispositiones quadratiformes punctorum in superficie sphæræ rigoroso sensu mathematico accipi non debent. Qui primus nævum doctrinæ Elementorum Euclidis de angulis solidis animadvertit geometra (*Hist. de l'Acad. royal. des Sc.* 1756. *Bermanni Commentatio de angulis solidis*); poteratne in gravem incidere errorem, quem commisisset, si expressiones ipsius ad discussionem physicam pertinentes mathematice essent interpretandæ? (Vid. *Wilckens Aufsäze mathemat. Inhalts.* Gött. 1790. *Kästners geometr. Abhandl.* Gött. 1791.)

(d) Dn. Le Sage non solum arduum gravitatis universalis phænomenon per actionem immediatam corpusculorum supra indicatorum explicat: sed eadem quoque mediatam plerorumque phænomenorum naturæ generalium causam esse ostendit, quatenus activitatem suam ipsis debeant fluida elastica, per quæ phænomena illa possunt explicari.

§. 225. Conceptus actionis discontinuae fluidi gravifici, physico sua se luculentia commendans, saepe etiam mathematico commodus est. Theoriae fluidorum continuorum, elasticorum pariter ac non elasticorum, admodum imperfectae adhuc sunt, quamvis summi hujus & superioris seculi mathematici sua illis studia impenderint; & quas a priori stabilire quidam aggressi sunt, experientià evertit. Consequentias principiorum, quae pro basibus sumuntur, persequi exercitatissimis tantum calculatoribus datur. Alia est ratio theoriae fluidorum discretorum. Principia ejus sunt indubitata; & consequentiae deductu faciles; quod calculi Dni. Le Sage comprobant, qui ne primas quidem calculorum superiorum notiones exigunt.

§. 226. Ex. gr. Galilei lex accelerationis gravium (physice intellecta) immediate ex actione fluidi ejusmodi consequitur. Cum incrementum velocitatis gravium debeatur fluido, tempusculis aequalibus per impulsus aequales agenti: celeritas, quam gravia tempore sensibili acquirunt, numero harum impulsionum; proinde tempori proportionalis est, quo corpora illas sunt experta. Immedite igitur consequitur, esse $v \backsim t$.

Hinc spatia tempusculis aequalibus successivis percursa, utpote celeritatibus acquisitis proportionalia, pariter crescunt uti tempora, proinde uti numeri naturales; & spatia inde ab initio motus percursa crescunt uti numeri triangulares numerorum momentorum illorum temporis. Sed numerorum triangularium ratio eo propius accedit ad rationem numerorum quadratorum correspondentium, quo majores sunt numeri, qui ipsorum sunt radices (§. 20.). Itaque spatia percursa eo accuratius proportionalia sunt quadratis temporum elapsorum, quo plura quodvis tempus sensibile continet tempuscula, duobus causae motricis impulsibus immediate sese consequentibus interjecta.

§. 227. Optabile sine dubio foret, ut mathematicis semper liceret eadem facilitate consequentias principii assumti deducere. Magis autem assueti exponentibus differentialibus $\frac{ds}{dt} = v$, $\frac{dv}{dt} = p$, aliisque similibus; quam expressionibus per differentias finitas, quae illis respondent: priores tractare expressiones, quam posteriores, magis fere (in investigationibus praesertim prolixis ac difficiliori-

lioribus) commodum ducunt; ideoque eas alteris ſubſtituunt, quibus propoſitum facilius conſequuntur. Qua quidem ſubſtitutione uti ipſis licet: dummodo rationis, qua ducti eam uſurpant, non obliviſcantur; neque exigant, ut, quas inde nectunt, conſequentiæ abſoluto ac mathematico rigore veræ eſſe cenſeantur.

Ex æquatione $v = t$, vel potius $v = gt$ (poſita g celeritate, quam corpus acquirit tempore pro unitate ſumto, ex. gr. uno minuto ſecundo) conſequitur $\frac{dv}{dt} = g$.

Cum autem ſit $\frac{ds}{dt} = v$; ſit $\frac{d^2 s}{dt^2} = \frac{dv}{dt} = g$.

Porro cum ſit $\frac{dv}{dt} = g$;

& $v = \frac{ds}{dt}$:

ſequitur $v\frac{dv}{dt} = g\frac{ds}{dt}$; & $v\frac{dv}{ds} = g$.

Hinc, ob g conſtantem, ſit $\frac{1}{2}v^2 = gs$; $v = \sqrt{2gs}$.
Sed $v = gt$. Ergo $gt = \sqrt{2gs}$; & $t = \sqrt{\frac{2s}{g}}$.

§. 228. Formula differentialis $\frac{dv}{dt} = g$ pariter applicatur ad caſum, quo g variabilis eſt; & qui, phyſice intellectus, ſupponit tempus lapſus in partes adeo exiguas dividi, ut, durante qualibet earum, actio cauſæ corpus impellentis ſit uniformis.

Exemplum primum. Gravitas variet uti diſtantia a centro.
Sit a diſtantia ab centro, a qua inde corpus delabitur;
g actio gravitatis ad hanc diſtantiam.
Corpore delapſo per ſpatium s, gravitas ad diſtantiam $a - s$ erit $g \times \frac{a-s}{a}$.

$$\text{Igitur } \frac{dv}{dt} = g \times \frac{a-s}{a}$$

$$\text{ſed } \frac{dt}{ds} = \frac{1}{v}$$

$$\text{Ergo } v\frac{dv}{ds} = g \times \frac{a-s}{a}$$

$$v^2 = C + g \times \frac{2as - s^2}{a}.$$

 Sed

Sed v evanescit, quando $s=0$. Proinde $C=0$; $v^2=g\times\frac{2as-s^2}{a}$.

Sit $s=a$; erit $v^2=ag$. Sed gravitate posita uniformi, erat $v^2=2ag$ (§. 227.) Quadratum igitur celeritatis acquisitæ per actionem gravitatis constantis duplum est quadrati celeritatis acquisitæ per actionem gravitatis variabilis juxta legem propositam.

Porro $\frac{dt}{ds}=\frac{1}{v}=\sqrt{\frac{a}{g}}\times\frac{1}{\sqrt{(2as-s^2)}}$; unde $t=\sqrt{\frac{a}{g}}\left(C-\text{arc.sin.}\frac{a-s}{a}\right)$

Sed $t=0$, dum $s=0$. Ergo $C=\text{arc.sin.}1=p$;

$$t=\sqrt{\frac{a}{g}}\left(p-\text{arc.sin.}\frac{a-s}{a}\right)=\sqrt{\frac{a}{g}}\times\text{arc.cosin.}\frac{a-s}{a}.$$

Sit $s=a$; erit $t=\sqrt{\frac{a}{g}}\times p$. Et cum sit $g \backsim a$: est $t \backsim p$; seu tempus lapsus ad centrum usque constans est, quodcunque sit spatium percurrendum: quod principium est isochronismi in cycloide.

Exemplum secundum. Gravitas sit in ratione inversa quadrati distantiæ.

Erit $\frac{dv}{dt}=g\times\frac{a^2}{(a-s)^2}$

$\frac{dt}{ds}=\frac{1}{v}$

Ideoque $v\frac{dv}{ds}=g\times\frac{a^2}{(a-s)^2}$; $\frac{1}{2}v^2=C+g\times\frac{a^2}{a-s}$.

Sit $v=0$, dum $s=0$: erit $C=-g$; $\frac{1}{2}v^2=g\left(\frac{a^2}{a-s}-a\right)=g\times\frac{as}{a-s}$

$$v=\sqrt{2ag}\times\sqrt{\frac{s}{a-s}}=\sqrt{\frac{2}{a}}\times\sqrt{\frac{s}{a-s}} \text{ ob } g=\frac{1}{a^2}.$$

Porro $\frac{dt}{ds}=\frac{1}{v}=\frac{1}{\sqrt{2ag}}\times\sqrt{\frac{a-s}{s}}=\frac{1}{\sqrt{2ag}}\times\frac{a-s}{\sqrt{(as-s^2)}}=\frac{1}{\sqrt{2ag}}\left(\frac{\frac{1}{2}a}{\sqrt{(as-s^2)}}+\frac{\frac{1}{2}a-s}{\sqrt{(as-s^2)}}\right)$

Unde $t=\frac{1}{\sqrt{2ag}}\left(\frac{1}{2}a\times\text{arc.sin.vers.}\frac{s}{\frac{1}{2}a}+\sqrt{(as-s^2)}+C\right)$

Sit $t=0$, dum $s=0$: erit $C=0$; $t=\frac{1}{\sqrt{2ag}}\left(\frac{1}{2}a\times\text{arc.sin.vers.}\frac{s}{\frac{1}{2}a}+\sqrt{s(a-s)}\right)$.

Sit $s=a$. Erit $t=\frac{1}{\sqrt{2ag}}\times\frac{1}{2}a\pi=\frac{1}{\sqrt{2ag}}\times ap$.

Atqui $g=\frac{1}{a^2}$. Ergo $t=\frac{a\sqrt{a}}{\sqrt{2}}p$; & $t^2=\frac{a^3}{2}p^2$.

Obser-

Obſervatio. Cum expreſſiones temporis t & celeritatis v imaginariæ fiant pro $s > a$; docemur, motum ultra centrum gravitationis locum habere non poſſe.

Sit $s = a$: erit $v = \sqrt{\frac{2}{a}} \times \sqrt{\frac{a}{a-a}} = \sqrt{\frac{2}{a}} \times \sqrt{\frac{a}{0}}$; quod eſt ſignum impoſſibilitatis (Cap. IX.): ideoque corpus non ſolum ultra centrum gravitationis progredi nequit; ſed ne ad centrum quidem usque poteſt pervenire. Quod geometrice oſtendi poteſt modo ſequente.

Cum ſit $v^2 = \frac{2}{a} \times \frac{s}{a-s} = \frac{2}{a}\left(\frac{a}{a-s} - 1\right) = \frac{2}{a-s} - \frac{2}{a} = \frac{2aC^2}{a-s} - 2C^2$;

eſt $v^2 + 2C^2 = \frac{2aC^2}{a-s}$.

Deſcripta igitur curva hyperbolica, cujus centrum ſit centrum gravitationis, & cujus abſciſſæ ab centro inde in aſymptoto alterutra ſumtæ ſint ſpatia percurrenda, cujus denique æquatio ſit $y^2 + b^2 = \frac{2ab^2}{x}$; quadrata celeritatum, aucta quantitate data, proportionalia erunt quadratis ordinatarum: pariterque ac ordinatæ nullum habent limitem magnitudinis, nec celeritas limitem magnitudinis habet; atque uti nullum eſt hyperbolæ punctum, cui ordinata reſpondeat occurrens axi abſciſſarum in centro; ita nec celeritas ulla eſt, quæ reſpondeat ſuppoſitioni, quod corpus ad centrum usque poſſit pervenire: proinde ſuppoſitio hæc impoſſibilis eſt.

Caſus hic plures occupavit mathematicos; quorum nonnulli valde paradoxa de eo ſtatuerunt. Exemplum, mea quidem ſententia, is præbet abuſus abſtractionum mathematicarum, objectis applicatarum realibus, cum quibus conſiſtere nequeunt.

Gravitatio, quam corpus ſphæricum exercet, ſequitur rationem inverſam duplicatam diſtantiæ ab centro corporis hujus, quoad corpus gravitans extra illud ſitum eſt. Poſito autem, corpus gravitans poſſe (per fictionem aliquam admiſſibilem) intra illud pervenire: tum lex gravitationis eadem, quæ prius, non ſubſiſtit; ſed gravitatio variatur in ratione ſimplici diſtantiæ a centro. Juxta hanc vero legem corpus ad centrum usque delabitur, & velocitatem determi-

terminatam acquirit: tum ultra centrum progreditur motu retardato ſimili ratione, qua acceſſus ipſius ad centrum fuerat acceleratus. Conſiderationes igitur phyſicæ ſententiam hoc caſu confirmant, quam de ſignificatione ſymboli $\frac{1}{0}$ Capite IX. profeſſus ſum. Conf. Diſs. Dn. Le Sage inſerta Diario *Journal de phyſique de l'Abbé Rozier. Janvier* 1776.

Formulæ præcedentes applicantur ad motus juxta legem quamcunque datam variatos; nominatim ad motus corporum in mediis reſiſtentibus. Sed ſcopus præſentis ſcriptionis applicationes has perſequi prohibet. Quare ad motus compoſitos progredior.

§. 229. Kepleri celeberrima lex arearum inſigne præbet exemplum facilitatis, quam inveſtigationibus phyſico-mathematicis conciliat ſuppoſitio actionis non continuæ cauſæ motricis. Quippe juxta eam demonſtratio legis hujus nonniſi ſimpliciſſimas requirit geometriæ elementaris propoſitiones, eas nimirum, quibus nititur æqualitas triangulorum æquealtorum ſuper eadem baſi. Re etiam ipſa omnes fere auctores, præeunte Newtono, eam juxta hanc ſuppoſitionem explicant; quicquid ceterum de conformitate ſuppoſiti hujus cum natura rei ſentiant: & tum ad ſemitas curvilineas applicant, quod in rectilineis locum habet, utcunque exigua ſint latera ipſarum.

Demonſtratio curvis ipſis immediate applicata æque facilis non eſt. Comparationis duarum methodorum inſtituendæ gratia poſteriorem tradam eo fere modo, quo ipſam expoſuit cel. de la Place in *Theoria* ſua *motus planetarum.*

Sit S focus gravitationis; SM radius vector, ſeu diſtantia, $= r$; M corpus gravitans; & gravitas ſit cuicunque diſtantiæ SM functioni $= \varphi r$ proportionalis.

Per focum S agantur duæ quæcunque rectæ SP, SQ, invicem normales; ad quas referatur motus corporis M. Sint $MP = SQ = x$, $MQ = y$.

Actio gravitatis ſecundum directionem MQ erit $\varphi r \times \frac{y}{r}$; ſecundum directionem SQ, $\varphi r \times \frac{x}{r}$: & celeritates planetæ juxta has directiones reſpective erunt $\frac{dy}{dt}$, $\frac{dx}{dt}$. Exponentes differentiales celeritatum harum, qui viribus $\varphi r \times \frac{y}{r}$, $\varphi r \times \frac{x}{r}$ proportionales ſunt, reſpective erunt $\frac{d^2y}{dt^2}$, $\frac{d^2x}{dt^2}$. Unde duæ conſequuntur

tur aequationes:

$$\frac{d^2y}{dt^2} = \varphi r \times \frac{y}{r}. \quad \text{Hinc} \quad x \times \frac{d^2y}{dt^2} = \varphi r \times \frac{xy}{r}$$

$$\frac{d^2x}{dt^2} = \varphi r \times \frac{x}{r} \qquad y \times \frac{d^2x}{dt^2} = \varphi r \times \frac{xy}{r}$$

$$\text{Ergo} \quad x \times \frac{d^2y}{dt^2} - y \times \frac{d^2x}{dt^2} = 0$$

$$\text{et} \quad x \times \frac{dy}{dt} - y \times \frac{dx}{dt} = C.$$

Sit angulus $MSQ = z$. Erunt $y = r\sin.z$; $\frac{dy}{dt} = r\cos.z\frac{dz}{dt}$

$$x = r\cos.z; \quad \frac{dx}{dt} = -r\sin.z\frac{dz}{dt}$$

$$x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\cos.^2 z\frac{dz}{dt} + r^2\sin.^2 z\frac{dz}{dt} = r^2\frac{dz}{dt} = C.$$

Posita autem area curvae, quam verrit radius vector, $= S$, est $\frac{dS}{dz} = \frac{1}{2}r^2$ (§.104.)

Quare etiam $\frac{dS}{dt} = \frac{1}{2}r^2\frac{dz}{dt} = C$

ideoque $\quad S = C' + Ct.$

Proinde si S evanescit, quando $t = 0$: fit $S = Ct$; seu area S est tempori proportionalis.

§. 230. Ex lege hac immediate duo corollaria sequentia fluunt:

1°. Celeritas mobilis est in ratione inversa perpendiculi, quod ab foco in directionem corporis demittitur; proinde in curva velocitas est in ratione inversa producti ex radio vectore ac sinu anguli, quem radius vector cum tangente curvae comprehendit. Ideoque expressio velocitatis est $\frac{\sqrt{\left(rr + \left(\frac{dz}{dr}\right)^2\right)}}{rr}$ (§. 49.)

2°. Celeritas angularis mobilis, relata ad focum gravitationis, sequitur rationem inversam duplicatam radii vectoris.

§. 231. In gratiam phyficorum, quibus calculi fuperiores & theoria curvarum minus funt familiares, Dn. LE SAGE confequentias actionis haud continuae caufae gravitationis & motus corporum in orbitis rectilineis perfequi fategit; ac fpeciatim in polygonis regularibus theoremata Hugenii ad vires centrales in circulo pertinentia demonftravit.

Cum in omni curva vis centralis refpondens tempori, quo mobile arcum aliquem curvae percurrit, fit in ratione compofita ex directa fimplice fegmenti radii vectoris ad unum arcus hujus extremum ducti, quod arcui huic & tangenti per alterum arcus extremum ductae interjacet, & ex inverfa ratione duplicata temporis ad arcum percurrendum impenfi: data curvae, quam mobile defcribit, aequatione ad focum gravitationis relata; ope formularum §§. 104. 196. determinatur lex gravitationis.

Fig. 57. Nempe pofita vi centrali $= g$; eft $g = \lim. \frac{M't}{MFM'^2}$

Atqui $\lim. \frac{\Delta z}{MFM'} = \frac{1}{r^2}$, & $\lim. \frac{\Delta z^2}{MFM'^2} = \frac{1}{r^4}$ (§. 104.)

$$\text{ac} \quad \lim. \frac{M't}{\Delta z^2} = \tfrac{1}{2}r + \frac{1}{r}\left(\frac{dr}{dz}\right)^2 - \frac{1}{1.2} \times \frac{ddr}{dz^2}$$

Igitur $$g\left(= \lim. \frac{Mt}{MFM'^2}\right) = \frac{1}{r^4}\left(\tfrac{1}{2}r + \frac{1}{r}\left(\frac{dr}{dz}\right)^2 - \frac{1}{1.2} \times \frac{ddr}{dz^2}\right).$$

Exemplum. Sit $r = \frac{bb}{a+e\,\text{cof.}z}$, quae eft aequatio focalis ellipfis;

itaque $$\frac{dr}{dz} = \frac{bbe\,\text{fin.}z}{(a+e\,\text{cof.}z)^2} = \frac{e\,\text{fin.}z}{bb}rr$$

$$\frac{ddr}{dz^2} = \frac{e\,\text{cof.}z}{bb}rr + \frac{2e\,\text{fin.}z}{bb}r\frac{dr}{dz} = \frac{e\,\text{cof.}z}{bb}rr + \frac{2ee\,\text{fin.}^2z}{b^4}r^3$$

erit $$g = \frac{1}{r^4}\left(\tfrac{1}{2}r + \frac{ee\,\text{fin.}^2z}{b^4}r^3 - \tfrac{1}{2} \times \frac{e\,\text{cof.}z}{b^2}r^2 - \frac{ee\,\text{fin.}^2z}{b^4}r^3\right) = \tfrac{1}{2}.\frac{1}{r^3}\left(1 - \frac{e\,\text{cof.}z}{bb}r\right)$$

Sed $er\,\text{cof.}z = bb - ar$.

Ergo $$g = \tfrac{1}{2} \times \frac{1}{r^3}\left(1 - \frac{bb - ar}{bb}\right) = \tfrac{1}{2} \times \frac{a}{bb} \times \frac{1}{rr}.$$

Pro-

Proinde in diverſis ejusdem ellipſis punctis gravitatio ſequitur rationem inverſam duplicatam diſtantiæ ab foco.

Eodem modo demonſtratur, eandem legem in ceteris ſectionibus conicis obtinere.

Viciſſim, data lege gravitationis, ex æquatione

$g = \frac{1}{r^4}\left(\frac{1}{2}r + \frac{1}{r}\left(\frac{dr}{dz}\right)^2 - \frac{1}{1.2} \times \frac{ddr}{dz^2}\right)$ deducitur natura curvæ deſcriptæ (quoad limites perfectionis calculi integralis concedunt).

Ex. gr. gravitate ſequente rationem inverſam duplicatam diſtantiæ, curva invenitur eſſe ſectio aliqua conica; quæ determinatur per angulum projectionis, & per rationem, quam celeritas projectionis habet ad eam, qua fieret, ut mobile circulum ad eandem diſtantiam deſcriberet.

Verum hæc fuſius deducere ab ſcopo libri hujus eſt alienum.

Corri-

Corrigenda.

pag.	lin.	loco	legatur
27.	3.	terminorum usque	terminorum poſt primum usque
ibid.	3 ſq.	terminum, cujus coëfficiens L.	terminum, qui præcedit eum, cujus coëfficiens eſt L, una cum hujus duplo.
28.	1.	n^n	x^n
ibid.	2.	$\frac{n-(m+r)}{m+r+1}$	$\frac{n-(m+r-1)}{m+r}$
29.	18.	x	y
33.	22.23.	n	$n-1$
52.	15.	P^{iv}	P^{N}
73.	ult.	$>$	$<$
74.	1.3.4.	$>$	$<$
ibid.	2.	$<$	$>$
ibid.	ibid.	tangentis creſcentis	tangentis decreſcentis
77.	15.	$x=C+\frac{m}{n}$	$x=C+\frac{m}{n}y$
81.	9.	TS	MT
96.	22.	$\frac{1}{5}v^6$	$\frac{1}{6}v^6$
108.	18.	(§.)	(§. *t. Introd.*)
113.	3.	$(z+\Delta)z$	$(z+\Delta z)$
118.	15.	2.2	3.2
163.	18.	b' & b	b' & b''
167.	ult.	$\sqrt{((aa+ee)^2-4ae\,\text{ſin.}^2 x)}$	$\sqrt{((a+e)^2-4ae\,\text{ſin.}^2 \tfrac{1}{2}x)}$
174.	25.	$2MP \times Mm'$	$2MP + Mm'$
189.	6.	B, B'	A', B'
190.	2.	$=$	$+$

Tab. I.

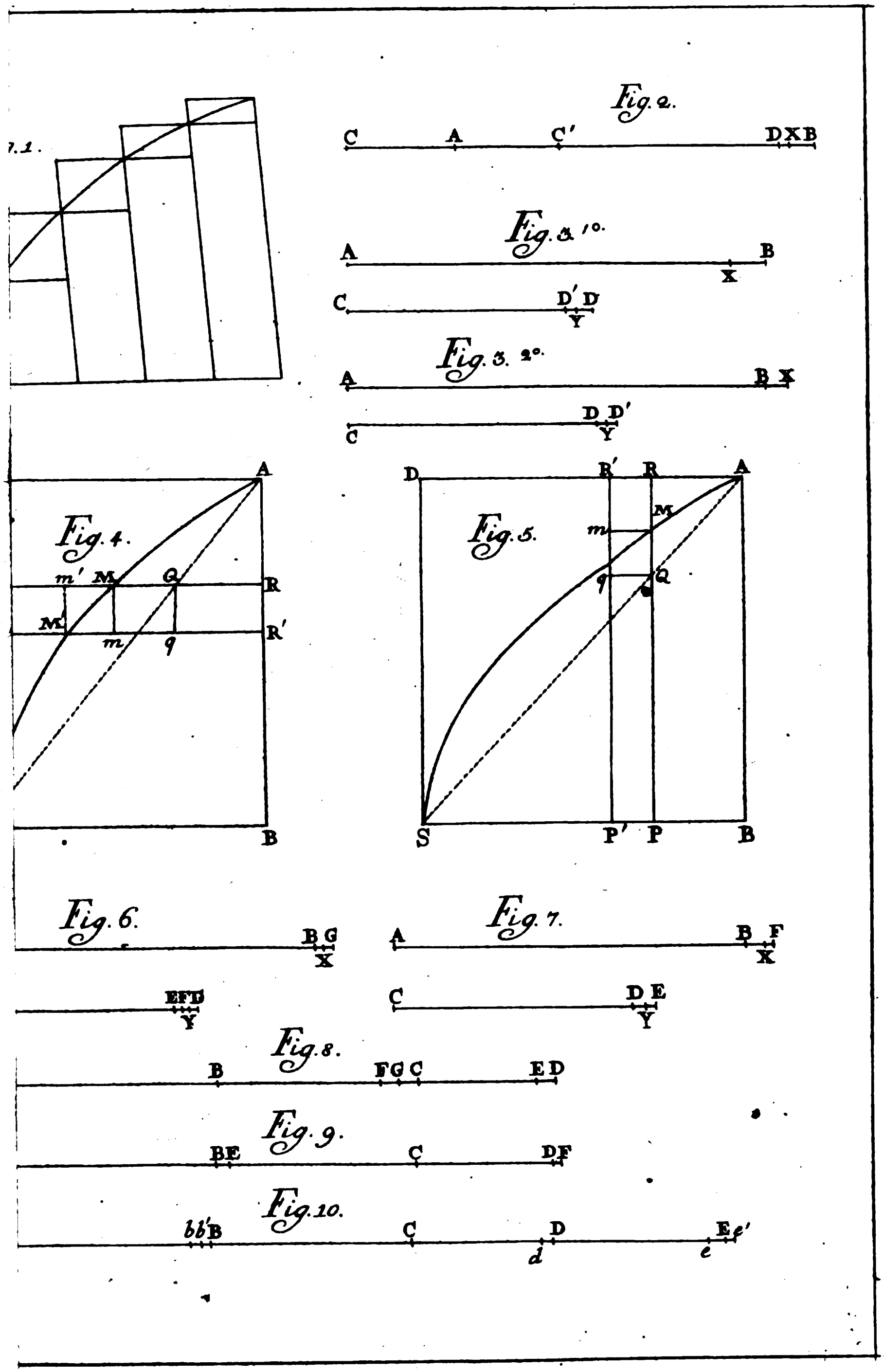
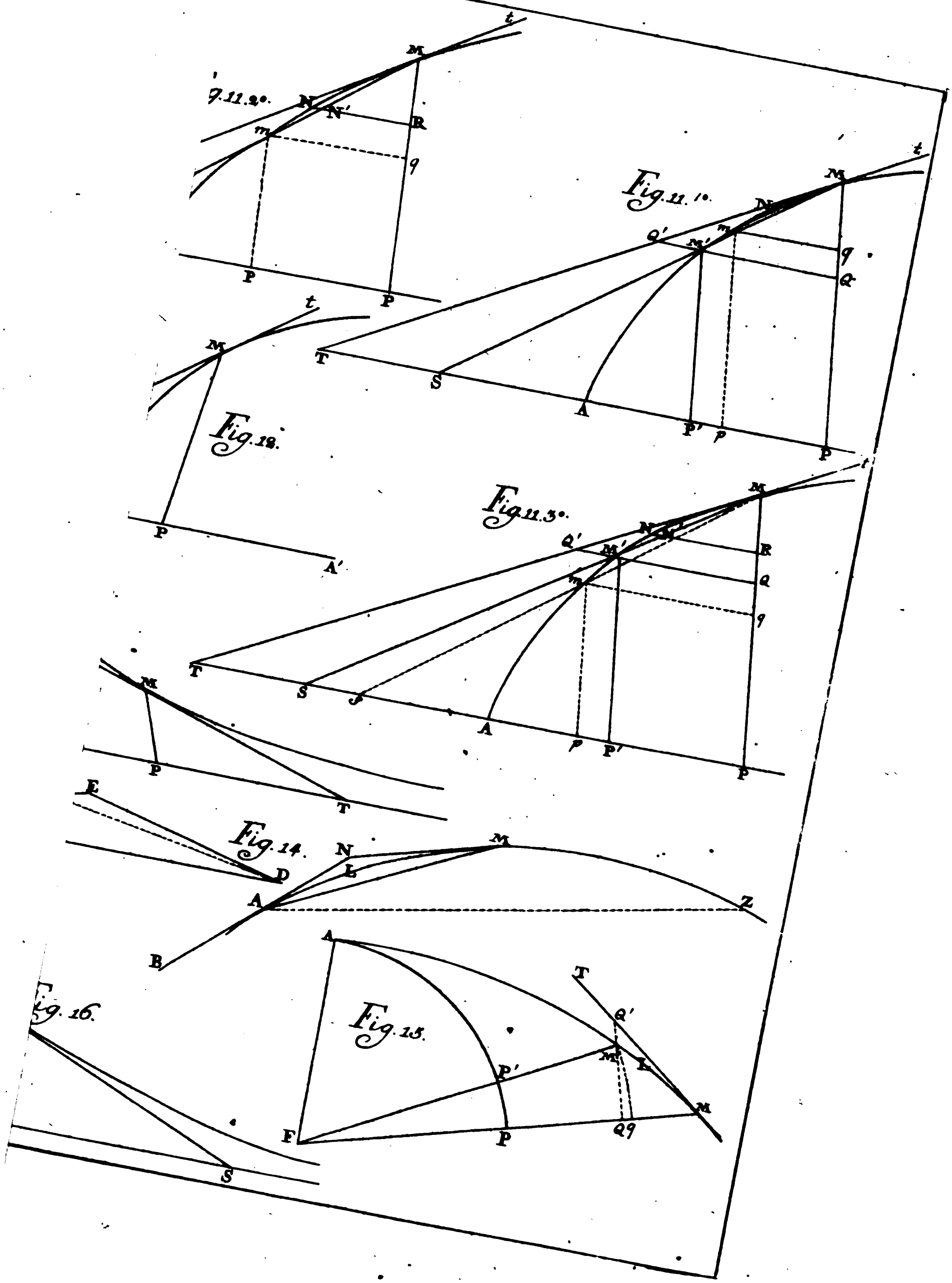


Fig. 1.
Fig. 2.
C A C′ DXB
Fig. 3. 1°.
A X B
C D′ D Y
Fig. 3. 2°.
A B X
C D D′ Y
Fig. 4.
A R R′ B m′ M Q M′ m q
Fig. 5.
D R′ R A M m q Q S P′ P B
Fig. 6.
B G X
E F D Y
Fig. 7.
A B F X
C D E Y
Fig. 8.
B F G C E D
Fig. 9.
B E C D F
Fig. 10.
b b′ B C D d E e′ e

Tab. II.



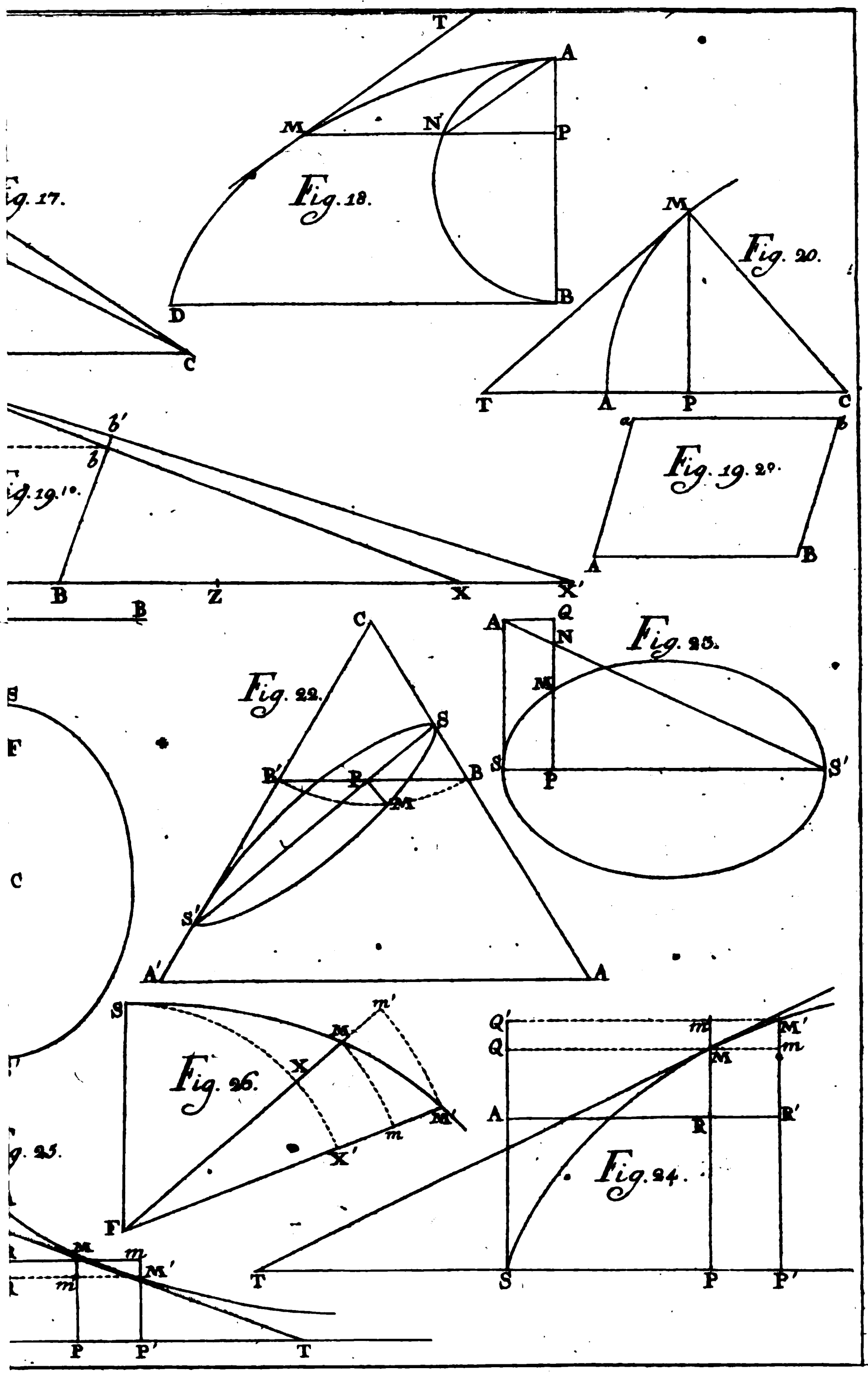


Fig. 17.
Fig. 18.
Fig. 20.
Fig. 19. 1°.
Fig. 19. 2°.
Fig. 22.
Fig. 23.
Fig. 26.
Fig. 24.
Fig. 25.

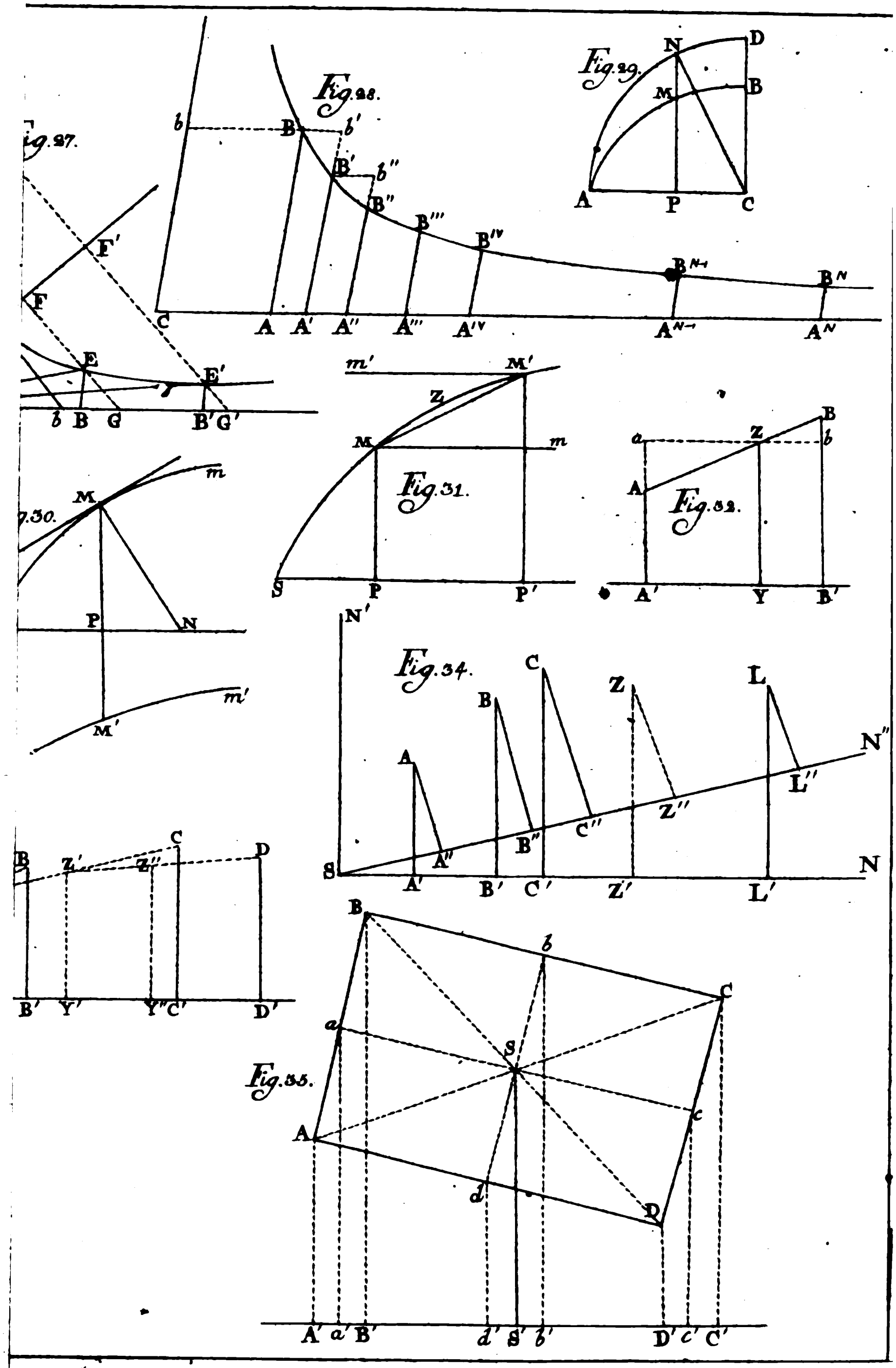
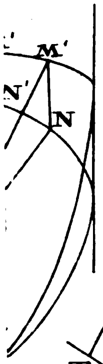


Fig. 27.
Fig. 28.
Fig. 29.
Fig. 30.
Fig. 31.
Fig. 32.
Fig. 34.
Fig. 35.

Tab. V.a



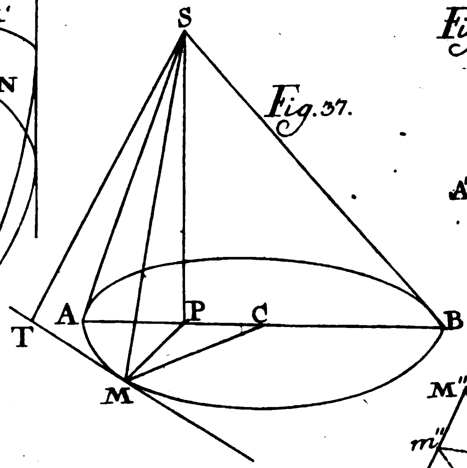


Fig. 37.

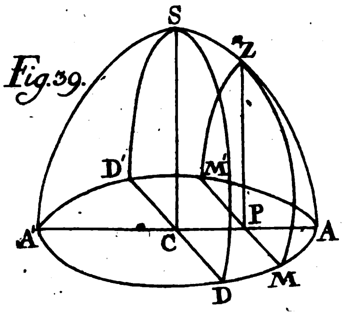


Fig. 39.

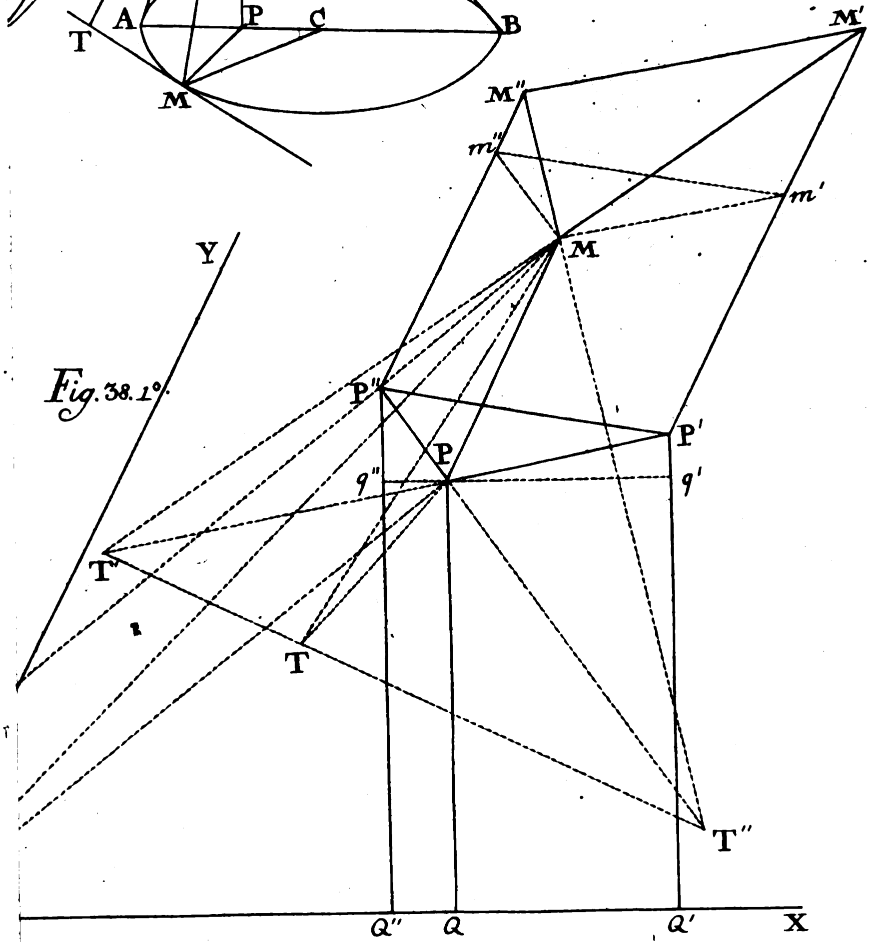
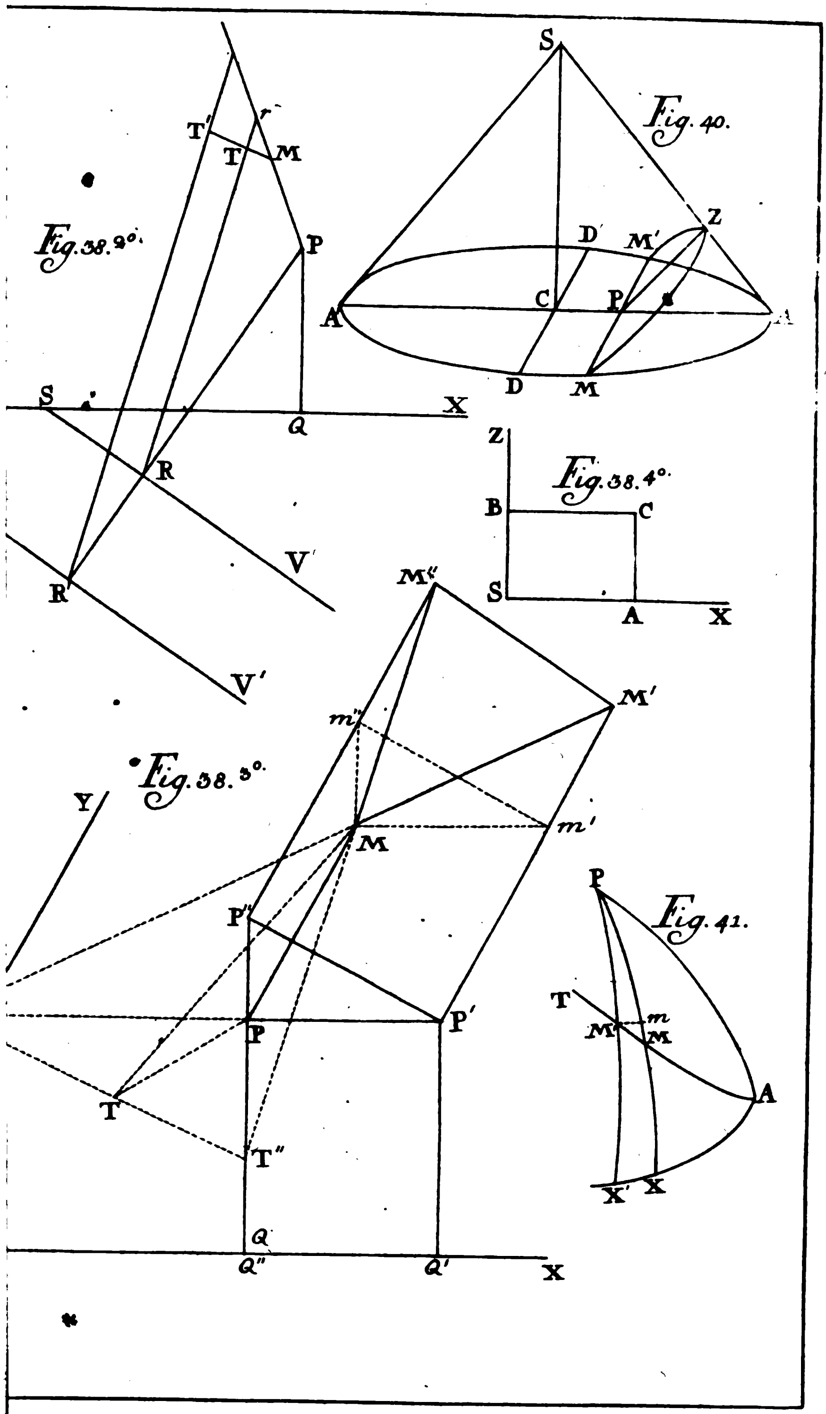
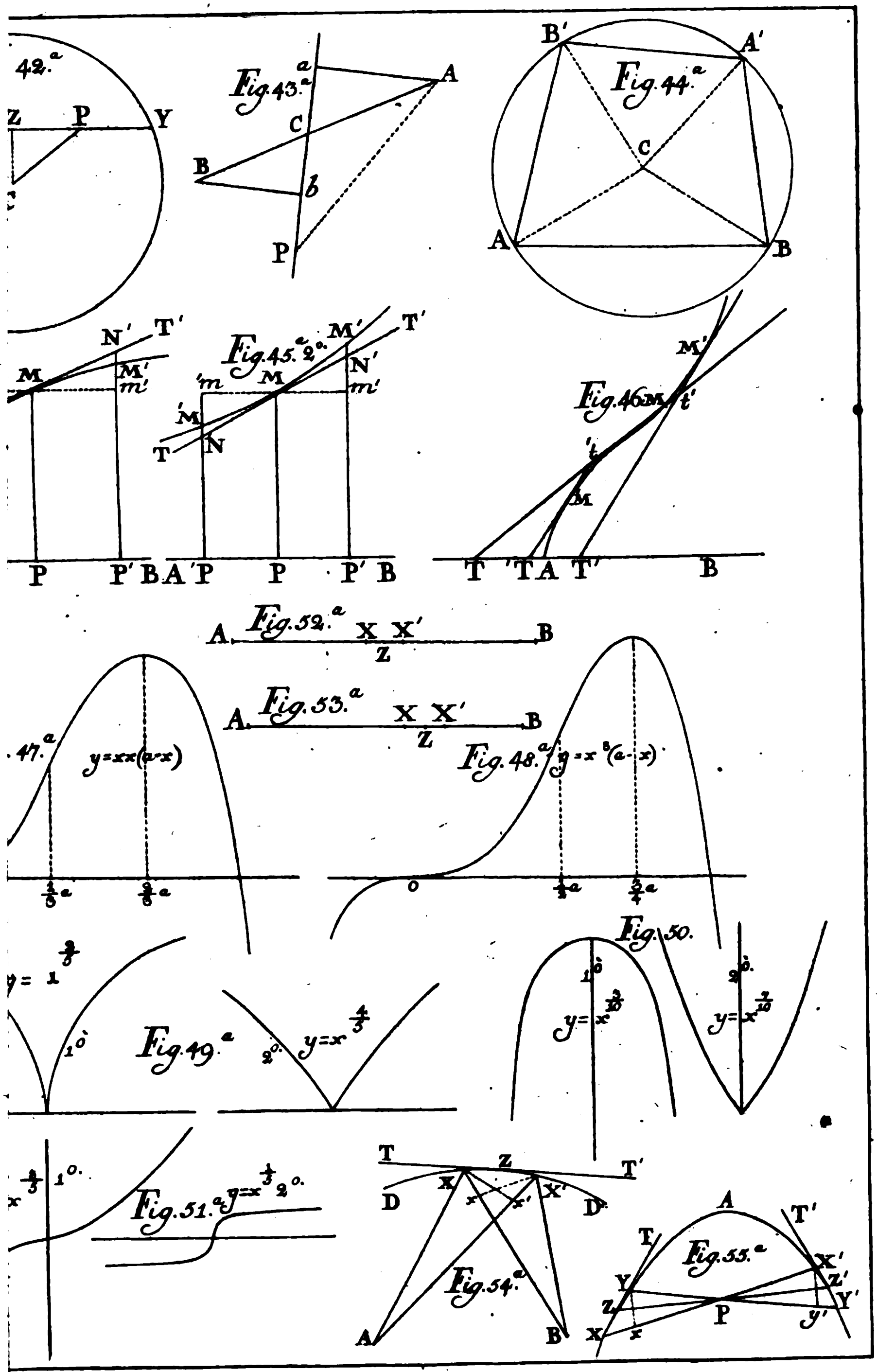


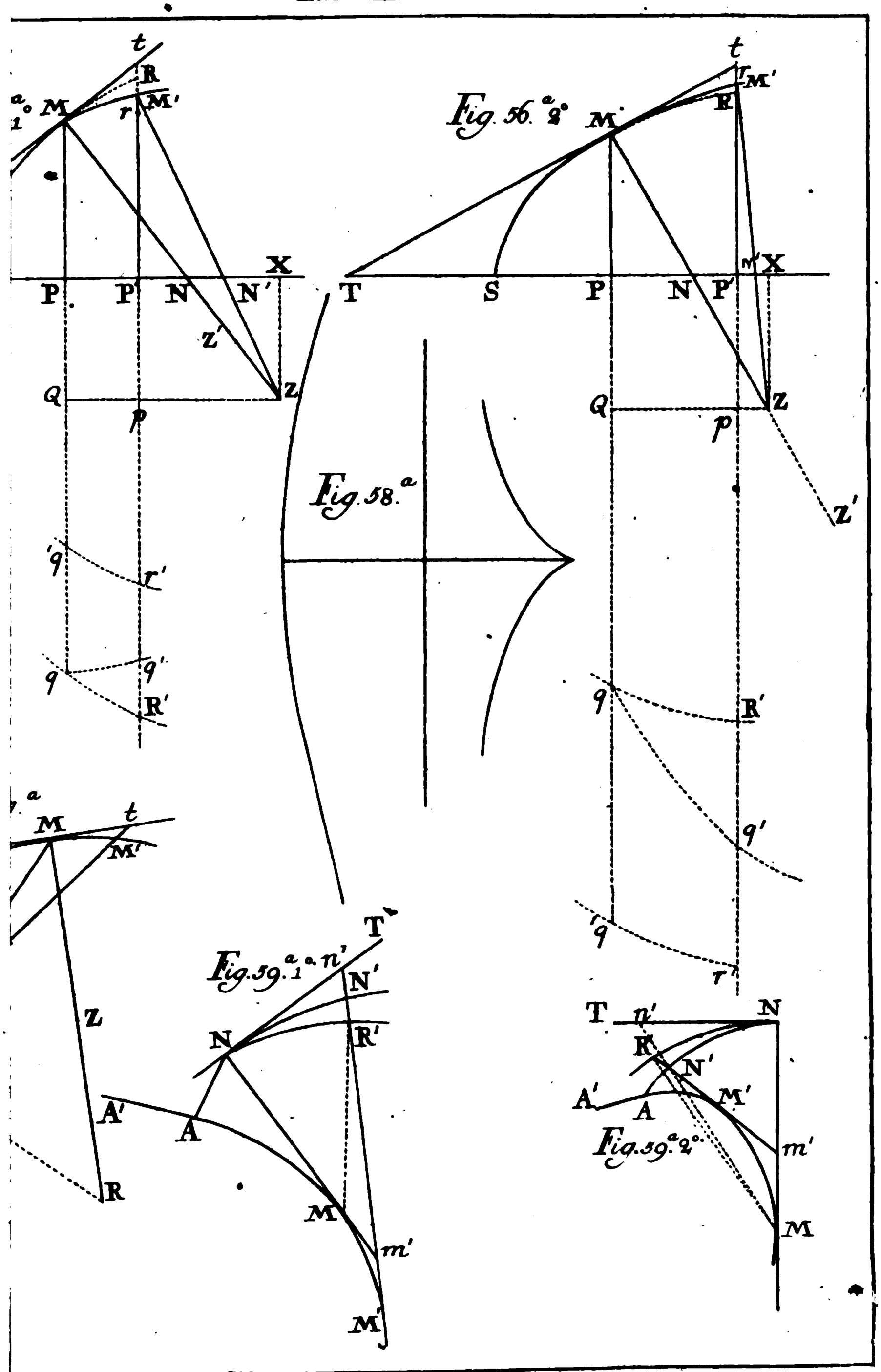
Fig. 38. 1°.

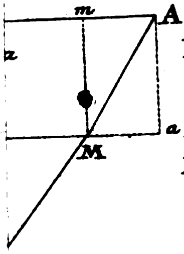
Tab. VI.a




42.a
Fig. 43.a
Fig. 44.a
Fig. 45.a 2o.
Fig. 46.
Fig. 47.a
y=xx(a-x)
Fig. 48.a
y=x³(a-x)
Fig. 49.a
y=x^{4/3}
Fig. 50.
Fig. 51.a
y=x^{1/3}
Fig. 52.a
Fig. 53.a
Fig. 54.a
Fig. 55.a

Tab. VIII.ª





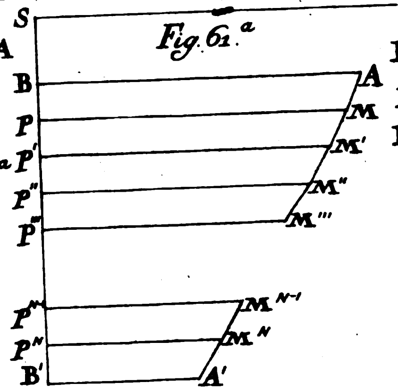


Fig. 61.ª

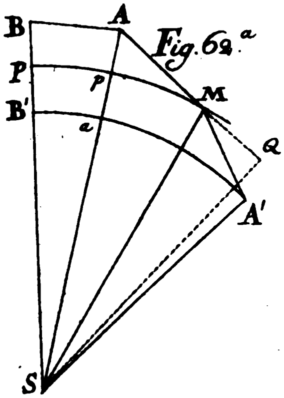
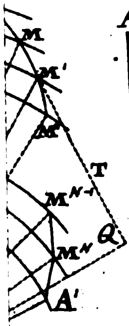


Fig. 62.ª



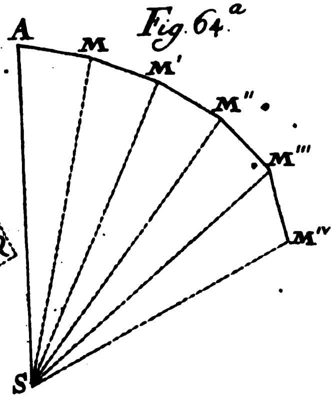


Fig. 64.ª

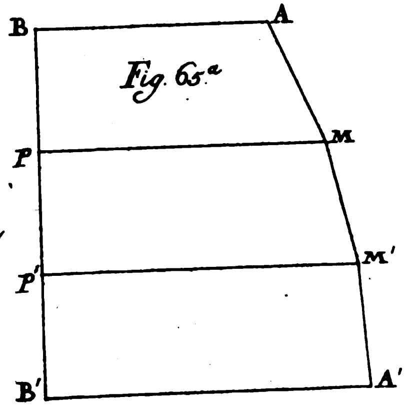


Fig. 65.ª

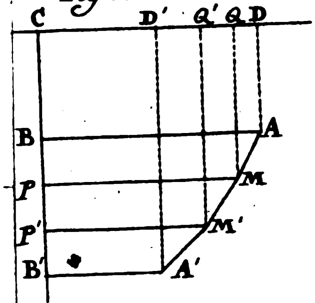


Fig. 66.ª

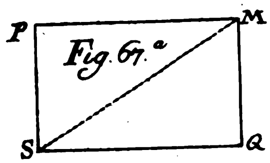


Fig. 67.ª

a-f.

eof—